Generalization of collinear factorization in perturbative QCD

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Based on arXiv:2402.14749 with Leandro Cieri and Germán Rodrigo

Resummation, Evolution, Factorization 2025

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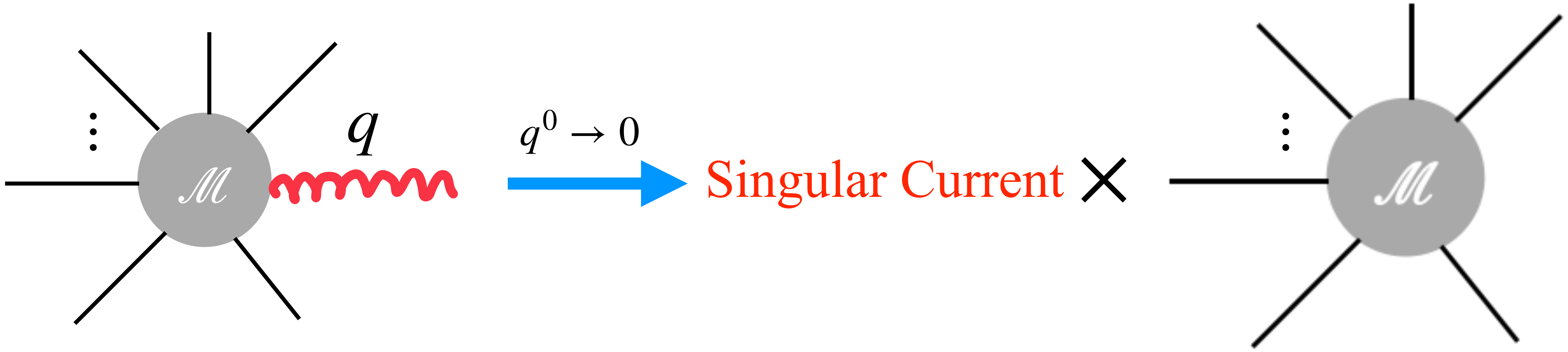
Universität Zürich[UZH]

Loop expansion

$$\mathcal{M}\left(p_1,\ldots,p_n\right) = \mathcal{M}^{(0)}\left(p_1,\ldots,p_n\right) + \sum_{i=1} \mathcal{M}^{(i)}\left(p_1,\ldots,p_n\right)$$

For a single **soft** gluon



quantitatively,

$$\langle a \,|\, \mathcal{M}^{(0)}(q,p_1,\ldots,p_m)\rangle = g_S\,\mu_0^{\epsilon}\,\varepsilon^{\mu}(q)\,J_{\mu}^{a\,(0)}(q)\,|\,\mathcal{M}^{(0)}(p_1,\ldots,p_m)\rangle + \cdots$$

Power suppressed contributions

where the current $\boxed{\boldsymbol{J}^{\mu\,(0)}(q) = \sum_{i=1}^{m} \boldsymbol{T}_i \frac{p_i^{\mu}}{p_i\cdot q}}$

$$q^{\mu}\boldsymbol{J}_{\mu}^{(0)}(q)\,|\,\mathcal{M}^{(0)}(p_1,\ldots,p_m)\rangle = \sum_{i=1}^{m} \boldsymbol{T}_i\,|\,\mathcal{M}^{(0)}(p_1,\ldots,p_m)\rangle = 0$$

Leading power contribution

$$\mathscr{M}\left(\lambda s, h\right)\Big|_{\lambda\to 0} \sim \frac{1}{\lambda^m}\mathrm{mod}\left(\ln^r \lambda\right) + \cdots$$

Set of hard particles

$$|\,\mathscr{M}(s,h)\rangle \simeq \boldsymbol{J}(s;h)\,|\,\mathscr{M}(h)\rangle$$

Set of soft particles

Reduced amplitude

Loop expansion: $\boldsymbol{J}(s;h) = \boldsymbol{J}^{(0)}(s;h) + \sum_{i=1} \boldsymbol{J}^{(i)}(s;h)$

Berends-Giele, Campbell-Glover (9710255), Catani-Grazzini (9908523), Bern-Chalmers (9503236), Bern-Del Duca-Kilgore-Schmidt (9810409, 9903516), Catani-Grazzini (0007142), Bierenbaum-Czakon-Mitov (1107.4384), Feige-Schwartz (1403.6472), Catani-Colferai-Torrini (1908.01616), Del Duca-Duhr-Haindl-Liu (2206.01584), Catani-Cieri-Colferai-Coradeschi (2210.09397), Zhu (2009.08919), Catani-Cieri (2108.13309), Czakon-Eschment-Schellenberger (2211.06465), Li-Zhu (1309.4391), Duhr-Gehrmann (1309.4393), Dixon-Herrmann-Yan-Zhu (1912.09370), Chen-Luo-Yang-Zhu (2309.03832), Herzog-Ma-Mistlberger-Suresh (2309.07874), Chen-Liu (2411.08795)

