On the N3LO contributions to the PDF scale evolution

Giulio Falcioni, in collaboration with
F. Herzog, **S. Moch**, **A. Pelloni**, **A. Vogt**

Resummation, Evolution, Factorization - 13th Oct 2025

The precision era at the LHC

Percent-level precision for LHC physics $\to$ N^3LO QCD corrections
E.g. Neutral current Drell-Yan [Duhr,Dulat,Mistlberger 2020] @ 100 GeV

K_{N^3LO}	δ(scale)	$\delta(\mathrm{PDF} + \alpha_s)$	δ(PDF-TH)
-2.1%	$^{+0.66\%}_{-0.79\%}$	$^{+1.8\%}_{-1.9\%}$	$\pm 2.5\%$

Unwanted uncertainty

δ(PDF-TH) = performing N^3LO calculations with NNLO PDFs [Anastasiou et al. 2016]

$$\delta(\text{PDF-TH}) = \frac{1}{2}\left|\frac{\sigma^{\text{NNLO}}(\text{NNLO PDF}) - \sigma^{\text{NNLO}}(\text{NLO PDF})}{\sigma^{\text{NNLO}}(\text{NNLO PDF})}\right|$$

What do we need to get **N^3LO PDFs**?

Scale evolution of the PDFs

The DGLAP equations control the evolution of the PDFs

$$\mu^2 \frac{d}{d\mu^2} \begin{pmatrix} f_{\mathrm{ns},(i)} \\ f_q \\ f_g \end{pmatrix} (x, \mu^2) = \begin{pmatrix} P_{\mathrm{ns},(i)} & 0 & 0 \\ 0 & P_{qq} & P_{qg} \\ 0 & P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} f_{\mathrm{ns},(i)} \\ f_q \\ f_g \end{pmatrix}$$

Perturbative information

$$P_{\mathrm{ij}}(\alpha_s, x) = \underbrace{a\, P_{\mathrm{ij}}^{(0)}}_{\mathrm{LO}} + \underbrace{a^2\, P_{\mathrm{ij}}^{(1)}}_{\mathrm{NLO}} + \underbrace{a^3\, P_{\mathrm{ij}}^{(2)}}_{\mathrm{NNLO}} + \underbrace{a^4\, P_{\mathrm{ij}}^{(3)}}_{\mathrm{N^3LO}}, \; a = \frac{\alpha_s}{4\pi}$$

Different behaviour according to **flavour** decomposition.

Non-singlet sector

Evolution of flavour-dependent combinations of PDFs

$$f^{ik}_{\mathrm{ns},(\pm)} = (f_i \pm f_{\bar{i}}) - (f_k \pm f_{\bar{k}}), \quad f_{\mathrm{ns},(V)} = \sum_i (f_i - f_{\bar{i}}).$$

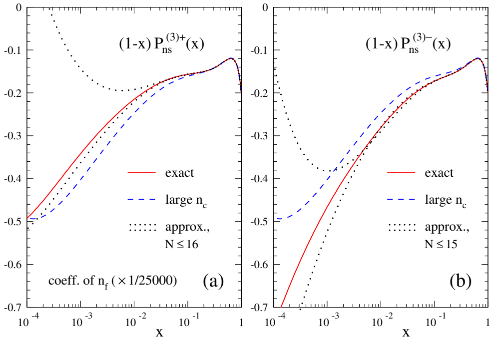
- $P^{(3)}_{\mathrm{ns}}$ in the **planar limit** [Moch,Ruijl,Ueda,Vermaseren,Vogt 2017]
- Large-n_f limit: leading terms [Gracey 1994-1998], terms $\mathcal{O}(n_f^2)$ [Davies,Vogt,Ruijl,Ueda,Vermaseren 2016] contributions
- Terms $\mathcal{O}(n_f C_F^3)$ [Gehrmann,von Manteuffel,Sotnikov,Yang 2023]

Complete n_f contribution

Complete terms $\mathcal{O}(n_f)$ [Kniehl,Moch,Velizhanin,Vogt 2025]

Non-singlet: exact results vs approximations

Approximation strategy: compute **fixed moments** of the splitting functions [Moch,Ruijl,Ueda,Vermaseren,Vogt 2017]



