From JIMWLK to CSS

Michael Lublinsky

Ben-Gurion University of the Negev, Israel

H. Duan, A. Kovner and M. Lublinsky, [arXiv:2407.15960 [hep-ph]], in PRD

High Energy Scattering

Target ($\rho^t = \rho^-;\ k^- > \Lambda$) **Projectile** ($\rho^p = \rho^+;\ k^+ > \Lambda$)

$$\langle \mathbf{T}| \quad \rightarrow \qquad\qquad \leftarrow \quad |\mathbf{P}\rangle \qquad\qquad \rho^+ \sim \int \mathbf{dk^+ a^\dagger\, T\, a}$$

S-matrix:

$$\mathbf{S(Y)} = \langle \mathbf{T} \langle \mathbf{P}|\ \hat{\mathbf{S}}(\rho^{\mathrm{t}},\ \rho^{\mathrm{p}})\ |\mathbf{P}\rangle\, \mathbf{T}\rangle \qquad\qquad \mathbf{Y} \sim \ln(\mathbf{s})$$

For any observable $\hat{\mathcal{O}}(\rho^{\mathrm{t}},\ \rho^{\mathrm{p}})$

$$\langle \hat{\mathcal{O}} \rangle_{\mathbf{Y}} = \langle \mathbf{T} \langle \mathbf{P}|\ \hat{\mathcal{O}}(\rho^{\mathrm{t}},\ \rho^{\mathrm{p}})\ |\mathbf{P}\rangle\, \mathbf{T}\rangle$$

The projectile perspective

Projectile averaged operators in Wigner-Weyl representation in terms of classical phase space variables.

$$\langle \mathbf{P}|\ \hat{\mathcal{O}}(\rho^{\mathrm{t}},\ \rho^{\mathrm{p}})\ |\mathbf{P}\rangle_{\mathbf{Y}}\ =\ \int \mathbf{D}\rho^{\mathrm{p}}\ \ \mathcal{O}(\rho^{\mathrm{t}},\ \rho^{\mathrm{p}})\ \ \mathbf{W}^{\mathbf{p}}_{\mathbf{Y}}[\rho^{\mathbf{p}}]$$

How do these averages change with increase in energy of the process?

Boosting the projectile $\quad |\mathbf{P}\rangle_{\mathbf{Y}}\ \rightarrow\ |\mathbf{P}\rangle_{\mathbf{Y}+\delta\mathbf{Y}}$

$$\partial_{\mathbf{Y}}\langle\mathbf{P}|\hat{\mathcal{O}}|\mathbf{P}\rangle_{\mathbf{Y}}\ =\ -\int \mathbf{D}\rho^{\mathrm{p}}\ \ \mathcal{O}(\rho^{\mathrm{t}},\ \rho^{\mathrm{p}})\ \mathcal{H}[\rho^{\mathrm{p}},\ \delta/\delta\rho^{\mathrm{p}}]\ \ \mathbf{W}^{\mathbf{p}}_{\mathbf{Y}}[\rho^{\mathbf{p}}]$$

$\mathcal{H}$ **defines the high energy limit of QCD and is universal**

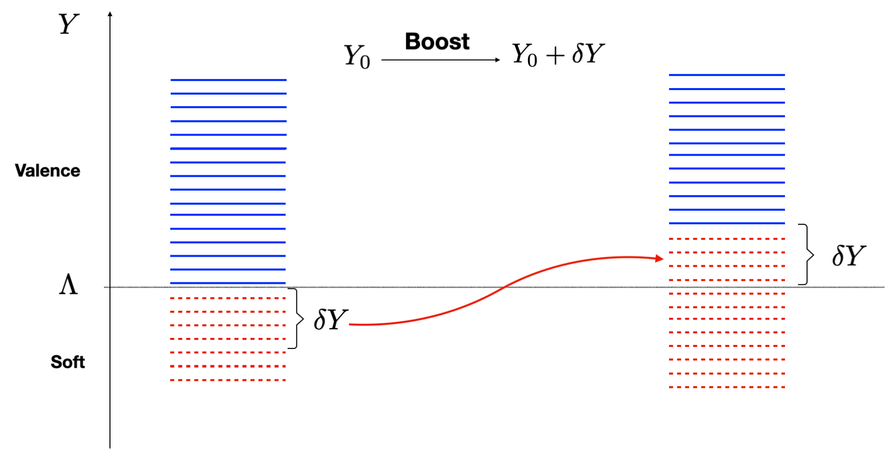
Dilute and Dense limits:

$$H^{KLWMIJ}\ =\ \mathcal{H}[\rho^{p}\ \rightarrow\ 0]\qquad \xleftrightarrow{DDD}\qquad H^{JIMWLK}\ =\ \mathcal{H}[\delta/\delta\rho^{p}\ \rightarrow\ 0]$$

Light Cone Wave Function in Born-Oppenheimer approximation

$$\mathbf{H}^{\mathrm{LC}}_{\mathrm{QCD}}\,|\Psi\rangle \;=\; \mathbf{E}\,|\Psi\rangle$$

BO: split the modes into hard and soft.
The hard (valence) modes with $k^+ \;>\; \Lambda$
They act as an external background current
$j^+_a \;=\; \delta(x^-)\,\rho^a$ for the soft modes.



$$\mathbf{H}^{\mathrm{LC}}_{\mathrm{QCD}} \;=\; \mathbf{H}[\rho,\,\mathbf{a},\,\mathbf{a}^\dagger] \;=\; \mathbf{H}_{\mathrm{V}}[\rho] \;+\; \mathbf{H}_{\mathrm{free}}[\mathbf{a},\,\mathbf{a}^\dagger] \;+\; \mathbf{H}_{\mathrm{int}}[\rho,\,\mathbf{a},\,\mathbf{a}^\dagger]$$

LCWF with no soft modes

$$|\Psi\rangle_{\mathrm{Y}_0} \equiv |\mathrm{Y}_0\rangle \;=\; |\rho\rangle_{\mathrm{valence}} \otimes |0_{\mathrm{a}}\rangle_{\mathrm{soft}}\,; \qquad \mathbf{H}_{\mathrm{V}}\,|\mathrm{Y}_0\rangle \;=\; \mathbf{E}_0\,|\mathrm{v}\rangle\,; \qquad \mathbf{a}\,|\mathrm{Y}_0\rangle \;=\; 0\,; \quad \mathbf{E}_0 \;=\; 0$$

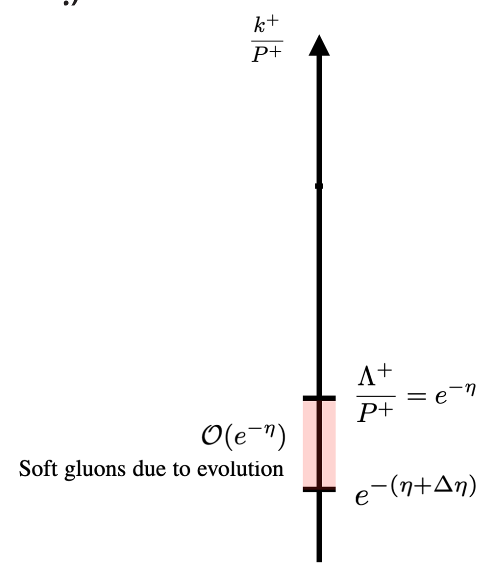
The evolved LCWF with soft gluon dressing

$$|\Psi\rangle_{\mathrm{Y}_0+\delta\mathrm{Y}} \;=\; \Omega_{\delta\mathrm{Y}}(\rho,\,\mathbf{a},\,\mathbf{a}^\dagger)\,|\Psi\rangle_{\mathrm{Y}_0}\,; \qquad\qquad \Omega^\dagger_{\delta\mathrm{Y}}\;(\mathbf{H}_{\mathrm{free}} + \mathbf{H}_{\mathrm{int}})\;\Omega_{\delta\mathrm{Y}} \;=\; \mathbf{H}_{\mathrm{diagonal}}$$

Given the evolution of the hadronic wave function one can calculate the evolution of an arbitrary observable $\hat{\mathcal{O}}[\rho]$ which depends on the TOTAL color charge density