Sudakov parametrization

$n^2 = 0$

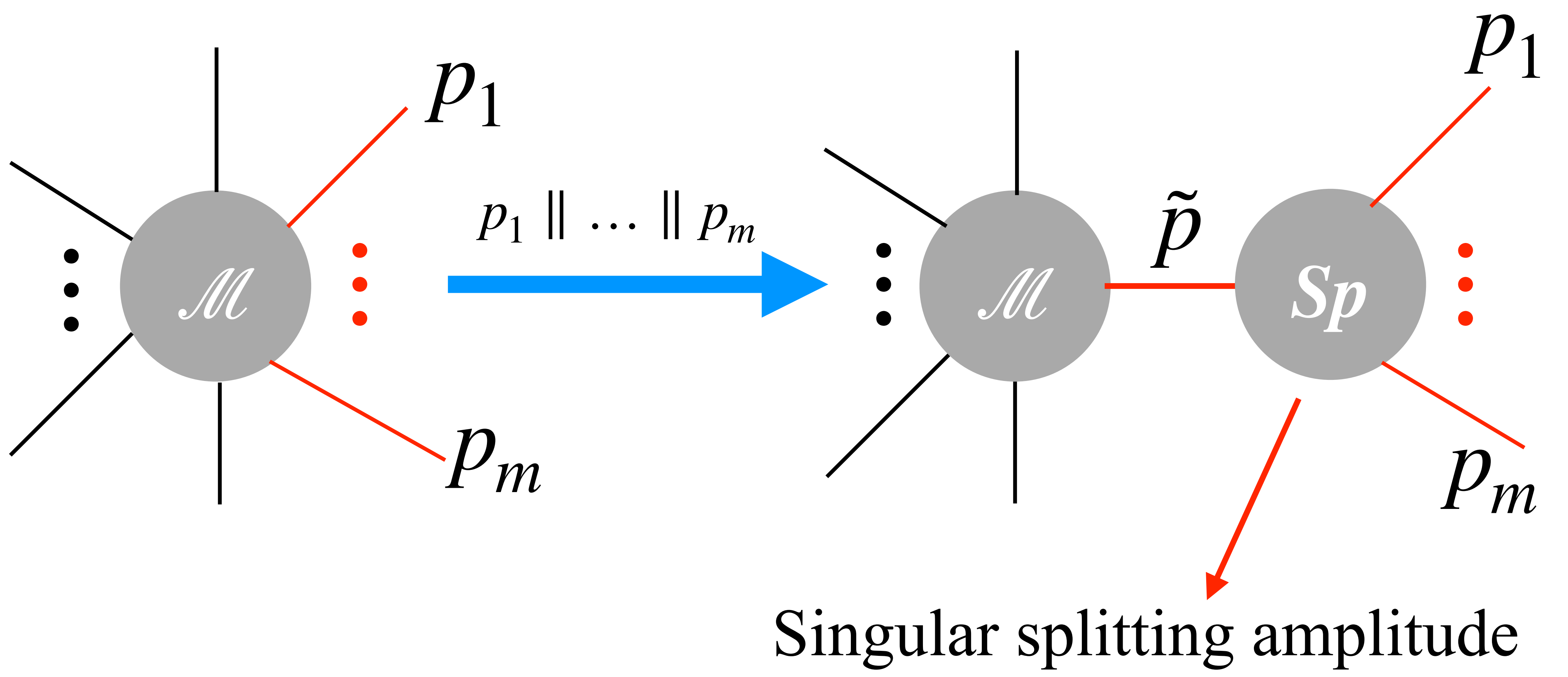
$$p_i^\mu = x_i\, \tilde{p}^\mu + k_{\perp i}^\mu - \frac{k_{\perp i}^2}{x_i} \frac{n^\mu}{2n \cdot \tilde{p}} \qquad i \in c \equiv \{1, \ldots, m\}$$

where the on-shell vector

$\tilde{p}^2 = 0$ $\qquad p_{1\ldots m}^2$

$$\tilde{p}^\mu = p_{1\ldots m}^\mu - \frac{s_{1\ldots m}}{2n \cdot p_{1\ldots m}} n^\mu$$

$$\sum_{i=1}^{m} p_i^\mu$$



Boost invariant quantities: $\quad z_i = \dfrac{x_i}{\sum_i x_i}$ $\qquad \tilde{k}_{\perp i}^\mu = k_{\perp i}^\mu - z_i \sum_i k_{\perp i}^\mu$

$$\sum_i z_i = 1 \qquad\qquad \sum_i \tilde{k}_{\perp i}^\mu = 0$$

Tree-level factorization

$$\mathscr{M}^{(0)}(p_1,\ldots,p_m,\ldots,p_n)\rangle = \boldsymbol{Sp}^{(0)}_{a_1\ldots a_m}(p_1,\ldots,p_m;\tilde{p})\,|\,\mathscr{M}^{(0)}(\tilde{p},p_{m+1},\ldots,p_n)\rangle + \cdots$$

Example: $1 \to 2$ splitting

$$\boldsymbol{Sp}^{(0)}_{q_1 g_2} = \frac{g_S\mu_0^{\epsilon}}{s_{12}}\,\boldsymbol{t}^{c_2}_{c_1 c}\,\bar{u}(p_1)\not{\varepsilon}(p_2)u(\tilde{p}) \qquad \boldsymbol{Sp}^{(0)}_{q_1\bar{q}_2} = \frac{g_S\mu_0^{\epsilon}}{s_{12}}\,\boldsymbol{t}^{c}_{c_1 c_2}\,\bar{u}(p_1)\not{\varepsilon}^*(\tilde{p})v(p_2)$$

To all perturbative orders

$$\boxed{|\,\mathscr{M}(c,h)\rangle \simeq \boldsymbol{Sp}(c;\tilde{p})\,|\,\mathscr{M}(\tilde{p},h)\rangle}$$

Set of collinear particles

$$\boldsymbol{Sp}(c;\tilde{p}) = \sum_{i=0} \boldsymbol{Sp}^{(i)}(c;\tilde{p})$$

Campbell-Glover (9710255), Catani-Grazzini (9908523), Catani-Grazzini (9810389), Bern-Del Duca-Kilgore-Schmidt (9810409, 9903516), Kosower-Uwer (9903515), Catani-de Florian-Rodrigo (1112.4405), Sborlini-de Florian-Rodrigo (1310.6841), Del Duca-Frizzo-Maltoni (9909464), Birthwright-Glover-Khoze-Marquard (0503063, 0505219), Del Duca-Duhr-Haindl-Lazopoulos (1912.06425, 2007.05345), Catani-de Florian-Rodrigo (0312067), Badger-Buciuni-Peraro (1507.05070), Czakon-Sapeta (2204.11801), Bern-Dixon-Kosower (0404293), Badger-Glover (0405236), Duhr-Gehrmann-Jaquier (1411.3587), Guan-Herzog-Ma-Mistlberger-Suresh (2408.03019)

Several questions come to our mind such as

- At which point does it become process dependent ?