Plot from [Kniehl,Moch,Velizhanin,Vogt 2025]

- Moments up to $N = 16$ available
- 1%-level precise evolution $x \gtrsim 10^{-4}$
- Comparison with exact result at $\mathcal{O}(n_f)$ now possible

Singlet sector

Less information on the singlet splitting functions: $\begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix}$

- Large-n_f limit: leading terms $\mathcal{O}(n_f^3)$ [Gracey 1996;+Bennett 1998] completed in [Davies,Vogt,Ruijl,Ueda,Vermaseren 2016]
- Terms $\mathcal{O}(n_f^2)$ in $P_{qq}^{(3)}$ [Gehrmann,von Manteuffel,Sotnikov,Yang 2023]
- Terms $\mathcal{O}(n_f^2)$ in $P_{gq}^{(3)}$ [**GF**,Herzog,Moch,Vermaseren,Vogt 2023]

Information from fixed moments

[Moch,Ruijl,Ueda,Vermaseren,Vogt 2023]: 6 moments of $P_{qq}^{(3)}$ and 5 moments of $P_{ij}^{(3)}$, with $i/j = g$. **Not enough** for percent precision!

Moment space

The evolution of the **moments** of the PDFs is determined by

$$\gamma_{\mathrm{ij}}^{(k)}(N) = -\int_0^1 dx\, x^{N-1}\, P_{\mathrm{ij}}^{(k)}(x)$$

Computational approaches

- Expansion of the DIS structure functions

  On-shell approach: theoretically very clean, computationally more challenging

  - Successfull at NNLO [Moch,Vermaseren,Vogt 2004]
  - $\gamma_{qq}^{(3)}(N \leq 12)$ and $\gamma_{ij}^{(3)}(N \leq 10)$ [Moch,Ruijl,Ueda,Vermaseren,Vogt 2023]

Moment space

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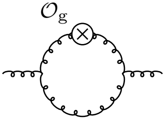
Computational approaches

- Operator Product Expansion (OPE)

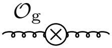
  Based on **off-shell** renormalisation: computationally simpler. Theoretical pitfalls

  - Local operator counterterms [**GF**,Herzog 2022]
  - All-N counterterms [Gehrmann,von Manteuffel,Yang 2023-2024]
  - Testing their agreement [**GF**,Herzog,Moch,Van Thurenhout 2024]

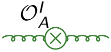
Computation in a nutshell



$$+ .. = -\frac{a}{\epsilon}\left(\gamma_3 + \gamma_{gg}^{(0)}\right)$$



$$-\frac{a}{\epsilon}\,\eta^{(0)}$$



- γ_{ij} are anomalous dimensions of gauge invariant operators
  - γ_{ij} extracted from the **poles** of 2-point functions
- Diagram generation `QGRAF` [Nogueira 1993] and processing in `FORM` [Vermaseren 2000;+Tentyuokov 2007;+Kuipers,Ueda,Vollinga 2012]
- Reduction to master integrals [Baikov,Chetyrkin 2010; Lee,Smirnov,Smirnov 2012] in `FORCER` [Ruijl,Ueda,Vermaseren 2017]
- Care required with **unphysical contributions**

Results - fixed moments

"Top row" $\gamma_{\mathrm{qq}}^{(3)}$ and $\gamma_{\mathrm{qg}}^{(3)}$

- $\gamma_{qq}^{(3)}(N \leq 20)$ 2302.07593 - [**GF**,Herzog,Moch,Vogt]
- $\gamma_{qg}^{(3)}(N \leq 20)$ 2307.04158 - [**GF**,Herzog,Moch,Vogt]

"Bottom row" $\gamma_{\mathrm{gq}}^{(3)}$ and $\gamma_{\mathrm{gg}}^{(3)}$

- $\gamma_{gq}^{(3)}(N \leq 20)$ 2404.09701 - [**GF**,Herzog,Moch,Pelloni,Vogt]
- $\gamma_{gg}^{(3)}(N \leq 20)$ 2410.08089 - [**GF**,Herzog,Moch,Pelloni,Vogt]

$$\gamma_{\mathrm{gg}}^{(3)}(N=2) = 654.463 n_f - 245.611 n_f^2 + 0.924991 n_f^3,$$

$$\dots$$

$$\gamma_{\mathrm{gg}}^{(3)}(N=20) = 90499.3 - 26132.3 n_f + 1178.50 n_f^2 + 25.6433 n_f^3.$$