The evolution of the expectation value

$$\frac{\mathbf{d}\,\langle\eta|\hat{\mathcal{O}}|\eta\rangle}{\mathbf{d}\eta} = \lim_{\boldsymbol{\Delta}\eta\to\mathbf{0}} \frac{\langle\eta|\boldsymbol{\Omega}^{\dagger}_{\boldsymbol{\Delta}\eta}\,\hat{\mathcal{O}}[\rho+\delta\rho]\,\boldsymbol{\Omega}_{\boldsymbol{\Delta}\eta}|\eta\rangle \;-\; \langle\eta|\hat{\mathcal{O}}[\rho]|\eta\rangle}{\boldsymbol{\Delta}\eta} = -\int \mathbf{D}\rho\,\mathbf{W}[\rho]\,\mathcal{H}[\rho]\;\mathcal{O}[\rho]$$



The charge density due to newly produced gluon

$$\delta\rho^{\mathbf{a}}(\mathbf{x}) \;=\; \rho^{\mathbf{a}}_{\mathbf{soft}}(\mathbf{x}) \;=\; \int_{\mathbf{e}^{-(\eta+\boldsymbol{\Delta}\eta)}\,\boldsymbol{\Lambda}^+}^{\mathbf{e}^{-\eta}\,\boldsymbol{\Lambda}^+} \mathbf{dk}^+\,\mathbf{a}^{\dagger\mathbf{b}}_{\mathbf{i}}(\mathbf{k}^+,\mathbf{x})\,\mathbf{T}^{\mathbf{a}}_{\mathbf{bc}}\,\mathbf{a}^{\mathbf{c}}_{\mathbf{i}}(\mathbf{k}^+,\mathbf{x})$$

The charge density shift operator

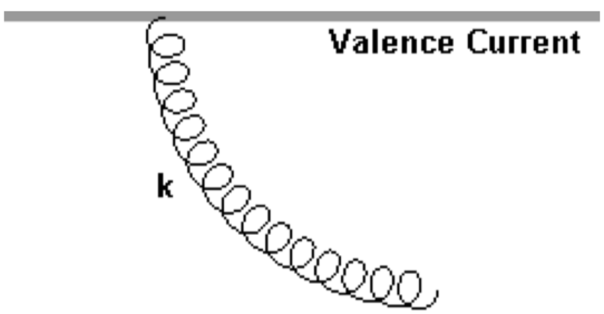
$$\hat{\mathbf{R}}_{\mathbf{a}} \;=\; \exp\left[\int \mathbf{d}^2\mathbf{z}\,\rho^{\mathbf{a}}_{\mathbf{soft}}(\mathbf{z})\,\frac{\delta}{\delta\rho^{\mathbf{a}}(\mathbf{z})}\right], \qquad\qquad \hat{\mathbf{R}}_{\mathbf{a}}\,\mathcal{O}[\rho] \;=\; \mathcal{O}[\rho\,+\,\rho_{\mathbf{soft}}]$$

$$\mathcal{H}\left[\rho,\frac{\delta}{\delta\rho}\right] \;=\; -\lim_{\boldsymbol{\Delta}\eta\to\mathbf{0}} \frac{\langle\mathbf{0_a}|\,\boldsymbol{\Omega}^{\dagger\,\mathbf{L}}_{\boldsymbol{\Delta}\eta}\;\hat{\mathbf{R}}_{\mathbf{a}}\;\boldsymbol{\Omega}^{\mathbf{R}}_{\boldsymbol{\Delta}\eta}\,|\mathbf{0_a}\rangle \;-\; \mathbf{1}}{\boldsymbol{\Delta}\eta}\,.$$

LCWF at LO

Eikonal coupling between valence and soft gluons due to separation of scales

$$\mathbf{H_{int}} = -\int \frac{dk^+}{2\pi}\frac{d^2k}{(2\pi)^2}\frac{g\,k_i}{\sqrt{2}\,|k^+|^{3/2}}\left[a_i^{\dagger a}(k^+,\,k)\;\rho^a(-k)\;+\;a_i^a(k^+,\,-k)\;\;\rho^a(k)\right]$$