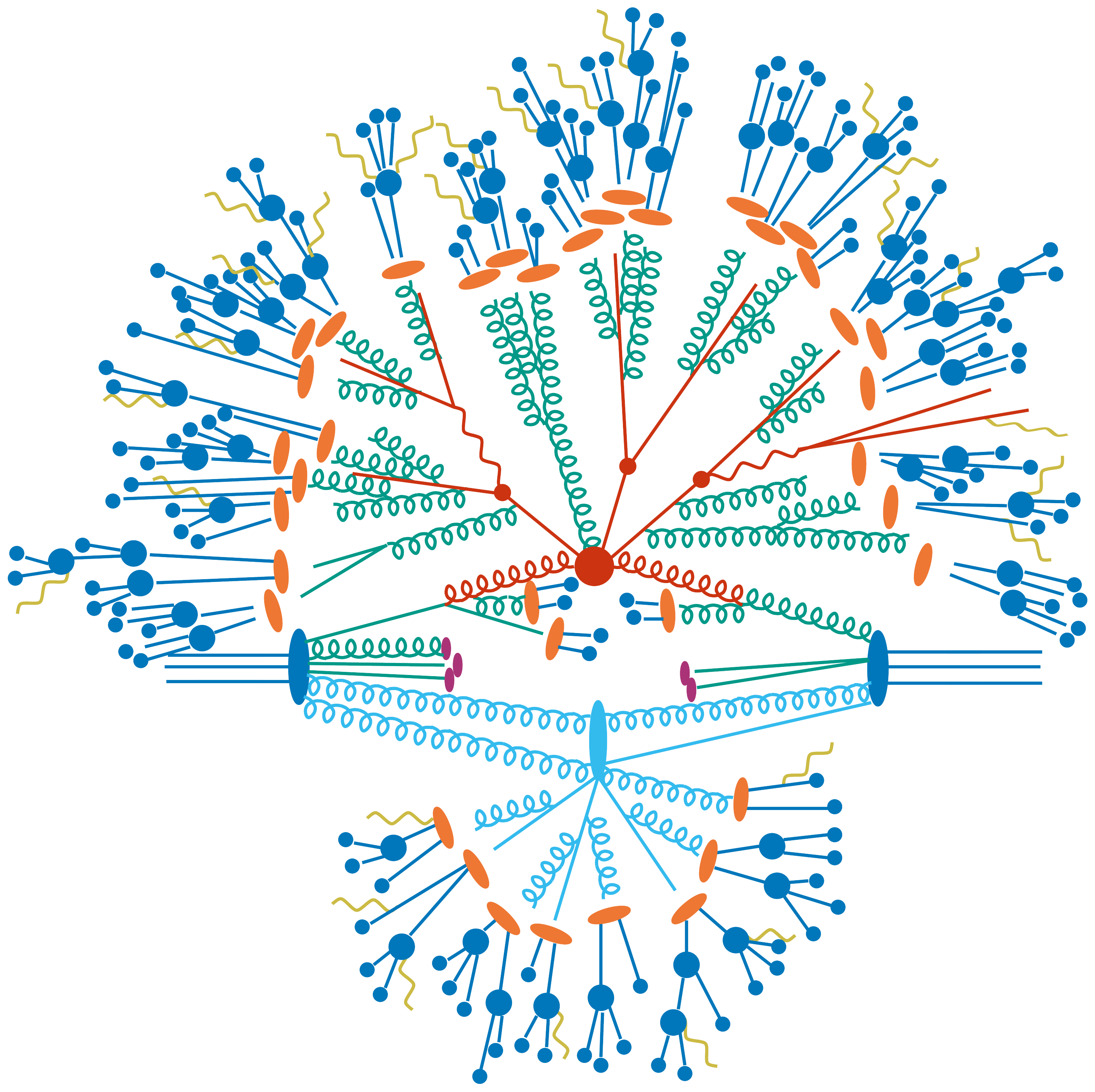
  Amplitude level?

  Squared amplitude level?

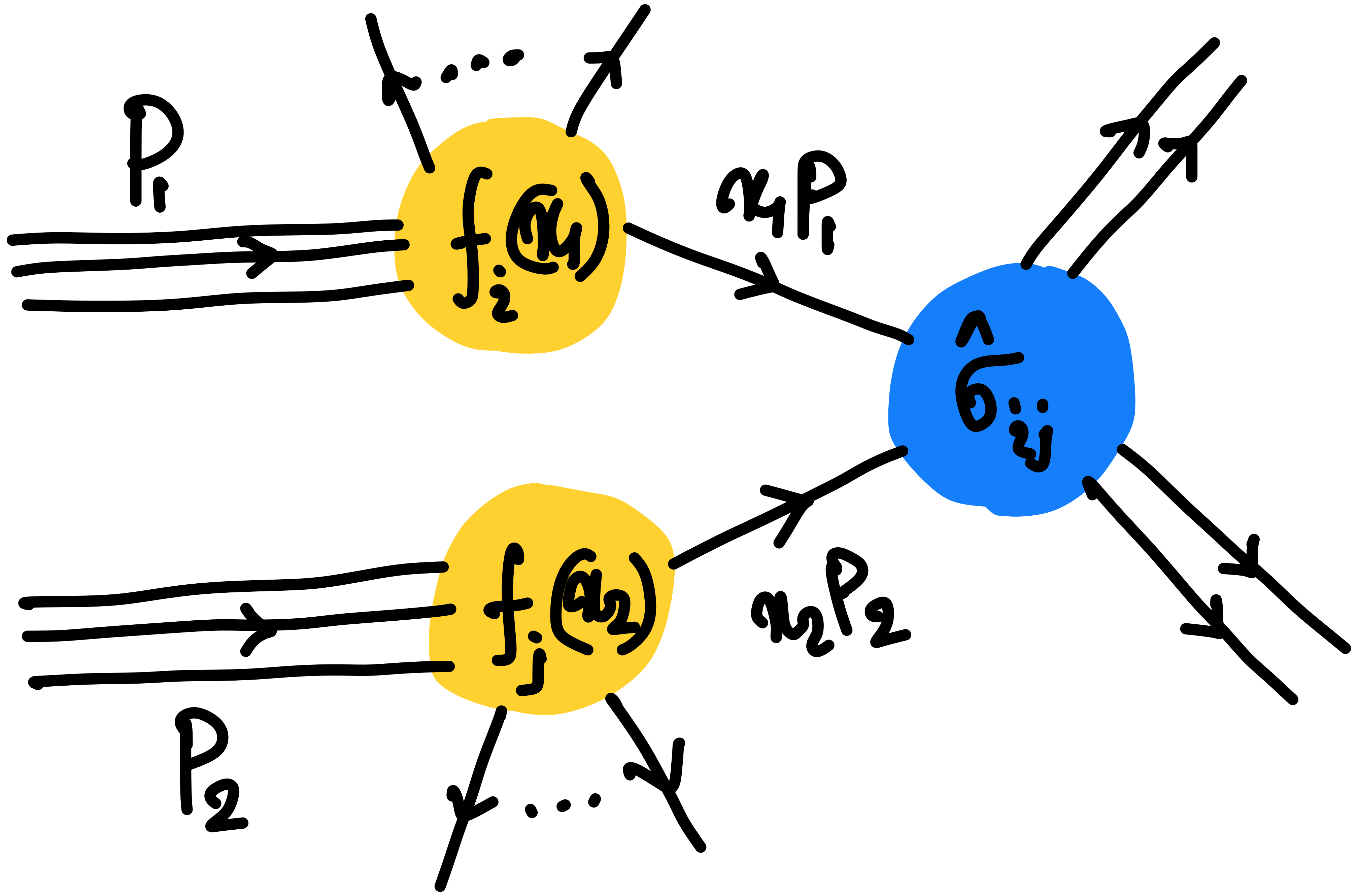
  Partonic cross section level?