Checks

- Correct limit at large-n_f [Gracey 1994,1998; Davies,Ruijl,Ueda,Vermaseren,Vogt 2016].
- Agreement with the known quartic Casimir contributions [Moch,Ruijl,Ueda,Vermaseren,Vogt 2018].
- ζ_4 terms predicted by the no-π^2 theorem [Jamin,Miravitllas 2018; Baikov,Chetyrkin 2018; Kotikov,Teber 2019].
- Agreement with published low-N moments, $\gamma^{(3)}_{\mathrm{qq}}(N \leq 12)$ and $\gamma^{(3)}_{\mathrm{qg,gq,gg}}(N \leq 10)$, obtained from the expansion of the DIS structure functions [Moch,Ruijl,Ueda,Vermaseren,Vogt 2021-2023]
- Agreement with the terms $\gamma^{(3)}_{\mathrm{qq}}\Big|_{n_f^2}$ computed exactly for *every* value of N [Gehrmann,Sotnikov,von Manteuffel,Yang 2023].

Beyond fixed moments

Analytic reconstruction

Assume the **function space** for the splitting functions

- Harmonic Polylogs (x-space) $\Rightarrow$ Harmonic Sums (N-space)

$$\gamma_{ij}^{(3)}(N) = c_{0,0} + \frac{c_{0,0}}{N} + \frac{c_{0,1}}{N+1} + \dots + c_{1,0} + c_{1,0}\frac{S_1}{N} + \dots$$

- Use moments to fix the coefficients $\rightarrow$ **Diophantine** system!

At four loops $\mathcal{O}(10^3)$ coefficients*! Only possible for

- Coefficients ζ_5 [GF,Herzog,Moch,Pelloni,Vogt 2024] and ζ_3 $\rightarrow$ reduce weight in S-sums
- Selected colour factors, e.g. $P_{gq}^{(3)}\Big|_{n_f^2}$ $\rightarrow$ simpler analytic structure. [GF,Herzog,Moch,Vermaseren,Vogt 2023]

Approximation setup

Construct an ansatz directly for $P_{ij}^{(3)}(x)$, i.e. in x-space, such that

- The first 10 moments **match** the results
- Has correct behaviour at the endpoints $P_{ij}(x \to 0)$ and $P_{ij}(x \to 1)$

Many important contributions to aN^{3}LO phenomenology

- MSHT collaboration [McGowan,Cridge,Harland-Lang,Thorne 2022-2025]
- NNPDF collaboration [Ball et al. 2024]
- Results from `xFitter` **Talk by F. Giuli later today**
- Work in progress by CTEQ
- Benchmarking [Cooper-Sarkar et al. 2024]

Example: $P_{\mathrm{ps}} = P_{qq} - P_{\mathrm{ns},(+)}$

Endpoint logarithms: $L_0 = \log x$, $L_1 = \log(1-x)$

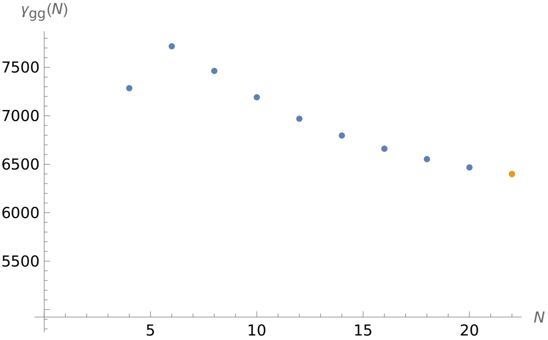
$$P_{\mathrm{ps}}^{(3)}(x \to 0) \simeq \underbrace{E_1 \frac{L_0^2}{x}}_{[1]} + E_2 \frac{L_0}{x} + \frac{E_3}{x} + \underbrace{F_1 L_0^6 + F_2 L_0^5 + F_3 L_0^4}_{[2]} + \cdots + F_6 L_0$$

$$P_{\mathrm{ps}}^{(3)}(x \to 1) \simeq (1-x)\left[\underbrace{G_1 L_1^4 + G_2 L_1^3}_{[3]} + G_3 L_1^2 + G_4 L_1\right]$$

[1] = [Catani,Hautmann 1994], [2] = [Davies,Kom,Moch,Vogt 2022],
[3] = [Soar,Moch,Vermaseren,Vogt 2009]

Construct 10-parameter ansatz including the 7 unknown coefficients
$E_2, E_3, F_4, F_5, F_6, G_3, G_4$ + 3-parameter trial functions.
The envelope of the curves estimates the **uncertainties**.

