

Conformal Small-x Evolution at the QCD Wilson-Fisher Point

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Classically, QCD is conformal invariant

⇒ LL evolution of light-ray QCD operator invariant under $SL(2, R)$

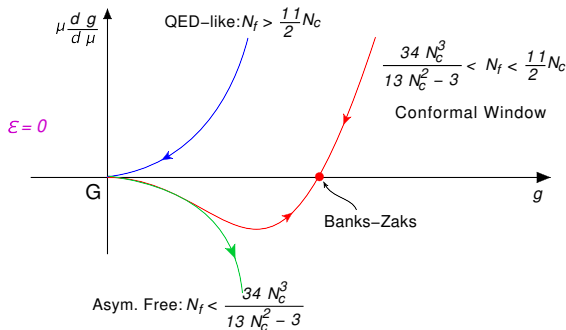
⇒ LO BFKL is invariant under $SL(2, C)$

Why Conformal invariance is useful

- At higher order in PT, the $\mathcal{N}=4$ SYM theory gives the most complicated part: Maximal Transcendentality (MT)
 - ▶ The MT part of 3-loop DGLAP kernel is exactly the 3-loop result in $\mathcal{N}=4$ SYM
 - ▶ The same is true for the NLO Pomeron intercept we expect to be true for the NNLO intercept as well.
- Another example: QCD at the Wilson-Fisher point

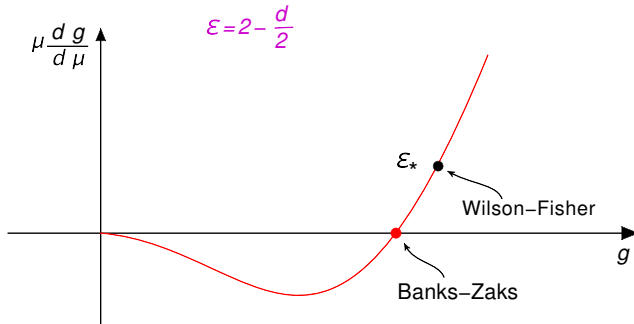
The Wilson-Fisher point

$$\frac{1}{g}\beta(g) = -\varepsilon - b_0 \frac{g^2}{(4\pi)^2} - b_1 \left(\frac{g^2}{(4\pi)^2} \right)^2 + \dots, \quad b_0 = \frac{11}{3}N_c - \frac{2}{3}N_f; \quad b_1 = \frac{34}{3}N_c^2 - \frac{10}{3}N_c N_f - 2C_f N_f$$



Banks-Zaks point: $b_0 \frac{g_{BZ}^2}{(4\pi)^2} + b_1 \left(\frac{g_{BZ}^2}{(4\pi)^2} \right)^2 + \dots = 0$

The Wilson-Fisher point

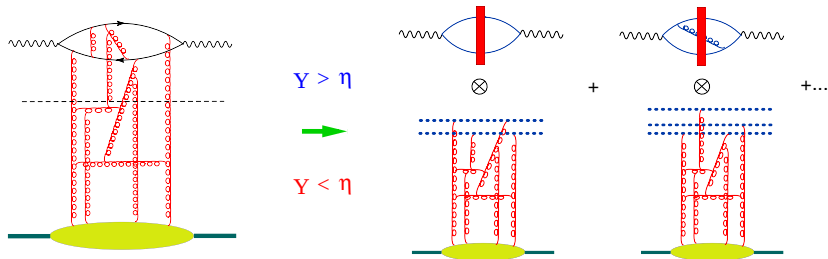


Wilson-Fisher point

$$\varepsilon_* = b_0 \left(\frac{g_{BZ}^2}{(4\pi)^2} - \frac{g^2}{(4\pi)^2} \right) + b_1 \left(\frac{g_{BZ}^2}{(4\pi)^2} - \frac{g^2}{(4\pi)^2} \right)^2 + \dots$$

After the calculation we can analytically continue the result to N_f outside the conformal window

High-energy expansion in color dipoles at the NLO

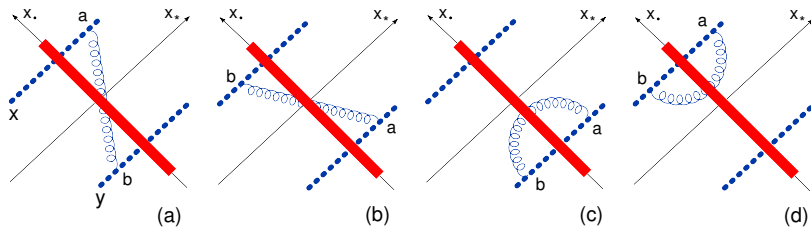


The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 I_{\mu\nu}^{\text{LO}}(z_1, z_2, x, y) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\ + \int d^2z_1 d^2z_2 d^2z_3 I_{\mu\nu}^{\text{NLO}}(z_1, z_2, z_3, x, y) [\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]$$

$$U_x = [\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} dx^+ A_+(x^+, x_\perp) \right\}$$

Leading order diagrams



$$x_{\bullet} = \sqrt{\frac{s}{2}} x^{-}$$

$$x_{*} = \sqrt{\frac{s}{2}} x^{+}$$

Non linear evolution equation: Balitsky-Kovchegov equation

$$\hat{U}(x, y) \equiv 1 - \frac{1}{N_c} \text{tr} \{ \hat{U}(x_\perp) \hat{U}^\dagger(y_\perp) \}$$

$$\frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z (x-y)^2}{(x-z)^2 (y-z)^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z) \hat{U}(z, y) \right\}$$

- LLA for DIS in pQCD $\Rightarrow$ BFKL
 - ▶ (LLA: $\alpha_s \ll 1, \alpha_s \eta \sim 1$): describes proliferation of gluons.
- LLA for DIS in semi-classical-QCD $\Rightarrow$ BK eqn
 - ▶ background field method: describes recombination process.

Alternative approach: JIMWLK evolution equation

Conformal invariance of the BK equation

Formally, a light-like Wilson line $U(x_\perp) \equiv U_x$

$$U(x_\perp) = [\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} dx^+ A_+(x^+, x_\perp) \right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

Indeed,

$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \rightarrow x_\perp/x_\perp^2$ and $x^+ \rightarrow x^+/x_\perp^2 \Rightarrow$

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \rightarrow \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2} A_+\left(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2}\right) \right\} = [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

$\Rightarrow$ The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta} \text{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int \frac{d^2 z}{z^4} \frac{(x-y)^2}{(x-z)^2(z-y)^2} [\text{Tr}\{V_x V_z^\dagger\} \text{Tr}\{V_z V_y^\dagger\} - N_c \text{Tr}\{V_x V_y^\dagger\}]$$

Conformal properties of light-like Wilson lines

Wilson lines are invariant under the (Möbius) $SL(2,C)$ group

$$\begin{aligned}\hat{S}_- &\equiv \frac{i}{2}(K^1 + iK^2), & \hat{S}_0 &\equiv \frac{i}{2}(D + iM^{12}), & \hat{S}_+ &\equiv \frac{i}{2}(P^1 - iP^2) \\ \bar{\hat{S}}_- &\equiv \frac{i}{2}(K^1 - iK^2), & \bar{\hat{S}}_0 &\equiv \frac{i}{2}(D - iM^{12}), & \bar{\hat{S}}_+ &\equiv \frac{i}{2}(P^1 + iP^2)\end{aligned}$$

form $SL(2,C)$ algebra

$$\begin{aligned}[\hat{S}_0, \hat{S}_\pm] &= \pm \hat{S}_\pm, & [\hat{S}_+, \hat{S}_-] &= 2\hat{S}_0, \\ [\bar{\hat{S}}_0, \bar{\hat{S}}_\pm] &= \pm \bar{\hat{S}}_\pm, & [\bar{\hat{S}}_+, \bar{\hat{S}}_-] &= 2\bar{\hat{S}}_0.\end{aligned}$$

Momentum operator $\hat{P}$, angular momentum operator $\hat{M}$, dilatation operator $\hat{D}$, and special conformal generator $\hat{K}$.