- Why should we care at all?
- Does it affect the way we make theoretical predictions for colliders?

  How do we make theoretical predictions for collider observables?



Sherpa: A typical collider event



$$d\sigma_{h_1 h_2 \to X} = \sum_{i,j} \int dx_1 dx_2 f_{i/h_1}(x_1, \mu_F^2) f_{j/h_2}(x_2, \mu_F^2) d\hat{\sigma}_{ij \to X} + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{Q}\right)$$

Proved only for sufficiently inclusive observables such as Drell-Yan like processes!

Collins, Soper, Sterman

Space-Like: collinear partons are in both physical initial and final states

Factorization is not exact:

$$|\mathscr{M}(c,h)\rangle \simeq \boldsymbol{Sp}(c;\tilde{p};h)\,|\mathscr{M}(\tilde{p},h)\rangle$$

Dependence on hard non-collinear partons

And the loop expansion

$$\boldsymbol{Sp}(c;\tilde{p};h) = \boldsymbol{Sp}^{(0)}(c;\tilde{p}) + \sum_{i=1} \boldsymbol{Sp}^{(i)}(c;\tilde{p};h)$$

Forshaw-Kyrieleis-Seymour (0604094, 0808.1269), Collins-Qiu (0705.2141), Keates-Seymour (0902.0477), Rogers-Mulders (1001.2977), Echevarria-Idilbi-Scimemi (1111.4996, 1211.1947), Rogers (1304.4251), Feige-Schwartz (1403.6472), Rothstein-Stewart (1601.04695), Schwartz-Yan-Zhu (1703.08572), Dixon-Herrmann-Yan-Zhu (1912.09370), Becher-Neubert-Shao (2107.01212), Henn-Ma-Xu-Yan-Zhang-Zhu (2406.14604), Becher-Hager-Jaskiewicz-Neubert-Schwienbacher (2408.10308), Duhr-Venkata-Zhang (2507.05355), Becher-Hager-Jaskiewicz-Neubert-Schwienbacher (2509.07082)

Double collinear splitting ($\tilde{p} \to p_1 + p_2$) at 1-loop

Time-like splitting process: momentum fractions z_1 and z_2 are both positive

$$\delta(p_1, p_2; \tilde{p}) = -\frac{1}{\epsilon}(C_{12} + C_2 - C_1) f(\epsilon; z_2)$$

$$f\left(\epsilon; \frac{1}{x}\right) = \frac{1}{\epsilon}\left[\,{}_2F_1(1, -\epsilon, 1-\epsilon, 1-x) - 1\right]$$

Space-like splitting process: one of the momentum fractions is negative, say z_2

$$\boldsymbol{\delta}(p_1, p_2; p_3, \ldots, p_n) = +\frac{2}{\epsilon}\sum_{j=3}^{n} \boldsymbol{T}_2 \cdot \boldsymbol{T}_j\, f(\epsilon; z_2 - i\,0\,s_{j2}), \qquad s_{j2} = (p_j + p_2)^2$$

$$= \delta_R(p_1, p_2; \tilde{p}) + i\,\boldsymbol{\delta}_I(p_1, p_2; p_3, \ldots, p_n)$$

$$\delta_R(p_1, p_2; \tilde{p}) = -\frac{1}{\epsilon}(C_{12} + C_2 - C_1) f_R(\epsilon; z_2)$$

$$\boldsymbol{\delta}_I(p_1, p_2; p_3, \ldots, p_n) = -\frac{2}{\epsilon}\boldsymbol{T}_2 \cdot \left[\sum_{j=3}^{n} \boldsymbol{T}_j \,\mathrm{sign}(s_{j2})\right] f_I(\epsilon; z_2)$$

Color conservation: $\sum_{j=3}^{n} \boldsymbol{T}_j = -(\boldsymbol{T}_1 + \boldsymbol{T}_2)$

Double collinear splitting ($\tilde{p} \to p_1 + p_2$) at 2-loops

Space-like splitting

No DIS

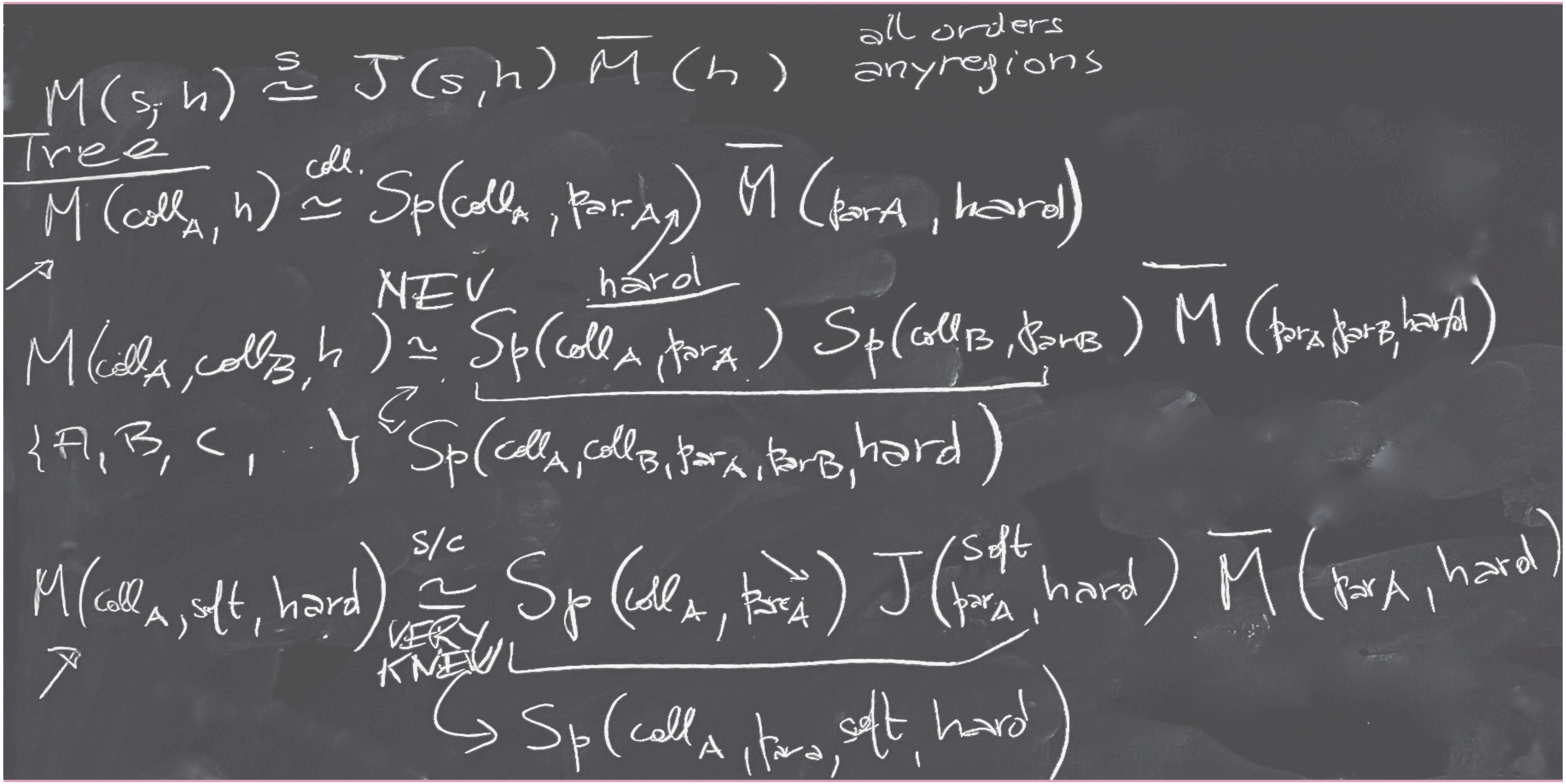
$$\boldsymbol{\Delta}_C^{(2)} \sim \pi f_{abc} \sum_{i=1,2} \sum_{\substack{j,k=3 \\ j \neq k}}^{n} T_i^a \, T_j^b \, T_k^c \, \Theta(-z_i) \, \mathrm{sign}(s_{ij}) \, \Theta(-s_{jk})$$