A cloud of classical Weizsaker-Williams gluons dressing the valence ones

$$b_i^a(z) = \frac{g}{2\pi}\int d^2x\frac{(z-x)_i}{(z-x)^2}\;\;\rho^a(x)$$

$$\Omega_{\Delta\eta}(\rho\;\rightarrow\;0)\;\equiv\;\mathcal{C}_{\Delta\eta}$$

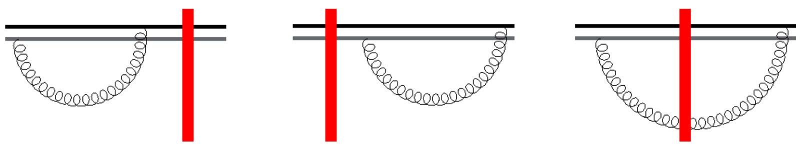
$$=\;\mathrm{Exp}\left\{i\int d^2z\,b_i^a(z)\int_{e^{-(\eta+\Delta\eta)}\Lambda^+}^{e^{-\eta}\,\Lambda^+}\frac{dk^+}{\pi^{1/2}|k^+|^{1/2}}\left[a_i^a(k^+,z)+a_i^{\dagger a}(k^+,z))\right]\right\}$$

KLWMIJ Hamiltonian (dilute limit)

$$\mathcal{H}[\rho \to 0;\ \Omega \ \to\ \mathcal{C}] \ \to\ \mathbf{H}^{\mathbf{KLWMIJ}} \ =\ \int_z [\mathbf{b}^a_{L\,i}(\mathbf{z})\,\mathbf{b}^a_{L\,i}(\mathbf{z}) + \mathbf{b}^a_{R\,i}(\mathbf{z})\,\mathbf{b}^a_{R\,i}(\mathbf{z}) - 2\mathbf{b}^a_{L\,i}(\mathbf{z})\,\mathbf{R}^{ab}_z\,\mathbf{b}^b_{R\,i}(\mathbf{z})]$$

Dual Wilson line (S-matrix of a projectile gluon scattering on a target)

$$\mathbf{R}_z \ =\ \mathbf{e}^{\tau(\mathbf{z})}\,; \qquad \tau(\mathbf{z}) \equiv \mathbf{T}^c \frac{\delta}{\delta\rho^c(\mathbf{z})}$$



The color field b_i^a

$$\mathbf{b}_{L,R} \ =\ \mathbf{b}[\rho_{L,R}]\,; \qquad \rho^a_{L,R} \ =\ \rho^b[\frac{\tau}{2}\coth\frac{\tau}{2} \pm \frac{\tau}{2}]^{ba}$$

$$\mathbf{b}^a_i(\mathbf{z}) \ =\ -\frac{\mathbf{i}}{\mathbf{g}}\,\mathbf{tr}[\mathbf{T}^a\,\mathbf{S}^\dagger(\mathbf{z})\,\partial^z_i\,\mathbf{S}(\mathbf{z})]$$

Wilson line (S-matrix of a target gluon scattering on the projectile)

$$\mathbf{S}(\mathbf{z}) \ \equiv\ \mathcal{P}\ \exp\left[\mathbf{ig}\int^z \mathbf{dy}_i\,\mathbf{b}^a_i(\mathbf{y})\,\mathbf{T}^a\right]$$

Gluon TMD

$$\mathbf{T}(\mathbf{k}, x; \zeta) = \langle\zeta|\, \hat{\mathbf{T}}(\mathbf{k}, x \equiv \mathbf{k}^+/\mathbf{P}^+)\, |\zeta\rangle\,; \qquad \hat{\mathbf{T}}(\mathbf{k}, x) = \int_{\mathbf{x},\mathbf{y}} \mathbf{e}^{\mathbf{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\, \mathbf{a}_{\mathbf{i}}^{\dagger\mathbf{a}}(\mathbf{k}^+, \mathbf{x})\, \mathbf{a}_{\mathbf{i}}^{\mathbf{a}}(\mathbf{k}^+, \mathbf{y})$$

ζ is a longitudinal momentum resolution: $P^+\, e^{-\zeta}$ is the minimal k^+ allowed in the wave-function ($\ln 1/x \, < \, \zeta$).