$$\begin{aligned}
 i[\hat{P}^\mu, \hat{A}^\alpha] &= \partial^\mu \hat{A}^\alpha, \quad i[\hat{D}, \hat{A}^\alpha] = (x_\mu \partial^\mu + 1) \hat{A}^\alpha, \\
 i[\hat{M}^{\mu\nu}, \hat{A}^\alpha] &= (x^\mu \partial^\nu - x^\nu \partial^\mu) \hat{A}^\alpha - (g^{\nu\alpha} \hat{A}^\mu - g^{\mu\alpha} \hat{A}^\nu) \\
 i[K^\mu, A^\alpha] &= (2x^\mu x_\nu \partial^\nu - x^2 \partial^\mu + 2x^\mu) A^\alpha - 2x_\nu (g^{\nu\alpha} A^\mu - g^{\mu\alpha} A^\nu)
 \end{aligned}$$

Complex notation

$$z \equiv z^1 + iz^2, \quad \bar{z} \equiv z^1 - iz^2, \quad \frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial z^1} - i \frac{\partial}{\partial z^2} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial z^1} + i \frac{\partial}{\partial z^2} \right)$$

$$\begin{aligned}
 [\hat{S}_-, \hat{U}(z, \bar{z})] &= z^2 \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_0, \hat{U}(z, \bar{z})] = z \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_+, \hat{U}(z, \bar{z})] = -\partial_z \hat{U}(z, \bar{z}) \\
 [\bar{\hat{S}}_-, \hat{U}(z, \bar{z})] &= \bar{z}^2 \partial_{\bar{z}} \hat{U}(z, \bar{z}), \quad [\bar{\hat{S}}_0, \hat{U}(z, \bar{z})] = \bar{z} \partial_{\bar{z}} \hat{U}(z, \bar{z}), \quad [\bar{\hat{S}}_+, \hat{U}(z, \bar{z})] = -\partial_{\bar{z}} \hat{U}(z, \bar{z})
 \end{aligned}$$

$U(z, \bar{z})$ lies in the standard representation of conformal group $SL(2, C)$

$$X \equiv x_{\perp} - z_{\perp} \quad Y \equiv y_{\perp} - z_{\perp}$$

$$d_{\perp} = 2 - 2\epsilon$$

$$\begin{aligned} \frac{d}{d\eta} \mathcal{U}(x, y) = & \frac{\alpha_s N_c \Gamma^2(1 - \epsilon)}{2\pi^{2-2\epsilon} \mu^{2\epsilon}} \int d^{2-2\epsilon} z \left[\frac{X_i}{(X_{\perp}^2)^{1-\epsilon}} - \frac{Y_i}{(Y_{\perp}^2)^{1-\epsilon}} \right] \left[\frac{X_i}{(X_{\perp}^2)^{1-\epsilon}} - \frac{Y_i}{(Y_{\perp}^2)^{1-\epsilon}} \right] \\ & \times \left[\mathcal{U}(x, z) + \mathcal{U}(z, y) - \mathcal{U}(x, y) - \mathcal{U}(x, z)\mathcal{U}(z, y) \right] \end{aligned}$$

Not conformal invariant $\Rightarrow$ BFKL cannot be solved by power-like eigenfunction

Idea: Go to Wilson-Fisher point and restore conformal invariance

$$\epsilon_* \simeq -\frac{\alpha_s}{4\pi} b_0 - \frac{\alpha_s^2}{16\pi^2} b_1 + \dots \Rightarrow \beta(\alpha_s) = 0$$

BK equation at Wilson-Fisher point

I. Balitsky and G.A.C (2024)

Expand in ε and $\alpha_s \Rightarrow$ Consider NLO corrections (for conf. comp. operator)

$$\frac{d}{d\eta}\mathcal{U}(x,y) = \frac{\alpha_s N_c \Gamma(1-\varepsilon)}{2\pi^{2-\varepsilon}} \int d^{2-2\varepsilon}z \left(\frac{(x-y)_\perp^2}{X^2 Y^2} \right)^{1-\varepsilon} \left\{ 1 + \right. \\ \left. + \varepsilon \frac{\alpha_s}{4\pi} \left(\ln \frac{(x-y)_\perp^2 \mu^2}{4} + 2\gamma_E - \frac{X^2 - Y^2}{(x-y)_\perp^2} \ln \frac{X^2}{Y^2} \right) \right\}$$

$$2a \frac{d}{da} [\mathcal{U}(x,y)]_a^{\text{conf}} = \frac{\alpha_s N_c}{2\pi^2} \int d^{2-2\varepsilon}z \left(\frac{(x-y)_\perp^2}{X^2 Y^2} \right)^{1-\varepsilon} \left\{ 1 + \right. \\ \left. + b_0 \frac{\alpha_s}{4\pi} \left(\ln \frac{(x-y)_\perp^2 \mu^2}{4} + 2\gamma_E - \frac{X^2 - Y^2}{(x-y)_\perp^2} \ln \frac{X^2}{Y^2} \right) + \frac{\alpha_s N_c}{4\pi} \left(\frac{67}{9} - \frac{\pi^2}{3} - \frac{10n_f}{9N_c} \right) \right\} \\ \times \left[\mathcal{U}(x,z) + \mathcal{U}(z,y) - \mathcal{U}(x,y) - \mathcal{U}(x,z)\mathcal{U}(z,y) \right] + \frac{\alpha_s^2}{16\pi^4} K_{\text{conf}} + \mathcal{O}(\alpha_s^3)$$

Notice: expansion in ε does not reproduce the term $\ln \frac{X^2}{(x-y)_\perp^2} \ln \frac{Y^2}{(x-y)_\perp^2}$

BK equation at Wilson-Fisher point

$$\varepsilon_* \simeq -\frac{\alpha_s}{4\pi} b_0$$

$$b_0 = \frac{11}{3}N_c - \frac{2}{3}n_f$$

$$\begin{aligned} \frac{d}{d\eta} \mathcal{U}(x, y) &= \frac{\alpha_s N_c \Gamma(1 - \varepsilon_*)}{2\pi^{2-\varepsilon_*}} \int d^{2-2\varepsilon} z \left(\frac{(x-y)_\perp^2}{X^2 Y^2} \right)^{1-\varepsilon} \left[1 + \right. \\ &\quad \left. + \frac{\alpha_s N_c}{4\pi} \left(\frac{67}{9} - \frac{\pi^2}{3} - \frac{10n_f}{9N_c} \right) \right] \left[\mathcal{U}(x, z) + \mathcal{U}(z, y) - \mathcal{U}(x, y) - \mathcal{U}(x, z)\mathcal{U}(z, y) \right] \\ &\quad \times + \frac{\alpha_s^2}{16\pi^4} K_{conf} + \mathcal{O}(\alpha_s^3, \alpha_s^2 \varepsilon_*, \alpha_s^2 \varepsilon_*^2) \end{aligned}$$

Reversing the argument $\Rightarrow$

We can obtain the NLO running coupling from the LO calculation!