Non-abelian

$$\times \ln\left(-\frac{s_{j\tilde{p}} \, s_{k\tilde{p}} \, z_1 \, z_2}{s_{jk} \, s_{12}} - i\,0 \right) \left[-\frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \ln\left(\frac{-z_i}{1 - z_i} \right) \right]$$

Real and positive

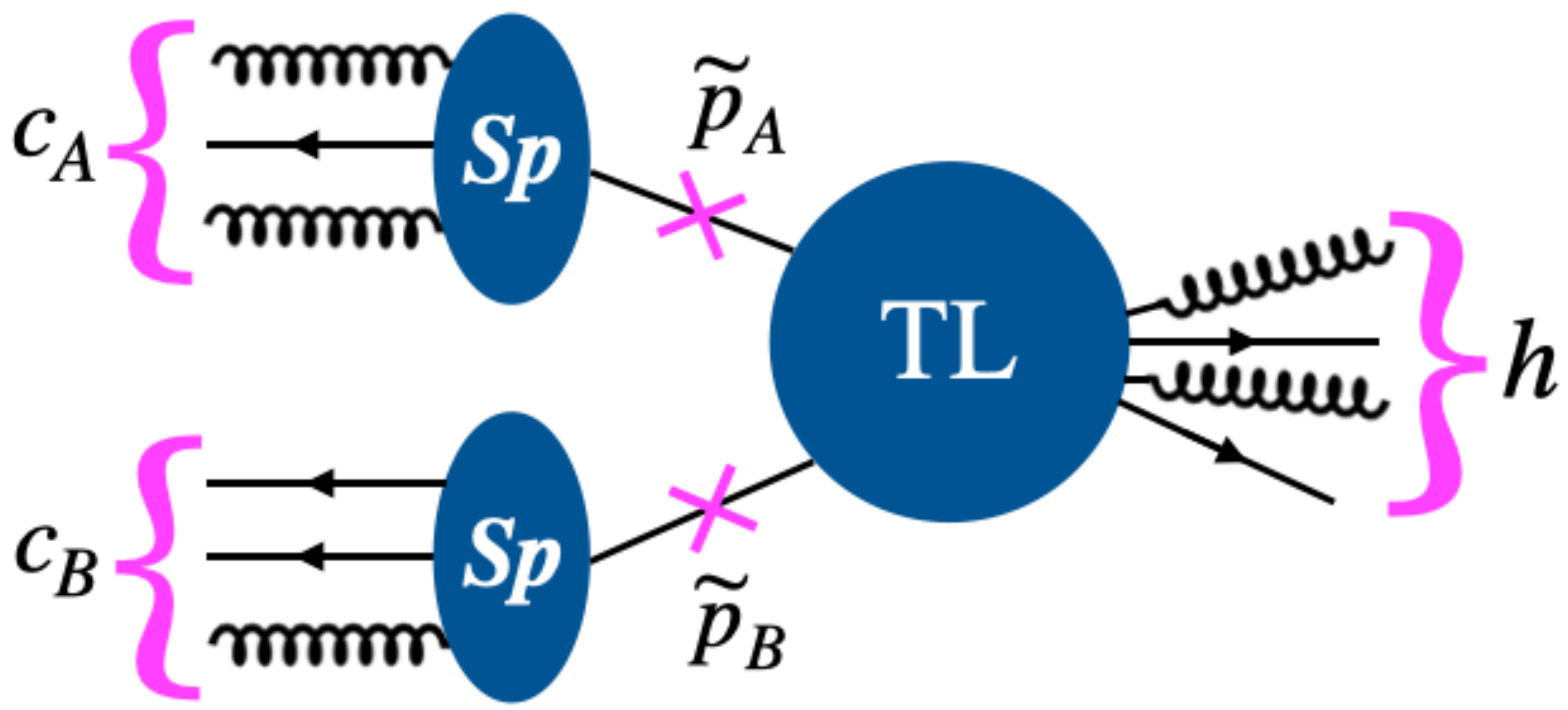
Cieri-PKD-Rodrigo (2402.14749)



Time-like

$$|\mathcal{M}(c_A, c_B, h)\rangle \simeq \boldsymbol{Sp}(c_A; \tilde{p}_A)\, \boldsymbol{Sp}(c_B; \tilde{p}_B)\, |\mathcal{M}(\tilde{p}_A, \tilde{p}_B, h)\rangle$$