Applications: predict the next moment



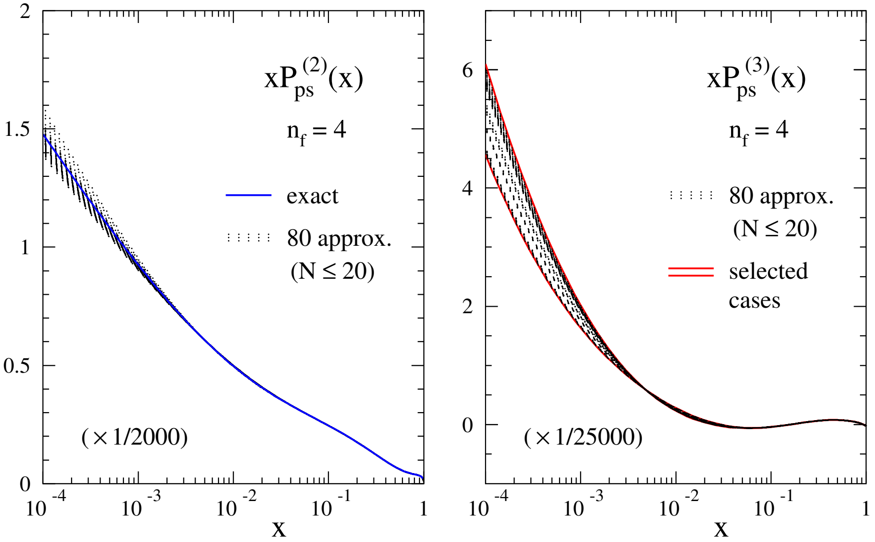
Error bars are not visible [GF,Herzog,Moch,Pelloni,Vogt 2024]

$$\gamma_{gg}^{(3)}(N=22)\,\big|_{n_f=4} = 6398.872080 \pm 20\cdot 10^{-6}$$

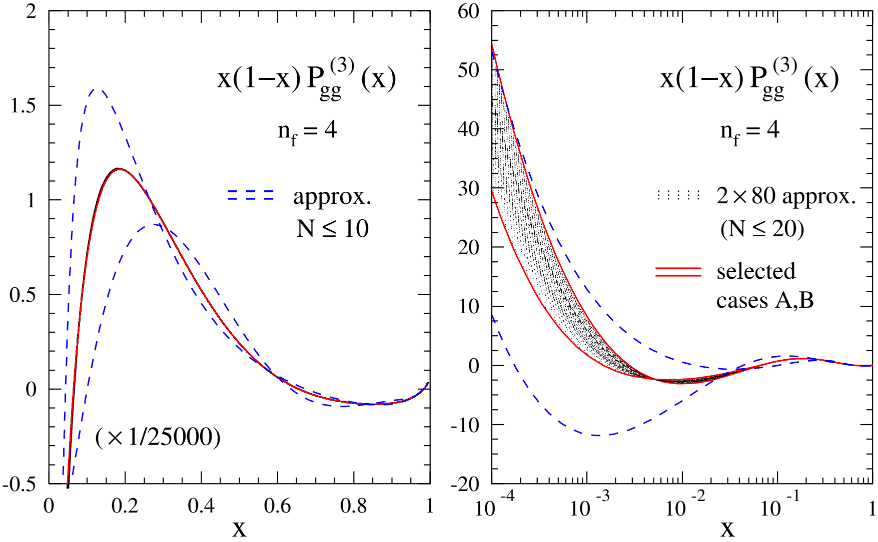
Check against **analytic** results [GF,Herzog,Moch,Pelloni,Vogt To appear]

$$\gamma_{gg}^{(3)}(N=22) = 93601.7 - 27096.9 n_f + 1218.73 n_f^2 + 26.3290 n_f^3$$
$$\to \gamma_{gg}^{(3)}(N=22)\,\big|_{n_f=4} = 6398.872078$$