$\Rightarrow$ Use NLO BK result to get the NNLO BK running coupling

Caveat: different choices for assembling in a conformal way the kernel at $d \neq 4$

$\Rightarrow$ ambiguity of terms prop to b_0

Pomeron intercept at Wilson-Fisher point

Forward linearized equation: **BFKL**

$$\begin{aligned} \frac{d}{d\eta} \mathcal{U}(z) = & \frac{\alpha_s N_c \Gamma(1 - \varepsilon_*)}{2\pi^{2-\varepsilon_*}} \int d^{2-2\varepsilon_*} z' \left[1 + \frac{\alpha_s N_c}{4\pi} \left(\frac{67}{9} - \frac{\pi^2}{3} - \frac{10n_f}{9N_c} \right) \right] \left(\frac{z^2}{(z - z')^2 z'^2} \right)^{1-\varepsilon_*} \\ & \times [2\mathcal{U}(z') - \mathcal{U}(z)] + \frac{\alpha_s^2 N_c^2}{4\pi^3} \int dz' \mathcal{K}_{\text{conf}}(z, z') \mathcal{U}(z') + O(\alpha_s^3, \alpha_s^2 \varepsilon_*, \alpha_s \varepsilon_*^2) \end{aligned}$$

Conformal invariance restored: Solve by power-like eigenfunctions

$$\mathcal{U}(|z|) = \frac{\Gamma(\frac{d_\perp}{2})}{\pi^{\frac{d_\perp}{2}}} \int d^{d_\perp} z' \int \frac{d\nu}{2\pi} (z'^2)^{-\frac{d_\perp}{2} - \frac{d_\perp}{4} + i\nu} (z^2)^{\frac{d_\perp}{2} - \frac{d_\perp}{4} - i\nu} \mathcal{U}(|z'|)$$

BK equation at Wilson-Fisher point

using the evolution equation in d-dimension

$$\mathcal{U}(|z|) = \frac{\Gamma\left(\frac{d_{\perp}}{2}\right)}{\pi^{\frac{d_{\perp}}{2}}} \int d^{d_{\perp}} z' \int \frac{d\nu}{2\pi} (z'^2)^{-\frac{d_{\perp}}{2} - \frac{d_{\perp}}{4} + i\nu} (z^2)^{\frac{d_{\perp}}{2} - \frac{d_{\perp}}{4} - i\nu} e^{\eta \bar{\kappa} \left(\frac{d_{\perp}}{4} - i\nu\right)} \mathcal{U}(|z'|)$$

$$\bar{\kappa}(\xi) = \frac{\alpha_s N_c}{\pi} \left[\bar{\chi}(\xi) + \frac{\alpha_s N_c}{4\pi} \bar{\delta}(\xi) + O(\alpha_s^2, \alpha_s \varepsilon_*, \varepsilon_*^2) \right]$$

$$\bar{\chi}(\xi) = \psi(1 - \varepsilon_*) - \gamma_E - \psi(\xi) - \psi(1 - \varepsilon_* - \xi)$$

Pomeron intercept (at $\nu = 0$)

$$\frac{\alpha_s N_c}{\pi} \chi\left(\frac{1}{2} - \frac{\varepsilon_*}{2}\right) = \frac{\alpha_s N_c}{\pi} \left[4 \ln 2 - \frac{\pi^2}{3} \varepsilon_* + O(\varepsilon_*^2) \right]$$

Pomeron intercept at $d \neq 4$ is different than the one at $d = 4$

What about the anomalous dimension of twist-2 operator?

Light-ray operator as analytic continuation of local operator

Gluon light-ray (LR) operator of twist 2

$$F_{-i}^a(x'_+ + x_\perp)[x'_+, x_+]^{ab} F_{-i}^b(x_+ + x_\perp)$$

Analytic continuation of local operator

$$\begin{aligned} F_{-i}^a \nabla_-^{j-2} F_-^{ai} &= \frac{\Gamma(j-1)}{2\pi i} \int_{\text{Hankel}} du u^{1-j} F_{-i}^a e^{-u\nabla_-} F_-^{ai}(0) \\ &= \frac{\Gamma(j-1)}{2\pi i} \int_{\text{Hankel}} dx_+ x_+^{1-j} F_{-i}^a(x_+)[x_+, 0] F_-^{ai}(0) \\ &= \frac{1}{\Gamma(2-j)} \int_0^{+\infty} dx_+ u_+^{1-j} F_{-i}^a(x_+)[x_+, 0] F_-^{ai}(0) \end{aligned}$$

Evolution equation for light-ray operator

Forward light-ray operator

$$F(L_+, x_\perp) = \int dx_+ F_{-i}^a(L_+ + x_\perp) [L_+ + x_+, x_+]_x^{ab} F_-^{bi}(x_+ + x_\perp)$$

Evolution equation

$$\begin{aligned} & \mu^2 \frac{d}{d\mu^2} F_{-i}^a(x'_+ + x_\perp) [x'_+, x_+]_x^{ab} F_-^b(x_+ + x_\perp) \\ &= \int_{x_+}^{x'_+} dz'_+ \int_{x_+}^{z'_+} dz_+ K(x'_+, x_+; z'_+, z_+; \alpha_s) F_{-i}^a(z'_+ + x_\perp) [z'_+, z_+]_x^{ab} F_-^b(z_+ + x_\perp) \end{aligned}$$

$$[x'_+, x_+]_x = [x'_+ + x_\perp, x_+ + x_\perp] = \text{Pexp} \left\{ ig \int_{x_+}^{x'_+} dw_+ A_-(w_+, x_\perp) \right\}$$

forward local operators

$$F(L_+, x_\perp) = \sum_{n=2}^{\infty} \frac{L_+^{n-2}}{(n-2)!} \mathcal{O}_n^g(x_\perp), \quad \mathcal{O}_n^g \equiv \int dx_+ F_{-i}^a \nabla_-^{n-2} F_-^{ai}(x_+, x_\perp)$$

$$\begin{aligned} \mu^2 \frac{d}{d\mu^2} F(L_+, x_\perp) &= \int_0^1 du K_{gg}(u, \alpha_s) F(uL_+, x_\perp) \\ \Rightarrow \mu^2 \frac{d}{d\mu^2} \mathcal{O}_N^g &= -\gamma_N(\alpha_s) \mathcal{O}_N^g, \quad \gamma_N(\alpha_s) = - \int_0^1 du u^{N-2} K_{gg}(u, \alpha_s) \end{aligned}$$

DGLAP kernel (only gluon) $(\bar{u} = 1 - u)$

$$u^{-1} K_{gg}(u) = \frac{2\alpha_s N_c}{\pi} \left(u\bar{u} + \left[\frac{1}{u\bar{u}} \right]_+ - 2 + \frac{11}{12} \delta(\bar{u}) \right) + \mathcal{O}(\alpha_s^2)$$

$$j = \frac{1}{2} + i\nu$$

$$\mathcal{F}_j(x_\perp) = \int_0^{+\infty} dL_+ L_+^{1-j} F(L_+, x_\perp)$$

Evolution equation

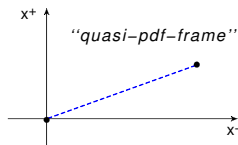
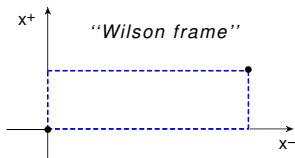
$$\mu^2 \frac{d}{d\mu^2} \mathcal{F}_j(z_\perp) = \int_0^1 du K_{gg}(u, \alpha_s) u^{j-2} \mathcal{F}_j(z_\perp)$$

$\Rightarrow \gamma_j(\alpha_s)$ is an analytic continuation of $\gamma_N(\alpha_s)$ from the “DGLAP side

How to get the anomalous dimension γ_j at $j \rightarrow 1$ from an operator language
“from BFKL side”

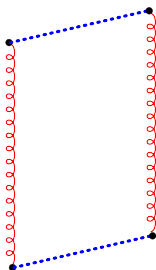
Correlation function of

- **Wilson frame** built from light-ray operator
 - ▶ I. Balitsky, V. Kazakov, E. Sobko (2013)
- **quasi-pdf frame** built from light-ray operator
 - ▶ G.A.C. (gluon (2019) and quark (2021))

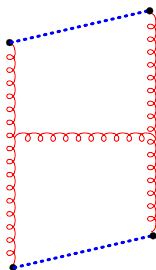


However, at $d_\perp \neq 2$ there are many integrals and they are complicated

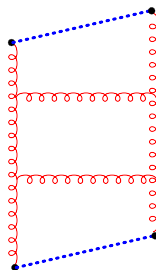
Example: Correlation Function of quasi-pdf LR operators



a)



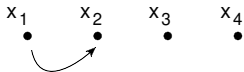
b)



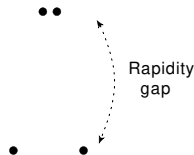
c)