Factorizable



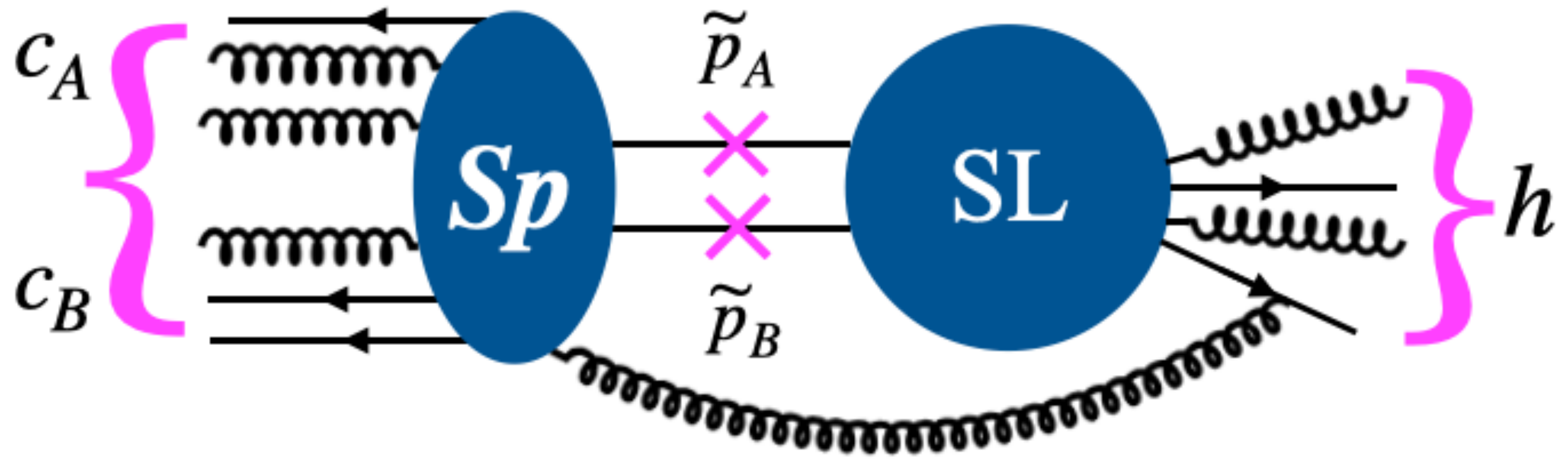
Space-like

Non-factorizable

Hard degrees of freedom

$$|\mathcal{M}(c_A, c_B, h)\rangle \simeq \boldsymbol{Sp}(c_A, c_B; \tilde{p}_A, \tilde{p}_B; h)\, |\mathcal{M}(\tilde{p}_A, \tilde{p}_B, h)\rangle$$

$$\boldsymbol{Sp}^{(0)}(c_A, c_B; \tilde{p}_A, \tilde{p}_B; h) = \boldsymbol{Sp}^{(0)}(c_A; \tilde{p}_A)\, \boldsymbol{Sp}^{(0)}(c_B; \tilde{p}_B)$$



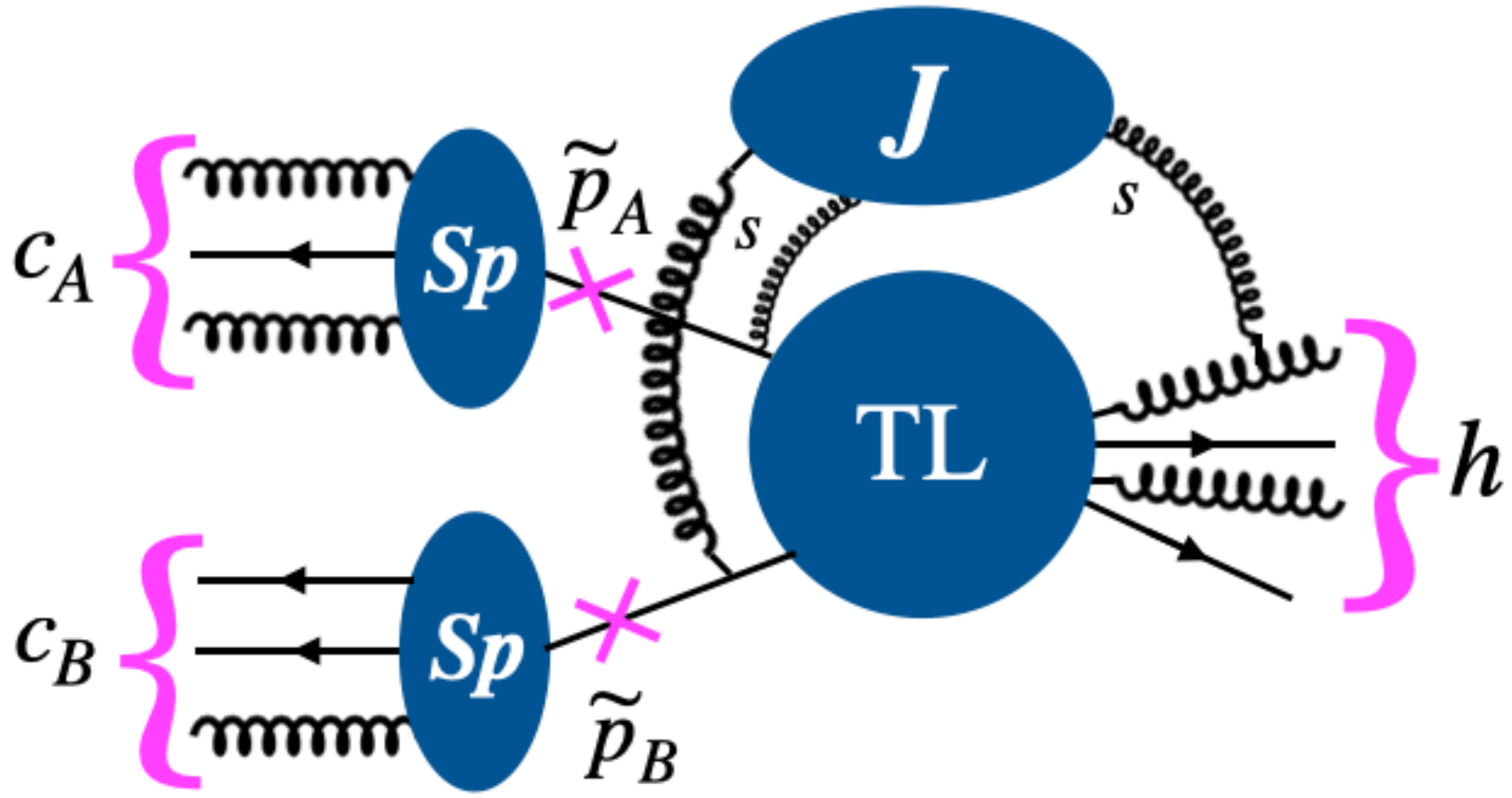
Cieri-PKD-Rodrigo (2402.14749)