Approximate quark-quark splitting: NNLO vs N^3LO



Approximate $P_{gg}^{(3)}(x)$: 5 moments vs 10 moments



Uncertainties at low-x

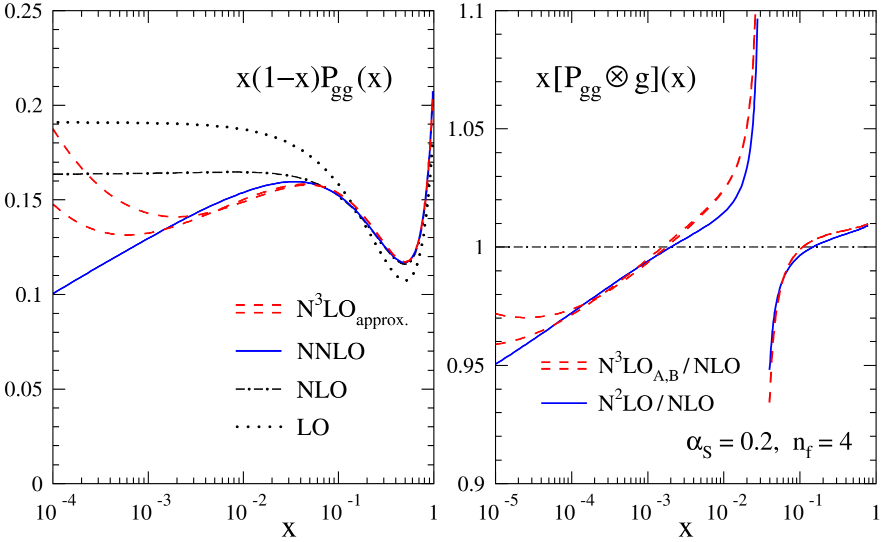
Can we rely on the approximations for PDF evolution?

$$\mu^2 \frac{d}{d\mu^2} f_i = P_{ij} \otimes f_j = \sum_j \int_x^1 \frac{dz}{z} \underbrace{P_{ij}(z)}_{\text{approx.}} f_j\left(\frac{x}{z}\right), \quad j = q, g$$

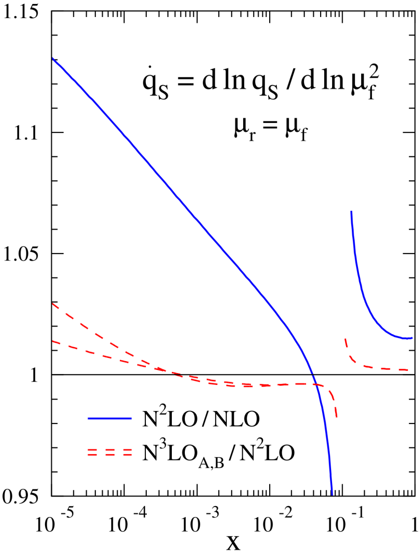
- $P_{ij}^{(3)}(z)$ have **large uncertainties** for $z \to x \ll 1$
- PDFs $f_j\left(\frac{x}{z}\right)$ **suppress** the contribution for $\frac{x}{z} \to 1$
  [Moch,Vermaseren,Vogt 2004]

$$\begin{aligned} x\,g(x) &= 1.6\,x^{-0.3}\,(\mathbf{1}-\mathbf{x})^{\mathbf{4.5}}\,(1-0.6x^{0.3}), \\ x\,q_{\rm s}(x) &= 0.6\,x^{-0.3}\,(\mathbf{1}-\mathbf{x})^{\mathbf{3.5}}\,(1+5.0x^{0.8}) \end{aligned}$$