One might naively confuse $\mathbf{T}(\mathbf{k}, x; \zeta)$ with $\mathbf{T}_{\mathbf{WW}}(\mathbf{k}, \zeta) \equiv \mathbf{T}(\mathbf{k}, x = \mathbf{e}^{-\zeta}; \zeta)$.

In the dilute limit, $\mathbf{T}_{\mathbf{WW}}$ is the Weizsaker-Williams TMD,

$$\mathbf{T}_{\mathbf{WW}}(\mathbf{k}, \zeta) \;=\; \langle\zeta|\, \mathbf{b}_{\mathbf{i}}^{\mathbf{a}}(\mathbf{k})\mathbf{b}_{\mathbf{i}}^{\mathbf{a}}(\mathbf{k})\, |\zeta\rangle \;=\; \frac{\mathbf{1}}{\mathbf{g}^2} \int_{\mathbf{x},\mathbf{y}} \mathbf{e}^{\mathbf{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\, \langle\zeta|\, \mathbf{tr}[\mathbf{S}^\dagger(\mathbf{x})\, \partial_{\mathbf{i}}^{\mathbf{x}}\, \mathbf{S}(\mathbf{x})\, \mathbf{S}^\dagger(\mathbf{y})\, \partial_{\mathbf{i}}^{\mathbf{y}}\, \mathbf{S}(\mathbf{y})]\, |\zeta\rangle$$

$\mathbf{T}_{\mathbf{WW}}$ depends on the total color charge density ρ. Hence $\mathbf{T}_{\mathbf{WW}}$ evolves with $\mathbf{H}^{\mathbf{KLWMIJ}}$.

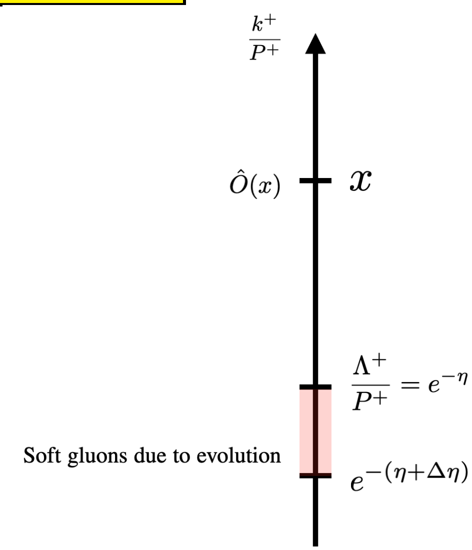
Q: Which quantities are we interested in? A: In many cases the gluon TMD $\mathrm{T}(\mathrm{k}, x; \zeta)$.

Q: How does such an object (a CSS-like observable) evolve with energy?

A: Because this observable depends not only on the total charge density, its evolution is Not given by $\mathrm{H}^{\mathrm{KLWMIJ}}$. We have to compute...

Energy evolution of CSS-like observables

What is the evolution of an observable $\hat{\mathrm{O}}(x)$ that involves gluons at fixed x (k^+) inside of the wave function, and not just at the very "edge" at η?



$$\frac{\mathbf{d}\langle\eta|\hat{\mathbf{O}}(x)|\eta\rangle}{\mathbf{d}\eta} = \lim_{\Delta\eta\to 0} \frac{\langle\eta|\Omega^\dagger_{\Delta\eta}\,\hat{\mathbf{O}}(x)\,\Omega_{\Delta\eta}|\eta\rangle \;-\; \langle\eta|\hat{\mathbf{O}}(x)|\eta\rangle}{\Delta\eta}$$

Since $\hat{\mathrm{O}}$ does not depend on the soft gluons, averaging over the soft modes is easy. When $\Omega = \mathcal{C}$,

$$\frac{\mathbf{d}\langle\eta|\hat{\mathbf{O}}(x)|\eta\rangle}{\mathbf{d}\eta} = \int_{\mathbf{z}}\langle\eta|\,[\mathbf{b}_{\mathrm{i}}^{\mathrm{a}}(\mathbf{z}),\ [\mathbf{b}_{\mathrm{i}}^{\mathrm{a}}(\mathbf{z}),\hat{\mathbf{O}}(x)]]\,|\eta\rangle$$

$$= -\frac{\mathbf{g}^2}{2\pi}\int_{\mathbf{u,v,z}}\frac{(\mathbf{u}-\mathbf{z})_{\mathrm{i}}(\mathbf{v}-\mathbf{z})_{\mathrm{i}}}{(\mathbf{u}-\mathbf{z})^2(\mathbf{v}-\mathbf{z})^2}\,\langle\eta|\,[\rho^{\mathbf{a}}(\mathbf{u}),\ [\rho^{\mathbf{a}}(\mathbf{v}),\ \hat{\mathbf{O}}(x)]]\,|\eta\rangle$$