Time-like

$$|\mathcal{M}(s,c,h)\rangle \simeq \boldsymbol{Sp}(c;\tilde{p})\,\boldsymbol{J}(s;\tilde{p};h)\,|\mathcal{M}(\tilde{p},h)\rangle$$

$$|\mathcal{M}(s,c_A,c_B,h)\rangle \simeq \boldsymbol{Sp}(c_A;\tilde{p}_A)\,\boldsymbol{Sp}(c_B;\tilde{p}_B)\,|\boldsymbol{J}(s;\tilde{p}_A,\tilde{p}_B;h)\,\mathcal{M}(\tilde{p}_A,\tilde{p}_B,h)\rangle$$

Factorizable



Space-like

$$|\mathcal{M}(s,c,h)\rangle \simeq \boldsymbol{Sp}(c;\tilde{p};s;h)\,|\mathcal{M}(\tilde{p},h)\rangle$$

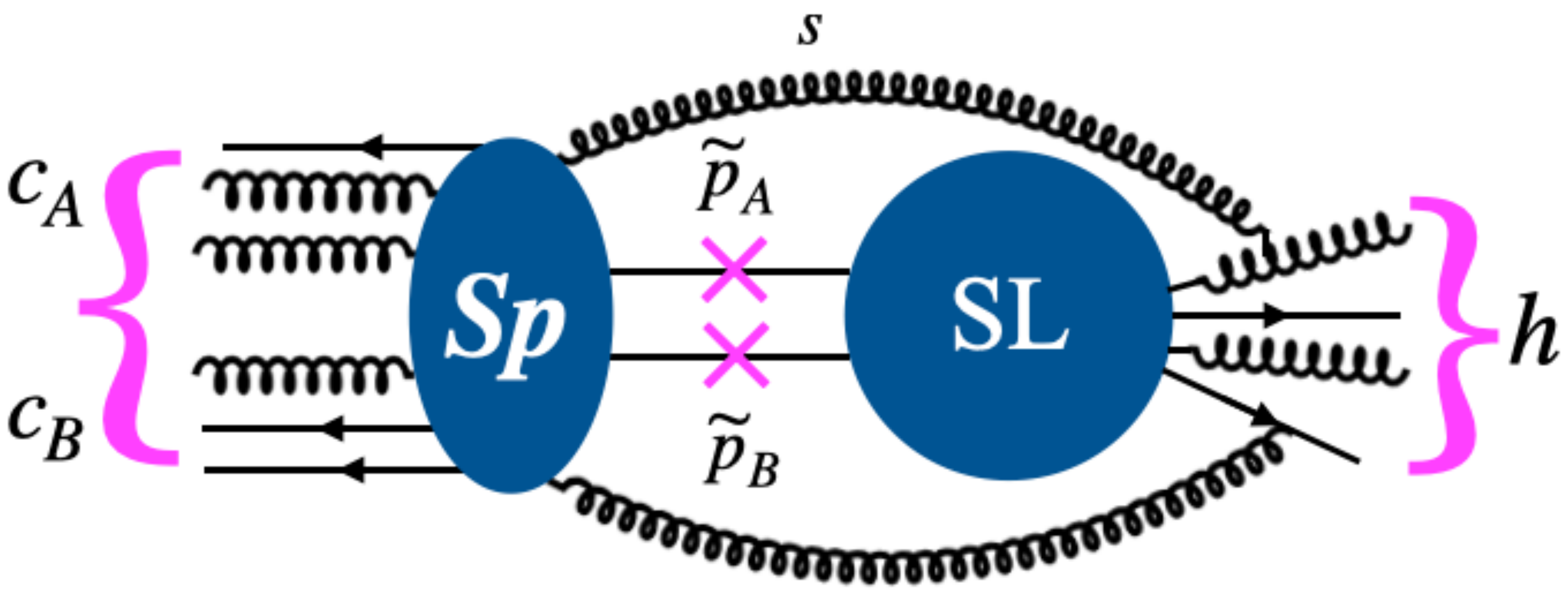
$$|\mathcal{M}(s\subset c,h)\rangle \simeq \boldsymbol{Sp}(s\subset c;\tilde{p};h)\,|\mathcal{M}(\tilde{p},h)\rangle$$

$$|\mathcal{M}(s,c_A,c_B,h)\rangle \simeq \boldsymbol{Sp}(c_A,c_B;\tilde{p}_A,\tilde{p}_B;s;h)\,|\mathcal{M}(\tilde{p}_A,\tilde{p}_B,h)\rangle$$

Non-factorizable

Hard degrees of freedom

$$\boldsymbol{Sp}^{(0)}(c_A,c_B;\tilde{p}_A,\tilde{p}_B;s;h) = \boldsymbol{Sp}^{(0)}(c_A;\tilde{p}_A)\,\boldsymbol{Sp}^{(0)}(c_B;\tilde{p}_B)\,\boldsymbol{J}^{(0)}(s;\tilde{p}_A,\tilde{p}_B;h)$$



Single soft gluon current at one-loop

$$\tilde{c}_\Gamma = c_\Gamma \Gamma(1-\epsilon)\Gamma(1+\epsilon), \quad c_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(1-2\epsilon)}$$

$$\boldsymbol{J}^{(1)a}(q) = -\tilde{g}_S{}^3 \frac{\tilde{c}_\Gamma}{\epsilon^2} i f^{abc} \sum_{\substack{i,j \in h \\ i \neq j}} \boldsymbol{T}_i^b \boldsymbol{T}_j^c \left[\frac{p_i \cdot \varepsilon(q)}{p_i \cdot q} - \frac{p_j \cdot \varepsilon(q)}{p_j \cdot q} \right] \frac{(-2p_i \cdot q - i0)^{-\epsilon}(-2p_j \cdot q - i0)^{-\epsilon}}{(-2p_i \cdot p_j - i0)^{-\epsilon}}$$

$g_S \mu^\epsilon$

Catani-Grazzini (0007142)