Easier way: Regge+DGLAP limit of CF of 4 scalar operators

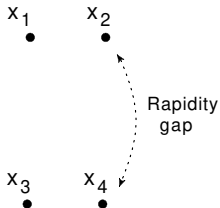
DGLAP limit



Regge limit



Regge limit



DGLAP limit



Easier way: Regge+DGLAP limit of CF of 4 scalar operators

Forward correlation function of 4 scalar operators (e.g. FF) in $d = 4 - \varepsilon_*$ in QCD

$$A(L_+, L_-; x_{1\perp}, x_{2\perp}, x_{3\perp}, x_{4\perp}) = (x_{12}^2 x_{34}^2)^{4-2\varepsilon_* + \gamma_O} (\mu^2)^{2\gamma_O} \\ \times \int dx_{2+} dx_{3+} \langle \mathcal{O}(L_+ + x_{2+}, x_{1\perp}) \mathcal{O}(x_{2+}, x_{2\perp}) \mathcal{O}(L_- + x_{4-}, x_{3\perp}) \mathcal{O}(x_{4-}, x_{4\perp}) \rangle$$

- Regge limit: $L_+ L_- \rightarrow \infty$
- DGLAP limit: $x_{12}^2 \rightarrow 0$

Plan:

Comparing Regge+light-cone vs light-cone+Regge limits to relate the BFKL Pomeron intercept to the anomalous dimensions of gluon LR operators in the limit

$$j - 1 = \omega \rightarrow 0, \quad \frac{\alpha_s}{\omega} \sim 1$$

In CFT the 4-points CF depends only on the 2 conformal ratios: $A(R, r)$

$$R = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2} \rightarrow \frac{\rho\rho'x_+x'_+y_-y'_-}{(x-x')^2_{\perp}(y-y')^2_{\perp}} \rightarrow \infty$$

$$r = \frac{[(x-y)^2(x'-y')^2 - (x'-y)^2(x-y')^2]^2}{(x-x')^2(y-y')^2(x-y)^2(x'-y')^2}$$

$$\rightarrow \frac{[(x'-y')^2_{\perp}x_+y_- + x'_+y'_-(x-y)^2_{\perp} + x_+y'_-(x'-y)^2_{\perp} + x'_+y_-(x-y')^2_{\perp}]^2}{(x-x')^2_{\perp}(y-y')^2_{\perp}x_+x'_+y_-y'_-}$$

Pomeron in CFT: “BFKL” representation of 4-point CF

Cornalba (2007); Cornalba, Costa, Penedones (2008); Costa, Gonsalves, Penedones (2012)

$$A(R, r) \stackrel{s \sim \rho \rho' \rightarrow \infty}{=} \frac{i}{2} \int d\nu f_+(\aleph(\alpha_s, \nu)) F(\alpha_s, \nu) \Omega(r, \nu) R^{\aleph(\alpha_s, \nu)/2}$$

$f_+(\omega) = \frac{e^{i\pi\omega} - 1}{\sin \pi\omega}$ is a signature factor

$$\Omega(r, \nu) = \frac{2\nu \sinh 2\pi\nu \Gamma(2 - \frac{d}{2} + 2i\nu) \Gamma(\frac{d}{2} - 1 - 2i\nu) \Gamma(d - 2)}{2^{d-1} \pi^{\frac{d+1}{2}} \Gamma(\frac{d}{2} - \frac{1}{2})} C_{-\frac{d}{2}+1+2i\nu}^{\frac{d}{2}-1} \left(\frac{1}{2\sqrt{r}} \right)$$

is a solution of the Laplace equation for H_{d-1} hyperboloid
 $(\partial_{H_{d-1}}^2 + \nu^2 + 1)\Omega(r, \nu) = 0$

Dynamics encoded in:

Pomeron intercept: $\aleph(g^2, \nu)$

Pomeron residue: $F(g^2, \nu)$

Light-cone limit to “BFKL” rep. of 4-point CF

$$R \xrightarrow{\text{Regge}+LC} \frac{x_{1+}x_{2+}x_{3-}x_{4-}}{x_{12\perp}^2 x_{34\perp}^2}, \quad r \xrightarrow{\text{Regge}+LC} \frac{x_{12+}^2 (x_{3-}x_{14\perp}^2 - x_{4-}x_{13\perp}^2)^2}{x_{1+}x_{2+}x_{3-}x_{4-}x_{12\perp}^2 x_{34\perp}^2}$$

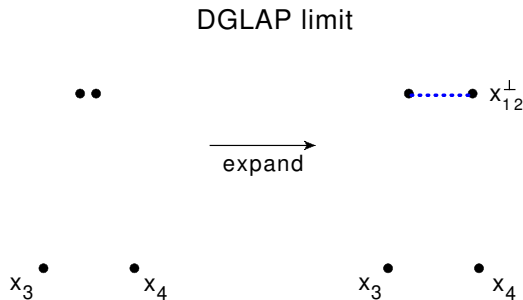
$$\begin{aligned} A(L_+, L_-; x_{1\perp}, x_{2\perp}, x_{3\perp}, x_{4\perp}) &= (x_{12}^2 x_{34}^2)^{4-2\varepsilon_* + \gamma_O} (\mu^2)^{2\gamma_O} \\ &\times \int dx_{2+} dx_{3+} \langle \mathcal{O}(L_+ + x_{2+}, x_{1\perp}) \mathcal{O}(x_{2+}, x_{2\perp}) \mathcal{O}(L_- + x_{4-}, x_{3\perp}) \mathcal{O}(x_{4-}, x_{4\perp}) \rangle \\ &\stackrel{\text{Regge}+LC}{=} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\xi}{2\pi} f_+(\bar{\aleph}(\xi)) F(\xi, \alpha_s) (L_+ L_-)^{1+\aleph(\xi)} (x_{12\perp}^2 x_{34\perp}^2)^{1-\varepsilon_* - \xi - \frac{\bar{\aleph}(\xi)}{2}} \\ &\times \int_0^1 du \frac{(\bar{u}u)^{1-\varepsilon_* - \xi + \frac{\aleph(\xi)}{2}}}{[x_{13\perp}^2 u + x_{14\perp}^2 \bar{u}]^{2-2\xi-2\varepsilon_*}} \end{aligned}$$

$$\xi \equiv \frac{d}{4} - i\nu = \frac{1}{2} - i\nu - \frac{\varepsilon_*}{2} \quad \bar{\aleph}(\xi, \alpha_s) \equiv \aleph(-i(\frac{1}{2} - \xi - \frac{\varepsilon_*}{2}), \alpha_s)$$

at $d_\perp = 2$ and in $\mathcal{N} = 4$ at NLO I.Balitsky and G.A.C. (2009)

but now we are in QCD!

“DGLAP” representation of amplitude + Regge limit



3-point correlation function is known in CFT

“DGLAP” representation of amplitude + Regge limit

$$\begin{aligned}
 A(L_+, L_-; x_{1\perp}, x_{2\perp}, x_{3\perp}, x_{4\perp}) &= [x_{12}^2 x_{34}^2]^{4-2\varepsilon_* + \gamma_O} (\mu^2)^{2\gamma_O} \\
 &\times \int dx_{2+} dx_{3-} \langle \mathcal{O}(L_+ + x_{2+}, x_{1\perp}) \mathcal{O}(x_{2+}, x_{2\perp}) \mathcal{O}(L_- + x_{4-}, x_{3\perp}) \mathcal{O}(x_{4-}, x_{4\perp}) \rangle \\
 &= \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{dj}{2\pi} (1 + e^{i\pi j}) (L_+ L_-)^j F(j, \alpha_s) (x_{12\perp}^2 x_{34\perp}^2)^{1-\varepsilon_* + \frac{\gamma_j}{2}} \\
 &\times \int_0^1 du \frac{(\bar{u}u)^{j+\frac{\gamma_j}{2}-\varepsilon_*}}{(x_{03\perp}^2 \bar{u} + x_{04\perp}^2 u)^{1+j+\gamma_j-2\varepsilon_*}}
 \end{aligned}$$

where

$$F(j, \alpha_s) = -i\pi c(j, \alpha_s) d(j, \alpha_s) \frac{\Gamma(1 + 2j - \gamma_j + 2\varepsilon_*)}{\Gamma^2(1 + j + \frac{\gamma_j}{2} - \varepsilon_*)}$$