We refer to this evolution as "CSS-JIMWLK".

This is a Lindblad equation for an open quantum system, where we follow only part of the degrees of freedom.

CSS-JIMWLK

From Hilbert to Wigner-Weyl:

$$\langle\eta|\,[\rho^{\mathbf{a}}(\mathbf{v}),\ \hat{\mathbf{O}}(x)]\,|\eta\rangle \ \longrightarrow \langle\,\left(\rho^{\mathbf{a}}_{\mathbf{L}}(\mathbf{v})\ -\ \rho^{\mathbf{a}}_{\mathbf{R}}(\mathbf{v})\right)\ \mathbf{O}(x)\,\rangle_\eta\ =\ \langle\,\left[\rho(\mathbf{v})\mathbf{T}^{\mathbf{a}}\frac{\delta}{\delta\rho(\mathbf{v})}\right]\ \mathbf{O}(x)\,\rangle_\eta$$

$\rho(\mathbf{v})\mathbf{T}^{\mathbf{a}}\frac{\delta}{\delta\rho(\mathbf{v})}$ **is the operator of color charge density rotation.**

The CSS-JIMWLK evolution can be represented in a formal way similar to JIMWLK - as generated by the "Hamiltonian"

$$\mathbf{H}_{\mathbf{CSS-JIMWLK}}\ =\ \int_{\mathbf{z}}\left[\mathbf{b}_{\mathbf{L}}(\mathbf{z})\ -\ \mathbf{b}_{\mathbf{R}}(\mathbf{z})\right]^2$$
$$=\ \frac{\mathbf{g}^2}{2\pi}\int_{\mathbf{x},\mathbf{y},\mathbf{z}}\frac{(\mathbf{x}-\mathbf{z})_{\mathbf{i}}(\mathbf{y}-\mathbf{z})_{\mathbf{i}}}{(\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{z})^2}\ \left[\rho(\mathbf{x})\mathbf{T}^{\mathbf{a}}\frac{\delta}{\delta\rho(\mathbf{x})}\right]\ \left[\rho(\mathbf{y})\mathbf{T}^{\mathbf{a}}\frac{\delta}{\delta\rho(\mathbf{y})}\right]$$

Emission of a gluon during the evolution rotates the color charge density.
For "CSS-JIMWLK" all that matters is how $\hat{O}$ changes due to this rotation, and not what the emitted gluon itself does.

Notice that the Hamiltonian is quadratic both in ρ and $\delta/\delta\rho$

Energy evolution of the Gluon TMD

$$\mathbf{T}(\mathbf{k},x;\zeta) = \langle\zeta|\,\hat{\mathbf{T}}(\mathbf{k},x)\,|\zeta\rangle\,; \qquad \hat{\mathbf{T}}(\mathbf{k},x) = \int_{\mathbf{x},\mathbf{y}} \mathbf{e}^{\mathbf{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\; \mathbf{a}_{\mathbf{i}}^{\dagger\mathbf{a}}(\mathbf{k}^+,\mathbf{x})\,\mathbf{a}_{\mathbf{i}}^{\mathbf{a}}(\mathbf{k}^+,\mathbf{y})$$