We find

Space-like collinear limit

Non-abelian

Depends on the color of non-collinear parton

$$\boldsymbol{Sp}_{\rm FB}^{(1)a}(q \subset c; \tilde{p}; h) \equiv \boldsymbol{J}_{\rm SL}^{(1)a}(q \subset c; \tilde{p}; h) - \boldsymbol{J}_{\rm TL}^{(1)a}(q \subset c; \tilde{p}) = -4\tilde{g}_S^3 \underbrace{f^{abc}\boldsymbol{T}_1^b\boldsymbol{T}_2^c}_{i\left[\boldsymbol{T}_1^a, \boldsymbol{T}_1 \cdot \boldsymbol{T}_2\right]} \frac{\tilde{c}_\Gamma \sin(\pi\epsilon)}{\epsilon^2} \frac{p_1 \cdot \varepsilon(q)}{p_1 \cdot q} (2p_1 \cdot q)^{-\epsilon} \left(\frac{n \cdot q}{n \cdot p_1} \right)^{-\epsilon}$$

Factorization breaking contribution

Time-like collinear limit

Can we find an example where the factorization breaking terms contribute at the squared amplitude level?

 Cieri-PKD-Rodrigo (2402.14749)

Double soft gluon current at one-loop

Zhu (2009.08919), Czakon-Eschment-Schellenberger (2211.06465)

$$\Delta \boldsymbol{Sp}_{\rm FB}^{(1)a_1a_2}(q_1,q_2\subset c;\tilde{p};h) = \tilde{g}_{\rm S}^4 f^{a_1bd}f^{a_2cd}\left(\boldsymbol{T}_1^b\boldsymbol{T}_2^c F_{\alpha_1\alpha_2}(q_1,q_2)+\boldsymbol{T}_2^b\boldsymbol{T}_1^c F_{\alpha_2\alpha_1}(q_2,q_1)\right)\varepsilon_1^{\alpha_1}(q_1)\varepsilon_2^{\alpha_2}(q_2)$$

Neglected lower order effects

where

Real contribution!

$$F_{\alpha_1\alpha_2}(q_1,q_2) = c_\Gamma e^{i\pi\epsilon}(2q_1\cdot q_2)^{-\epsilon}\left[\left(\frac{2i\pi}{\epsilon}+2\pi^2\right)J^{(0)}_{\alpha_1\alpha_2}+i\pi(\cdots)+\mathcal{O}(\epsilon)\right]$$

$$\Delta \boldsymbol{J}_{\rm SL/TL}^{(0)a_1a_2}(q_1,q_2\subset c;\tilde{p}) = -\,\tilde{g}_{\rm S}^2\frac{f^{a_1bd}f^{a_2cd}}{C_A}\left(\boldsymbol{T}_1^b\boldsymbol{T}_1^c J^{(0)}_{\alpha_1\alpha_2}(q_1,q_2)+\boldsymbol{T}_1^c\boldsymbol{T}_1^b J^{(0)}_{\alpha_2\alpha_1}(q_2,q_1)\right)\varepsilon_1^{\alpha_1}(q_1)\varepsilon_2^{\alpha_2}(q_2)$$

To our knowledge, this is the first instance where a FB term $\propto \pi^2$ at $\mathcal{O}(\epsilon^0)$ is identified which contributes at the level of squared amplitude!

Double spacelike collinear limits from multi-Regge kinematics

Claude Duhr,[1] Aniruddha Venkata,[1] and Chi Zhang (张驰)[1]
[1] *Bethe Center for Theoretical Physics, Universität Bonn, D-53115, Germany*
(Dated: July 9, 2025)

We study scattering amplitudes and form factors in planar $\mathcal{N} = 4$ Super Yang-Mills theory in the limit where two pairs of gluons become collinear. We find that, when the virtualities of both collinear pairs are spacelike, the collinear factorisation of the amplitude involves a generalised splitting amplitude that correlates the two collinear directions, confirming a recent proposal in the literature. Remarkably, we find that our generalised splitting amplitude agrees with the Bern-Dixon-Smirnov (BDS) subtracted six-point amplitude in multi-Regge kinematics. The latter can be explicitly evaluated using integrability to all orders in the coupling. We also present compelling evidence for the universality of the generalised splitting amplitude, by showing that the same function governs the double spacelike collinear limit of scattering amplitudes with up to 8 particles and 3 loops and form factors with up to 6 particles and 2 loops.

See the talk by Aniruddha Venkata on Friday

$$|\mathscr{M}|^2 \simeq \langle \mathscr{M}(\tilde{p}, h) | \boldsymbol{Sp}^{\dagger}(c; \tilde{p}; h) \, \boldsymbol{Sp}(c; \tilde{p}; h) | \mathscr{M}(\tilde{p}, h) \rangle$$

Purely imaginary contributions drops away

- Factorization breaking effects for double collinear splitting processes at 1-loop is purely imaginary, and hence gets cancelled. Catani-de Florian-Rodrigo (1112.4405)

- Factorization breaking effects for IR pole parts of double collinear splitting processes at 2-loops gets cancelled for tree level amplitudes generated in pure QCD.
  Catani-de Florian-Rodrigo (1112.4405), Forshaw-Seymour-Siodmok(1206.6363), Dixon-Herrmann-Yan-Zhu (1912.09370)

- Factorization breaking effects for double collinear splitting processes up to 2-loops gets cancelled in $\mathcal{N} = 4$ SYM theory. Henn-Ma-Xu-Yan-Zhang-Zhu (2406.14604)

Dedicated studies are required beyond this point!

- Factorization breaking effects are observed in mostly for non-inclusive (+ non-global such as gap-between-jets) observables. Still conclusive statements are yet to be made!

  Bacchetta-Bomhof-Mulders-Pijlman (0406099, 0505268, 0601171), Forshaw-Kyrieleis-Seymour (0604094, 0808.1269), Collins-Qiu (0705.2141), Gaunt (1405.2080), Zeng (1507.01652)

- Super-leading logarithms in gap-between-jets are one of the implications of factorization breaking.

  Forshaw-Kyrieleis-Seymour (0604094, 0808.1269), Becher-Neubert-Shao-Stillger (2107.01212, 2307.06359), Boer-Hager-Neubert-Stillger-Xu (2405.05305)

- Recently, it has been found a contribution at 3-loops that has the required form to turn the double super leading logarithms to single-logarithmic evolution.

  Becher-Hager-Jaskiewicz-Neubert-Schwienbacher (2408.10308, 2509.07082)

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Collinear factorization breaking doesn't necessarily imply PDF factorization breaking!!!

See the next talk by Dominik Schwienbacher

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Thank You Very Much!