Anomalous dimensions at $j \rightarrow 1$ from comparison of the two limits

$$\begin{aligned}
 & A(L_+, L_-; x_{1\perp}, x_{2\perp}, x_{3\perp}, x_{4\perp}) \\
 & \stackrel{LC+Regge}{=} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{dj}{2\pi} (1 + e^{i\pi j}) (L_+ L_-)^j F(j, \alpha_s) (x_{12\perp}^2 x_{34\perp}^2)^{1-\varepsilon_* + \frac{\gamma_j}{2}} \\
 & \quad \times \int_0^1 du \frac{(\bar{u}u)^{j + \frac{\gamma_j}{2} - \varepsilon_*}}{(x_{03\perp}^2 \bar{u} + x_{04\perp}^2 u)^{1+j+\gamma_j-2\varepsilon_*}} \\
 & \stackrel{Regge+LC}{=} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\xi}{2\pi} f_+(\bar{\aleph}(\xi)) F(\xi, \alpha_s) (L_+ L_-)^{1+\aleph(\xi)} (x_{12\perp}^2 x_{34\perp}^2)^{1-\varepsilon_* - \xi - \frac{\aleph(\xi)}{2}} \\
 & \quad \times \int_0^1 du \frac{(\bar{u}u)^{1-\varepsilon_* - \xi + \frac{\aleph(\xi)}{2}}}{[x_{13\perp}^2 u + x_{14\perp}^2 \bar{u}]^{2-2\xi-2\varepsilon_*}}
 \end{aligned}$$

The two representations are the same if we make the following identifications

$$\omega \equiv j - 1 = \bar{\aleph}(\xi, \alpha_s), \quad \gamma_j(\alpha_s) = -2\xi - \bar{\aleph}(\xi) = -2\xi - \omega$$

Anomalous dimensions at $j \rightarrow 1$ from comparison of the two limits

$$\omega \equiv j - 1 = \bar{\mathcal{N}}(\xi, \alpha_s), \quad \gamma_j(\alpha_s) = -2\xi - \bar{\mathcal{N}}(\xi) = -2\xi - \omega$$

to put the above identification as the one in Lipatov-Fadin and Camici-Ciafaloni,

use $\tilde{\gamma}_\omega = -\frac{\gamma_j}{2}$

$$\Rightarrow \omega = \bar{\mathcal{N}}\left(\tilde{\gamma}_\omega - \frac{\omega}{2}\right)$$

Solve at small ω and $\tilde{\gamma} \simeq 0$. With the NLO accuracy

$$\omega \simeq \bar{\mathcal{N}}(\tilde{\gamma}_\omega) - \frac{\omega}{2} \bar{\mathcal{N}}'(\tilde{\gamma}_\omega) \simeq \bar{\mathcal{N}}(\tilde{\gamma}_\omega) - \frac{1}{2} \bar{\mathcal{N}}(\tilde{\gamma}_\omega) \bar{\mathcal{N}}'(\tilde{\gamma}_\omega)$$

which can be rewritten as

$$\omega = \frac{\alpha_s N_c}{\pi} \left\{ \bar{\chi}(\tilde{\gamma}_\omega) + \frac{\alpha_s N_c}{4\pi} \left[\bar{\delta}(\tilde{\gamma}_\omega) - 2\bar{\chi}(\tilde{\gamma}_\omega) \bar{\chi}'(\tilde{\gamma}_\omega) \right] \right\}$$

Anomalous dimensions at $j \rightarrow 1$ from comparison of the two limits

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In Lipatov-Fadin's formalism the term $-2\bar{\chi}(\tilde{\gamma}_\omega) \bar{\chi}'(\tilde{\gamma}_\omega)$ came from changing the scale from sym to anti-sym.

Here it came from sym. energy scale R through the identification $\gamma_\omega = \xi + \frac{\bar{\mathfrak{N}}}{2}$

Anomalous dimensions at $j \rightarrow 1$ from comparison of the two limits

In the BFKL limit

$$j - 1 = \omega \rightarrow 0, \quad \frac{\alpha_s}{\omega} \sim 1$$

the anomalous dimensions are represented as a sum of series in $\frac{\alpha_s}{\omega}$

$$\tilde{\gamma}_\omega = \sum a_n \left(\frac{\alpha_s}{\omega}\right)^n + \omega \sum b_n \left(\frac{\alpha_s}{\omega}\right)^n + \omega^2 \sum c_n \left(\frac{\alpha_s}{\omega}\right)^n + \dots$$

So we expand in power of $\tilde{\gamma}$

$$\chi(\gamma) = \psi(1 - \varepsilon_*) - \gamma_E - \psi(\gamma) - \psi(1 - \varepsilon_* - \gamma) = \frac{1}{\gamma} + 2\varepsilon_* \zeta(3)\gamma + \dots = \frac{1}{\gamma} + \mathcal{O}(\gamma)$$

$$\omega = \frac{\alpha_s N_c}{\pi} \left\{ \frac{1}{\tilde{\gamma}_\omega} + \frac{\alpha_s N_c}{4\pi} \left[-\frac{11}{3} \frac{1}{\tilde{\gamma}_\omega^2} + 2\zeta(3) - \frac{395}{27} + \frac{11}{18} \pi^2 + \mathcal{O}(\tilde{\gamma}_\omega) \right] \right\}$$

which is exactly equation (23) of Fadin-Lipatov NLO BFKL paper (1998)

the corresponding anomalous dimension is

$$\begin{aligned}\tilde{\gamma}_\omega &= \frac{\alpha_s N_c}{\omega} + \mathcal{O}\left(\frac{\alpha_s}{\omega}\right)^4 \\ &+ \omega \left[-\frac{11}{12} \frac{\alpha_s N_c}{\pi \omega} - \left(\frac{\alpha_s N_c}{\pi \omega}\right)^3 \frac{1}{4} \left(\frac{395}{27} - \frac{11}{18} \pi^2 - 2\zeta(3) \right) + \mathcal{O}\left(\frac{\alpha_s}{\omega}\right)^4 \right]\end{aligned}$$

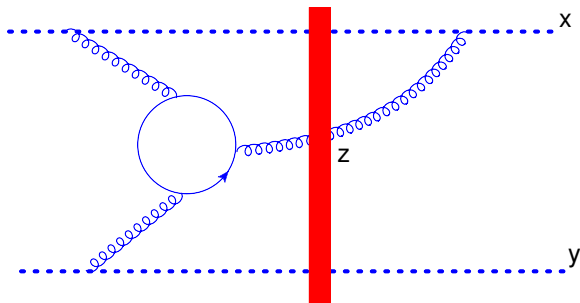
We see that at the critical point $d = 4 - 2\varepsilon_*$ we get the same anomalous dimensions as at $d = 4$ in accordance with analysis of Braun, Manashov, Moch, Strohmaier (2016-2017)

- BK in d -dimension is useful if we go to the Wilson-Fisher point
 - ▶ BFKL/BK are conformal invariant and eigenfunctions are power-like
 - ▶ We recovered the NLO running coupling from LO BK
- At Wilson-Fisher point $d = 4 - 2\varepsilon_*$,
 - ▶ the intercept is different than that at $d = 4$, but
 - ▶ BFKL gives the correct anomalous dimensions of light-ray operator $F_{+i} \nabla^{j-2}$ as $j \rightarrow 1$ (in all orders in α_s)
- This formalism allows one to calculate the running coupling part of the NNLO BFKL/BK (caveat: ambiguity mentioned above)

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- This formalism allows one to calculate the running coupling part of the NNLO BFKL/BK (caveat: ambiguity mentioned above)
- The Wilson-Fisher point allows exploiting the correspondence NGLs in jets and the BK/JIMWLK equation
 - ▶ *High-energy evolution in planar QCD to three loops: the non-conformal contribution* 2508.03794 [hep-ph] (2025) planar contribution
 - ▶ In this case there is an ambiguity to be resolved from the NGLs side

Diagrammatic calculation of the NNLO-BK/BFKL running coupling

Typical diagram for the NNLO-BK running coupling



Work in progress (I. Balitsky and G. A. C.)