Consider $\zeta > \ln 1/x$. Evolve with $\mathbf{H}_{\mathbf{CSS-JIMWLK}}$ from $\eta_0 = \ln 1/x$ to $\eta = \zeta$

$$\frac{\mathbf{d}\mathbf{T}(\mathbf{k},x;\eta)}{\mathbf{d}\eta} = -\frac{\mathbf{g}^2}{2\pi}\int_{\mathbf{u},\mathbf{v},\mathbf{z}} \frac{(\mathbf{u}-\mathbf{z})_{\mathbf{i}}(\mathbf{v}-\mathbf{z})_{\mathbf{i}}}{(\mathbf{u}-\mathbf{z})^2(\mathbf{v}-\mathbf{z})^2}\,\langle\eta|\,[\hat{\rho}^{\mathbf{a}}(\mathbf{u}),\,[\hat{\rho}^{\mathbf{a}}(\mathbf{v}),\,\hat{\mathbf{T}}(\mathbf{k},x)]]\,|\eta\rangle$$

$$= -\frac{\alpha_{\mathbf{s}}\mathbf{N_c}}{\pi}\,\ln\frac{\mathbf{\Lambda}^2}{\mathbf{k}^2}\,\mathbf{T}(\mathbf{k},x;\eta)$$

This is exactly the (longitudinal) CSS equation with extra large transverse resolution scale $\mu^2 = \mathbf{\Lambda}^2$ -UV cutoff.

The solution yields the doubly logarithmic Sudakov suppression factor

$$\mathbf{T}(\mathbf{k},x;\zeta) = \mathbf{e}^{-\frac{\alpha_{\mathbf{s}}\mathbf{N_c}}{\pi}\ln\frac{\mathbf{\Lambda}^2}{\mathbf{k}^2}[\zeta-\ln 1/x]}\,\mathbf{T}(\mathbf{k},x;\ln 1/x) = \mathbf{e}^{-\frac{\alpha_{\mathbf{s}}\mathbf{N_c}}{\pi}\ln\frac{\mathbf{\Lambda}^2}{\mathbf{k}^2}[\zeta-\ln 1/x]}\,\mathbf{T}_{\mathbf{WW}}(\mathbf{k};\ln 1/x)$$

with the initial condition for $\mathbf{T}$ as the WW TMD.

Dense Projectile

In the dense regime: $\quad \Omega(\rho \sim 1/\alpha_s) = \mathcal{C}\,\mathrm{B}$ $\qquad$ $\mathbf{B}$ is a Bogolyubov operator

$$\mathbf{B} = \exp[\,\boldsymbol{\Lambda}(\rho)\,(\mathbf{a}^2 + \mathbf{a}^{\dagger\,2}) + \cdots]$$

$\mathbf{B}$ defines quasiparticles above the WW background

$$\mathbf{H}^{\mathbf{dense}}_{\mathbf{CSS-JIMWLK}} = -\lim_{\Delta\eta\to 0}\frac{\langle \mathbf{0_a}|\,\boldsymbol{\Omega}^{\dagger\,\mathbf{L}}_{\Delta\eta}\;\boldsymbol{\Omega}^{\mathbf{R}}_{\Delta\eta}\,|\mathbf{0_a}\rangle - \mathbf{1}}{\boldsymbol{\Delta}\eta}\,\Big|_{\frac{\delta}{\delta\rho}\to\mathbf{0}}\,.$$

We find that $\mathrm{H}^{\mathrm{dense}}_{\mathrm{CSS-JIMWLK}} = \mathrm{H}^{\mathrm{dilute}}_{\mathrm{CSS-JIMWLK}}$. Hence the solution (Sudakov form factor) is the same.

Only the initial condition changes

$$\mathbf{T}(\mathbf{k}, x; \zeta = \ln \mathbf{1}/x) = \int \mathbf{e}^{\mathbf{ik}\cdot(\mathbf{x}-\mathbf{y})}\langle\zeta|\,\mathbf{b_i}\,\mathbf{N}^{\dagger}_{\perp}\;\mathbf{N}_{\perp}\mathbf{b_i}\,(\mathbf{x},\mathbf{y})\,|\zeta\rangle \neq \mathbf{T}_{\mathbf{WW}}$$

Summary/Outlook

- One can evolve observables at fixed x to high energy (or to high longitudinal resolution). But not with $\mathrm{H}^{\mathrm{KLWMIJ/JIMWLK}}$ but rather with $\mathrm{H}_{\mathrm{CSS-JIMWLK}}$.