Example: $P_{gg} \otimes g$



Evolution of the singlet PDFs

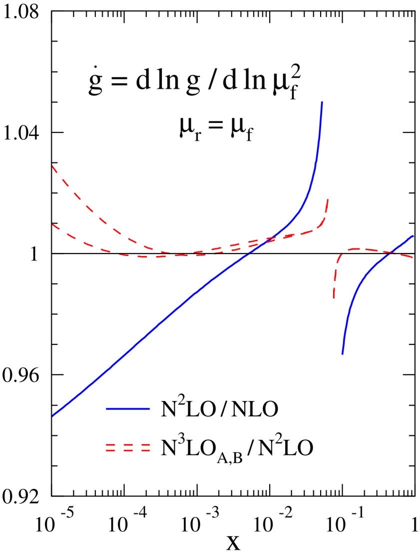


Evolution of the quark (S) PDF

$$\mu^2 \frac{d}{d\mu^2} f_q = P_{qq} \otimes f_q + P_{qg} \otimes f_g$$

- N^3LO effect is **small** compared to NNLO
  - **subpercent** for $x > 10^{-4}$
  - $\left.\frac{\dot{q}_s^{N^3LO}}{\dot{q}_s^{NNLO}}\right|_{x=10^{-5}} \sim (2 \pm 1)\%$
- Uncertainties
  - **Negligible** for $x \gtrsim 10^{-4}$
  - $\delta \dot{q}_s = \pm 1\%$ at $x = 10^{-5}$

Evolution of the singlet PDFs



Evolution of the gluon PDF

$$\mu^2 \frac{d}{d\mu^2} f_g = P_{gq} \otimes f_q + P_{gg} \otimes f_g$$

- NNLO effects already small. Same picture at N^{3}LO
  - **subpercent** for $x > 10^{-4}$
  - $\left.\frac{\dot{g}^{\mathrm{N^3LO}}}{\dot{g}^{\mathrm{NNLO}}}\right|_{x=10^{-5}} \sim (2 \pm 1)\%$
- Uncertainties
  - **Negligible** for $x \gtrsim 10^{-4}$
  - $\delta \dot{g} = \pm 1\%$ at $x = 10^{-5}$

Conclusion

- The OPE provides an efficient method to compute the moments of the DGLAP splitting functions to N^3LO.
- Analytic results were obtained through the moment $N = 20$ for all the splitting functions.
- The computed moments were employed to **approximate** the scale evolution of the PDFs to N^3LO.
- The precision of the approximation becomes worse at low-x, but it is estimated of $\mathcal{O}(1\%)$ or better for $x \gtrsim 10^{-4}$
  - **High precision across the LHC kinematics**

Outlook

- Validate the approximation at higher values of N.
- Investigation of the small-x region, where errors are larger
  - Higher-order BFKL logarithms are needed!

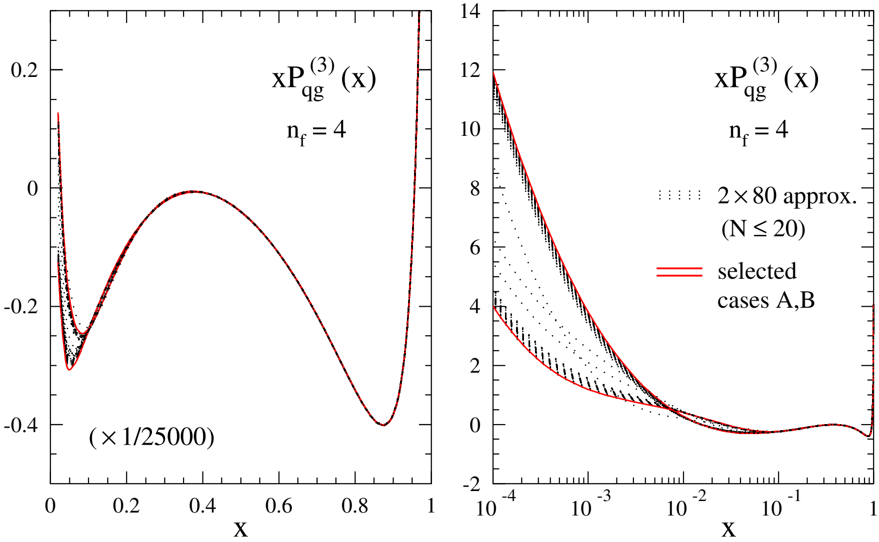
Can one get the *exact* results? Many developments by several groups