Workshop at ECT* Trento, Italy

Bridging TMD Frameworks: Intersections, Tensions, and Applications

09 - 13 March 2026

Organizers:

Aleksandra Lelek (University of Antwerp)

Pia Zurita (Universidad Complutense de Madrid)

Alexey Vladimirov (Universidad Complutense de Madrid)

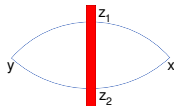
Giovanni A. Chirilli (University of Salento & INFN, Lecce)

Appendix

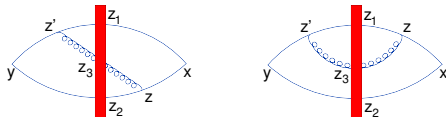
LO and NLO Impact Factor

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 I_{\mu\nu}^{\text{LO}}(z_1, z_2, x, y) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\ + \int d^2z_1 d^2z_2 d^2z_3 I_{\mu\nu}^{\text{NLO}}(z_1, z_2, z_3, x, y) [\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]$$

LO Impact Factor diagram: I^{LO}



NLO Impact Factor diagrams: I^{NLO}



$$[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A]^{\text{LO}} = \int \frac{d^2 z_1 d^2 z_2}{z_{12}^4} I_{\mu\nu}^{\text{LO}}(x, y; z_1, z_2) \langle \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}\rangle_A$$

$$[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A]^{\text{NLO}} = \int \frac{d^2 z_1 d^2 z_2}{z_{12}^4} d^2 z_3 \left[I_1^{\mu\nu}(z_1, z_2, z_3) + I_2^{\mu\nu}(z_1, z_2, z_3) \right] \\ \times [\text{tr}\{U_{z_1} U_{z_3}^\dagger\} \text{tr}\{U_{z_3} U_{z_2}^\dagger\} - N_c \text{tr}\{U_{z_1} U_{z_2}^\dagger\}]$$

where $I_2^{\mu\nu}(z_1, z_2, z_3)$ is finite and conformal, while

$$I_1^{\mu\nu}(z_1, z_2, z_3) = \frac{\alpha_s}{2\pi^2} I_{\mu\nu}^{\text{LO}} \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} e^{i\frac{s\alpha}{4} z_3}$$

is **rapidity divergent**.

$$\alpha \propto k^+$$

How to get the NLO Impact factor

$$\begin{aligned} \langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A &= \int \frac{d^2 z_1 d^2 z_2}{z_{12}^4} I_{\mu\nu}^{\text{LO}}(x, y; z_1, z_2) \langle \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}\rangle_A \\ &+ \int \frac{d^2 z_1 d^2 z_2}{z_{12}^4} d^2 z_3 I_{\mu\nu}^{\text{NLO}}(x, y; z_1, z_2, z_3; \eta) [\text{tr}\{U_{z_1} U_{z_3}^\dagger\} \text{tr}\{U_{z_3} U_{z_2}^\dagger\} - N_c \text{tr}\{U_{z_1} U_{z_2}^\dagger\}] + \dots \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} &[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A]^{\text{NLO}} - \int \frac{d^2 z_1 d^2 z_2}{z_{12}^4} I_{\mu\nu}^{\text{LO}}(x, y; z_1, z_2) [\langle \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}\rangle_A]^{\text{LO}} \\ &= \int \frac{d^2 z_1 d^2 z_2}{z_{12}^4} d^2 z_3 I_{\mu\nu}^{\text{NLO}}(x, y; z_1, z_2, z_3; \eta) [\text{tr}\{U_{z_1} U_{z_3}^\dagger\} \text{tr}\{U_{z_3} U_{z_2}^\dagger\} - N_c \text{tr}\{U_{z_1} U_{z_2}^\dagger\}] \end{aligned}$$

$$[\langle \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}\rangle_A]^{\text{LO}} = \frac{\alpha_s}{2\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} [\text{tr}\{U_{z_1} U_{z_3}^\dagger\} \text{tr}\{U_{z_3} U_{z_2}^\dagger\} - N_c \text{tr}\{U_{z_1} U_{z_2}^\dagger\}] \int_0^{e^\eta} \frac{d\alpha}{\alpha}$$

How to get the NLO Impact factor

$$\begin{aligned}
 & \int \frac{d^2 z_1 d^2 z_2}{z_{12}^4} d^2 z_3 I_{\mu\nu}^{\text{NLO}}(x, y; z_1, z_2, z_3; \eta) [\text{tr}\{U_{z_1} U_{z_3}^\dagger\} \text{tr}\{U_{z_3} U_{z_2}^\dagger\} - N_c \text{tr}\{U_{z_1} U_{z_2}^\dagger\}] \\
 &= \int \frac{d^2 z_1 d^2 z_2}{z_{12}^4} d^2 z_3 \left\{ I_2^{\mu\nu}(z_1, z_2, z_3) + \frac{\alpha_s}{2\pi^2} I_{\mu\nu}^{\text{LO}} \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[\int_0^{+\infty} \frac{d\alpha}{\alpha} e^{i\frac{\sigma\alpha}{4} \mathcal{Z}_3} - \int_0^{e^\eta} \frac{d\alpha}{\alpha} \right] \right\} \\
 & \quad \times [\text{tr}\{U_{z_1} U_{z_3}^\dagger\} \text{tr}\{U_{z_3} U_{z_2}^\dagger\} - N_c \text{tr}\{U_{z_1} U_{z_2}^\dagger\}]
 \end{aligned}$$

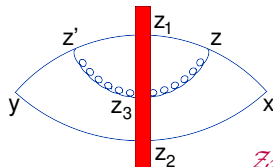
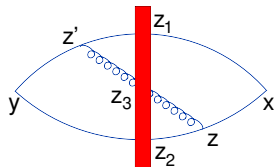
$$\left[\int_0^{+\infty} \frac{d\alpha}{\alpha} e^{i\frac{\sigma\alpha}{4} \mathcal{Z}_3} - \int_0^{e^\eta} \frac{d\alpha}{\alpha} \right] \rightarrow -\ln \frac{\sigma s}{4} \mathcal{Z}_3 - \frac{i\pi}{2} + C$$

where $\sigma = e^\eta$ and C is the Euler constant

$$\mathcal{Z}_3 \equiv \frac{(x - z_3)_\perp^2}{x^+} - \frac{(y - z_3)_\perp^2}{y^+}$$

$\mathcal{Z}_3$ is not conformal invariant in the transverse 2-d coordinate space, but QCD at tree level has to be conformal invariant.

NLO Impact Factor

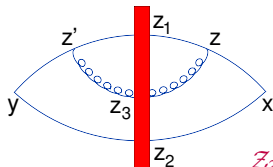
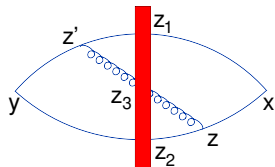


$$\mathcal{Z}_3 \equiv \frac{(x-z_3)_\perp^2}{x^+} - \frac{(y-z_3)_\perp^2}{y^+}$$

$$I_{\mu\nu}^{\text{NLO}}(x, y; z_1, z_2, z_3; \eta) = -I_{\mu\nu}^{\text{LO}} \times \frac{\alpha_s}{2\pi} \frac{z_{13}^2}{z_{12}^2 z_{23}^2} \ln \frac{\sigma s}{4} \mathcal{Z}_3 + \text{conf.}$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln \sigma$ is not invariant.

NLO Impact Factor



$$\mathcal{Z}_3 \equiv \frac{(x-z_3)_+^2}{x^+} - \frac{(y-z_3)_+^2}{y^+}$$

$$I_{\mu\nu}^{\text{NLO}}(x, y; z_1, z_2, z_3; \eta) = -I_{\mu\nu}^{\text{LO}} \times \frac{\alpha_s}{2\pi} \frac{z_{13}^2}{z_{12}^2 z_{23}^2} \ln \frac{\sigma s}{4} \mathcal{Z}_3 + \text{conf.}$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln \sigma$ is not invariant.

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$\begin{aligned} [\text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} &= \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\ &+ \frac{\alpha_s}{4\pi} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[\frac{1}{N_c} \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \right] \ln \frac{a z_{12}^2}{z_{13}^2 z_{23}^2} + O(\alpha_s^2) \end{aligned}$$

the impact factor becomes conformal at the NLO.