- $\mathrm{H}_{\mathrm{CSS-JIMWLK}}$ is universal in the same sense as JIMWLK- it is applicable not only to TMD, but to any observable that depends on gluonic degrees of freedom at fixed x . It is even more universal - it has the same form for dilute and dense projectile.

- The BFKL/KLWMIJ cascade allows emission of gluons with arbitrarily high transverse momentum - therefore the transverse resolution scale in the CSS-JIMWLK evolution is UV cutoff. This is unwelcome, but unavoidable in the k^+ scheme.

  For applications it means that we either have to "cheat" and take the UV cutoff as the hard scale in the physical process (if there is one), or be prepared that the Sudakov form factor will be split between the TMD calculated in the wave function and the "hard part".

S-matrix

S-matrix operator:

$$\hat{\mathbf{S}}(\rho^{\mathrm{t}},\,\rho^{\mathrm{p}}) = \mathrm{Exp}\left[\int_{\mathbf{x}} \rho^{\mathrm{p}}(\mathbf{x})\,\alpha^{\mathrm{t}}(\mathbf{x})\right]$$

Projectile averaged S-matrix:

$$\mathbf{\Sigma}^{\mathbf{p}}_{\mathbf{Y}-\mathbf{Y_0}}(\rho^{\mathrm{t}}) \;=\; \langle\,\mathbf{P}|\;\hat{\mathbf{S}}(\rho^{\mathrm{t}},\,\rho^{\mathrm{p}})\;|\mathbf{P}\rangle \;=\; \int \mathbf{D}\rho^{\mathrm{p}}\;\;\hat{\mathbf{S}}(\rho^{\mathrm{t}},\,\rho^{\mathrm{p}})\;\;\mathbf{W}^{\mathbf{p}}_{\mathbf{Y}-\mathbf{Y_0}}[\rho^{\mathrm{p}}]$$

S-matrix:

$$S(Y) \;=\; \int D\rho^{t}\;\;\Sigma^{p}_{Y-Y_0}[\rho^{t}]\;\;W^{t}_{Y_0}[\rho^{t}]$$

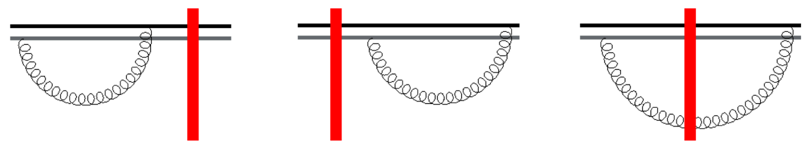
S-matrix evolution:

$$\frac{dS}{dY} \;=\; \int D\rho^{t}\;\;\Sigma^{p}_{Y-Y_0}[\rho^{t}]\;\;\mathcal{H}[\rho^{t}]\;\;W^{t}_{Y_0}[\rho^{t}]$$

LO JIMWLK Hamiltonian

$$\mathcal{H}^{\mathrm{JIMWLK}}_{\mathrm{LO}} = \int_{x,y,z} \mathrm{K}_{\mathrm{LO}} \left\{ J^a_L(x) J^a_L(y) + J^a_R(x) J^a_R(y) - 2 J^a_L(x) S^{ab}_A(z) J^b_R(y) \right\}$$

$$\mathrm{K}_{\mathrm{LO}}(x,y,z) = \frac{\alpha_s}{2\,\pi^2} \frac{(x-z)_i (y-z)_i}{(x-z)^2 (y-z)^2} \equiv \frac{\alpha_s}{2\,\pi^2} \frac{X_i Y_i}{X^2 Y^2}$$



$$S_A{}^{cd}(z) = \mathcal{P} \exp \left\{ i \int dx^+ \, T^a \, \alpha^a_t(z, x^+) \right\}^{cd} .$$

Here $\rho^p \to J_L$ and $\hat{S}\rho^p \to J_R$ are left and right SU(N) generators:

$$J^a_L(x) S^{ij}_A(z) = (T^a S_A(z))^{ij} \, \delta^2(x-z) \qquad J^a_R(x) S^{ij}_A(z) = (S_A(z) T^a)^{ij} \, \delta^2(x-z)$$

JIMWLK is valid for dilute-on-dense collisions only ($Q^P_s \ll Q^T_s$)