- New approaches
  - **See the talk by Y. J. Zhu**
  - Euclidean correlation functions [Cheng,Huang,Li,Li,Ma 2024]
- Progress in all-N analytic calculations [Gehrmann,von Manteuffel,Sotnikov,Yang 2024]
- Insight into the structures that appear for general values of N
  - Highly constraining ansätze [Kniehl,Moch,Velizhanin,Vogt 2025]
  - Progress in understanding the all-N structure of the operators [**GF**,Herzog,Moch,Van Thurenhout 2024]

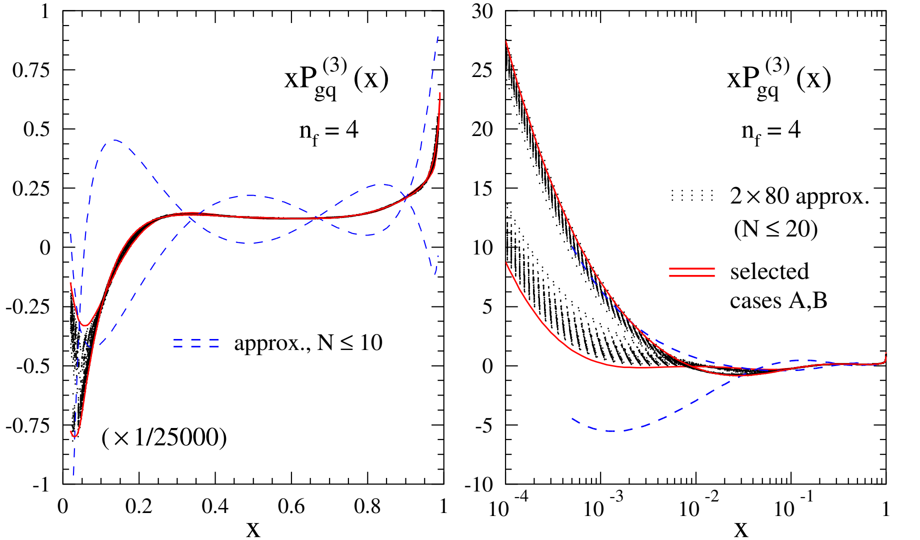
Thank you!