Operator expansion in conformal dipoles

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 I_{\mu\nu}^{\text{LO}}(z_1, z_2, x, y) \text{tr}[\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} \\ + \int d^2z_1 d^2z_2 d^2z_3 I_{\mu\nu}^{\text{NLO}}(z_1, z_2, z_3, x, y) \left[\frac{1}{N_c} \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \right]$$

$$I_{\mu\nu}^{\text{NLO}} = - I_{\mu\nu}^{\text{LO}} \frac{\alpha_s N_c}{4\pi} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2} \mathcal{Z}_3^2 + \text{conf.}$$

the energy dependence matrix element of Wilson-line operators

$$\Rightarrow a = -\frac{2x^+ y^+}{(x-y)^2}$$

The new NLO impact factor is conformally invariant.

NLO IF in coordinate space for BK I. Balitsky and G.A.C. (2010)

NLO IF in momentum space for BFKL I. Balitsky and G.A.C. (2012)

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 I_{\mu\nu}^{\text{LO}}(z_1, z_2, x, y) \text{tr}[\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} \\ + \int d^2z_1 d^2z_2 d^2z_3 I_{\mu\nu}^{\text{NLO}}(z_1, z_2, z_3, x, y) \left[\frac{1}{N_c} \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \right]$$

$$I_{\mu\nu}^{\text{NLO}} = - I_{\mu\nu}^{\text{LO}} \frac{\alpha_s N_c}{4\pi} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2} \mathcal{Z}_3^2 + \text{conf.}$$

The new NLO impact factor is conformally invariant.

NLO IF in coordinate space for BK I. Balitsky and G.A.C. (2010)

NLO IF in momentum space for BFKL I. Balitsky and G.A.C. (2012)

proton-Nucleus collision at NLO G.A.C., BW Xiao, F. Yuan (2011 - 2012)

to cure negativity of cross-section perhaps one could use similar constraint as the one used in NLO impact-factor

In conformal $\mathcal{N} = 4$ SYM theory one can construct the composite conformal dipole operator order by order in perturbation theory.

$$a \rightarrow e^{-2\eta} a \Rightarrow \left[\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} \right]_a^{\text{conf}} = 0$$

$$2a \frac{d}{da} [\text{Tr}\{U_x U_y^\dagger\}]_a^{\text{conf}} \sim \alpha_s K_{LO} \left(1 + \left(\frac{11}{3} N_c - \frac{2}{3} n_f \right) \text{run.coup.} \right) + \alpha_s^2 K_{NLO}^{\text{conf}}$$

Evolution equation for color dipoles in $\mathcal{N} = 4$

$$\begin{aligned}
 & \frac{d}{d\eta} \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\
 &= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ 1 - \frac{\alpha_s N_c}{4\pi} \left[\frac{\pi^2}{3} + 2 \ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{12}^2} \right] \right\} \\
 &\times [\text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^a \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}] \\
 &- \frac{\alpha_s^2}{4\pi^4} \int \frac{d^2 z_3 d^2 z_4}{z_{34}^4} \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{14}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \\
 &\times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^{a'} T^{b'} \hat{U}_{z_2}^{\dagger\eta} + T^b T^a \hat{U}_{z_1}^\eta [T^{b'}, T^{a'}] \hat{U}_{z_2}^{\dagger\eta}\} (\hat{U}_{z_3}^\eta)^{aa'} (\hat{U}_{z_4}^\eta - \hat{U}_{z_3}^\eta)^{bb'}
 \end{aligned}$$

NLO kernel = **Non-conformal term** + **Conformal term**.

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.

$$b = \frac{11}{3}N_c - \frac{2}{3}n_f$$

$$X = x - z \quad Y = y - z$$

$$\begin{aligned} \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\ &\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s}{4\pi} \left(b \ln(x-y)^2 \mu^2 - b \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} + \left(\frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} n_f \right) \right] \right. \\ &\quad \left. \left. - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \right\} \right. \\ &+ \frac{\alpha_s}{4\pi^2} \int d^2z' \left\{ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\ &\quad \left. - (z' \rightarrow z)] \frac{1}{(z-z')^4} \left[-2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \right. \\ &+ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z)] \\ &\quad \times \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Big\} + n_f\text{-terms} \Big) \end{aligned}$$

NLO kernel = **Running coupling terms** + **Non-conformal term** + **Conformal term**

$$\begin{aligned}
 & 2a \frac{d}{da} [\text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]_a^{\text{conf}} \\
 &= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ 1 - \frac{\alpha_s N_c}{4\pi} \left[\frac{\pi^2}{3} \right] \right\} \\
 &\times [\text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^a \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}] \\
 &- \frac{\alpha_s^2}{4\pi^4} \int \frac{d^2 z_3 d^2 z_4}{z_{34}^4} \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{14}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \\
 &\times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^{a'} T^{b'} \hat{U}_{z_2}^{\dagger\eta} + T^b T^a \hat{U}_{z_1}^\eta [T^{b'}, T^{a'}] \hat{U}_{z_2}^{\dagger\eta}\} (\hat{U}_{z_3}^\eta)^{aa'} (\hat{U}_{z_4}^\eta - \hat{U}_{z_3}^\eta)^{bb'}
 \end{aligned}$$

NLO kernel = Conformal

NLO evolution of composite “conformal” dipoles in QCD

$$\begin{aligned}
 2a \frac{d}{da} [\text{tr}\{\hat{U}_{z_1} U_{z_2}^\dagger\}]^{\text{conf}} &= \frac{\alpha_s}{2\pi^2} \int d^2 z_3 \left([\text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_2}^\dagger\} - N_c \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_2}^\dagger\}]^{\text{conf}} \right. \\
 &\times \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[1 + \frac{\alpha_s}{4\pi} \left(b \ln z_{12}^2 \mu^2 + b \frac{z_{13}^2 - z_{23}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{13}^2}{z_{23}^2} + N_c \left(\frac{67}{9} - \frac{\pi^2}{3} \right) - \frac{10}{9} n_f \right) \right] \\
 &+ \frac{\alpha_s}{4\pi^2} \int \frac{d^2 z_4}{z_{34}^4} \left\{ \left[-2 + \frac{z_{14}^2 z_{23}^2 + z_{24}^2 z_{13}^2 - 4z_{12}^2 z_{34}^2}{2(z_{14}^2 z_{23}^2 - z_{24}^2 z_{13}^2)} \ln \frac{z_{14}^2 z_{23}^2}{z_{24}^2 z_{13}^2} \right] \right. \\
 &\times [\text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} \{\hat{U}_{z_4} \hat{U}_{z_2}^\dagger\} - \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger \hat{U}_{z_4} \hat{U}_{z_2}^\dagger \hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} - (z_4 \rightarrow z_3)] \\
 &+ \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[2 \ln \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \left(1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right) \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right] \\
 &\times [\text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} \text{tr}\{\hat{U}_{z_4} \hat{U}_{z_2}^\dagger\} - \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_4}^\dagger \hat{U}_{z_3} \hat{U}_{z_2}^\dagger \hat{U}_{z_4} \hat{U}_{z_3}^\dagger\} - (z_4 \rightarrow z_3)] + n_f\text{-terms} \Big)
 \end{aligned}$$

$$b = \frac{11}{3} N_c - \frac{2}{3} n_f$$

I. Balitsky and G.A.C

$K_{\text{NLO BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part
 + Conformal analytic ($\mathcal{N} = 4$) part

Linearized $K_{\text{NLO BK}}$ reproduces the known result for the forward NLO BFKL kernel Fadin and Lipatov (1998).