Backup slides

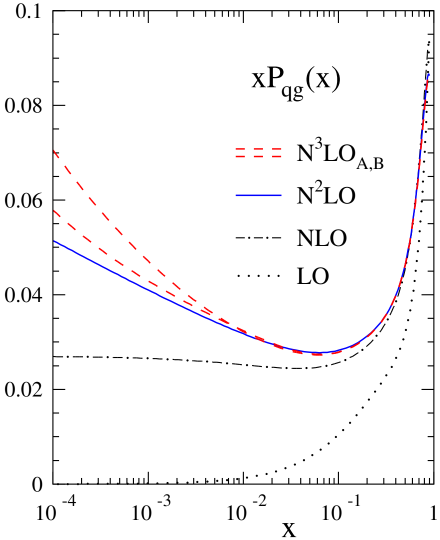
Approximate $P_{qg}^{(3)}(x)$

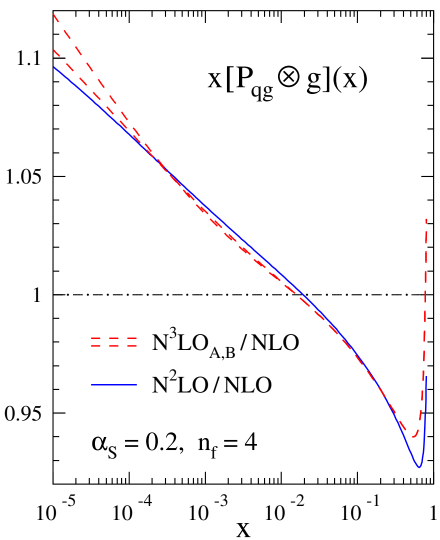


Approximate $P_{gq}^{(3)}(x)$

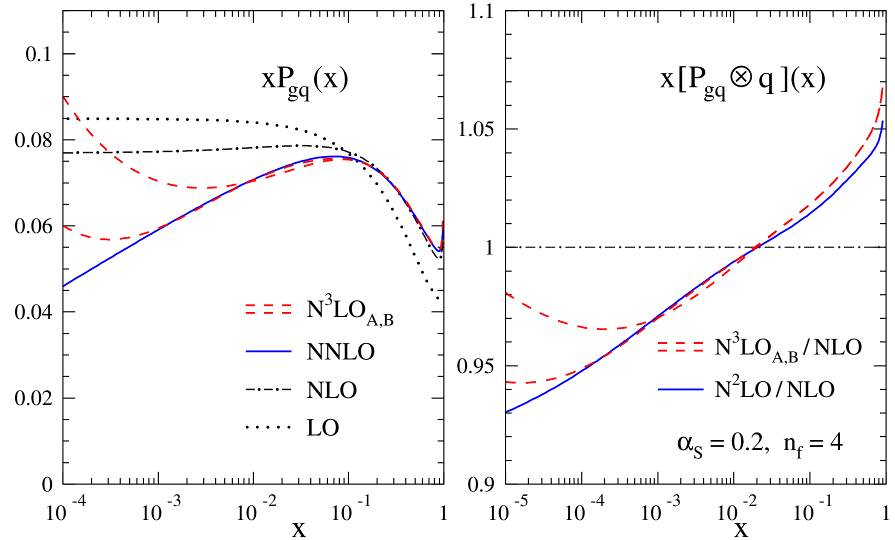


Quark evolution: gluon-to-quark contribution

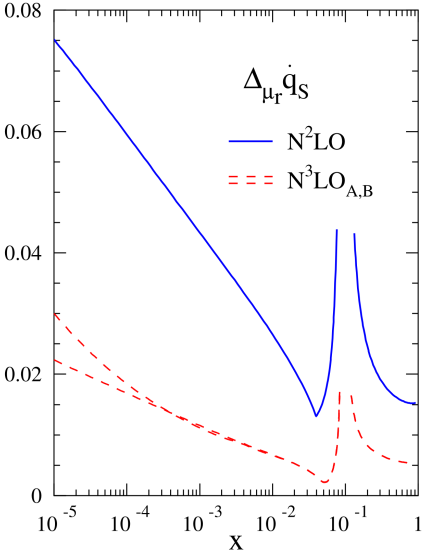




Gluon evolution: quark-to-gluon contribution



Scale stability: quark PDF



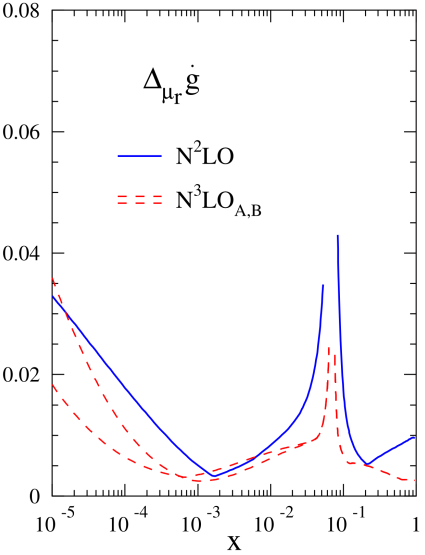
Variations of the renorm. scale:

- $M = \max\left[\dot{q}_s(\mu_r^2 = \lambda\mu_f^2)\right]$, $\lambda \in \left(-\frac{1}{4}; \frac{1}{4}\right)$
- $m = \min\left[\dot{q}_s(\mu_r^2 = \lambda\mu_f^2)\right]$

$$\Delta\dot{q}_s = \frac{1}{2}\frac{M - m}{\text{average}\left[\dot{q}_s(\mu_r^2 = \lambda\mu_f^2)\right]}$$

- $\Delta\dot{q}_s < 2\%$ for $x > 10^{-4}$
- $\Delta\dot{q}_s \sim (2.5 \pm 0.5)\%$ at $x = 10^{-5}$

Scale stability: gluon PDF



Same procedure for $\dot{g}$

- Scale stability already **small** at NNLO
- Moderate improvements at N^3LO
  - $\Delta\dot{g} < 1\%$ for $x > 10^{-4}$
  - $\Delta\dot{g} \sim (2.5 \pm 1)\%$ at $x = 10^{-5}$