DIS at high-energy $-q^2 = Q^2 \gg P^2 \quad s = (P + q)^2 \gg Q^2$

$$\sigma^{\gamma^* p}(x_B, Q^2) = \int d\nu F(\nu) x_B^{-\aleph(\nu)-1} \left(\frac{Q^2}{P^2} \right)^{\frac{1}{2} + i\nu}$$

$\aleph(\gamma)$ is the BFKL pomeron intercept.

$$\gamma = \frac{1}{2} = i\nu$$

Saddle point approximation:

$$\sigma^{\gamma^* p}(x_B, Q^2) \sim \left(\frac{1}{x_B} \right)^{\bar{\alpha}_s 4 \ln 2}$$

From local operators to Light-ray operators

N -th moment of the structure function is

$$M_N = \int_0^1 dx_B x_B^{N-1} \sigma^{\gamma^* p}(x_B, Q^2) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\gamma \frac{F(\gamma)}{N-1-\aleph(\gamma)} \left(\frac{Q^2}{P^2} \right)^\gamma$$

$$\aleph(\gamma) = \bar{\alpha}_s \left(2\psi(1) - \psi(\gamma) - \sum_{m=1}^N \frac{1}{m-\gamma} - \psi(N+1-\gamma) \right)$$

The BFKL is given as a sum over all the residues

- Leading residue: $\aleph(\gamma) \rightarrow \frac{\bar{\alpha}_s}{\gamma-1} \quad \bar{\alpha}_s = \frac{\alpha_s N_c}{\pi}$
- Next-to-Leading residue: $\aleph(\gamma) \rightarrow \frac{\bar{\alpha}_s}{\gamma-2}$

Closing the contour on the poles we get the anomalous dimensions of the leading and higher twist operators at the *nonphysical point* $N = 1$.

$$\int_0^1 dx_B x_B^{n-1} \sigma^{\gamma^* P}(x_B, Q^2) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\gamma \frac{F(\gamma)}{\omega - \aleph(\gamma)} \left(\frac{Q^2}{P^2} \right)^\gamma$$

Analytic continuation: $N - 1 \rightarrow \omega$ complex continuous variable

⇒ Residues $\omega = \aleph(\gamma) \simeq \frac{\bar{\alpha}_s}{\gamma-1}$;

Leading – Twist $\gamma(\alpha_s, \omega) = \frac{\bar{\alpha}_s}{\omega} + \mathcal{O}(\alpha_s^2), \quad \sigma(\omega, Q^2) \sim \left(\frac{Q^2}{P^2} \right)^{\frac{\bar{\alpha}_s}{\omega}}$

$\alpha_s \ln \frac{1}{x_B} \sim 1 \rightarrow \frac{\alpha_s}{\omega} \sim 1 \quad x_B \rightarrow 0 \Leftrightarrow \omega \rightarrow 0 \quad \Rightarrow$ resummation: BFKL eq.

Thus, we get the analytic continuation of anomalous dimension at the nonphysical point $N \rightarrow 1$ of twist-2 gluon operator $F_{\xi+}^a \nabla^{N-2} F_{+}^{\xi a}$

twist-2 gluon operator $F_{\xi+}^a \nabla^{N-2} F_{+}^{\xi a}$

Anomalous dimension (gluon only)

$$\gamma_N = \frac{\alpha_s N_c}{\pi} \left(-\frac{1}{N(N-1)} - \frac{1}{(N+1)(N+2)} + \psi(N+1) + \gamma_E - \frac{11}{12} \right) + \mathcal{O}(\alpha_s^2)$$

BFKL gives $\gamma(N, \alpha_s)$ at $N \rightarrow 1$

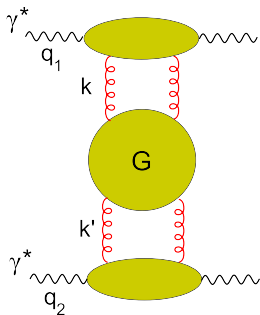
$$\gamma_N = \left[A_N^{\text{LO BFKL}} + (N-1) B_N^{\text{NLO BFKL}} + \dots \right] \left(\frac{\alpha_s N_c}{\pi(N-1)} \right)^n$$

- LO BFKL: Jaroszewicz (1982)
- NLO BFKL: Fadin and Lipatov; Camici, Ciafaloni (1998)
- NNLO BFKL in $\mathcal{N} = 4$ SYM: Gromov *et al* (2016)

at $N=1$ $F_{-i}^a \nabla_{-}^{-1} F_{-}^{ai}$ is a non-local operator

We want an operator language

Balitsky-Fadin-Kuraev-Lipatov equation



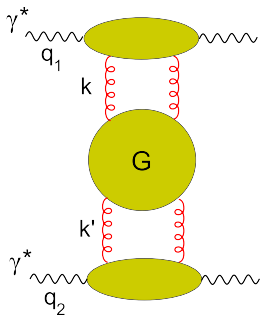
$$\frac{\partial}{\partial Y} G(k, k', Y) = \int d^2 q K(k, q) G(q, k', Y)$$

$$G(k, k', Y = 0) = \frac{1}{2\pi k} \delta(k - k')$$

$$k \equiv |\vec{k}_\perp| \quad \text{and} \quad k' \equiv |\vec{k}'_\perp|$$

$$Y = \ln \frac{s}{k k'} \quad \text{and} \quad s = (q_1 + q_2)^2$$

Balitsky-Fadin-Kuraev-Lipatov equation



$$\frac{\partial}{\partial Y} G(k, k', Y) = \int d^2 q K(k, q) G(q, k', Y)$$

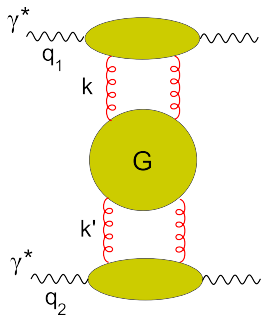
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- Resum $(\alpha_s Y)^n \longrightarrow$ LO BFKL eq.

Balitsky-Fadin-Kuraev-Lipatov equation



$$\frac{\partial}{\partial Y} G(k, k', Y) = \int d^2 q K(k, q) G(q, k', Y)$$

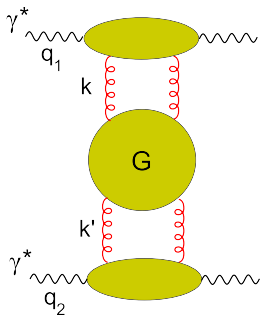
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- Resum $(\alpha_s Y)^n \longrightarrow$ LO BFKL eq.
- Resum $\alpha_s (\alpha_s Y)^n \longrightarrow$ NLO BFKL eq.

Balitsky-Fadin-Kuraev-Lipatov equation



$$\frac{\partial}{\partial Y} G(k, k', Y) = \int d^2 q K(k, q) G(q, k', Y)$$

$$G(k, k', Y = 0) = \frac{1}{2\pi k} \delta(k - k')$$

$$k \equiv |\vec{k}_\perp| \quad \text{and} \quad k' \equiv |\vec{k}'_\perp|$$

$$Y = \ln \frac{s}{k k'} \quad \text{and} \quad s = (q_1 + q_2)^2$$

- Resum $(\alpha_s Y)^n \longrightarrow$ LO BFKL eq.
- Resum $\alpha_s (\alpha_s Y)^n \longrightarrow$ NLO BFKL eq.
- The kernel is real and symmetric: $K(k, k') = K(k', k) \Rightarrow K(k, k')$ is Hermitian and the eigenvalues are real.

$$\frac{\partial}{\partial Y} G(k, k', Y) = \int d^2 q K^{\text{LO}}(k, q) G(q, k', Y)$$

$$\int d^2 q K^{\text{LO}}(k, q) (q^2)^{-1+\gamma} = \bar{\alpha}_\mu \chi_0(\gamma) (k^2)^{-1+\gamma} \quad \bar{\alpha}_\mu \equiv \frac{\alpha_\mu N_c}{\pi}$$

- $(k^2)^{-1+\gamma}$ are eigenfunctions.

$$\frac{\partial}{\partial Y} G(k, k', Y) = \int d^2 q K^{\text{LO}}(k, q) G(q, k', Y)$$

$$\int d^2 q K^{\text{LO}}(k, q) (q^2)^{-1+\gamma} = \bar{\alpha}_\mu \chi_0(\gamma) (k^2)^{-1+\gamma} \quad \bar{\alpha}_\mu \equiv \frac{\alpha_\mu N_c}{\pi}$$

- $(k^2)^{-1+\gamma}$ are eigenfunctions.
- For $\gamma = \frac{1}{2} + i\nu$ and ν real parameter $\Rightarrow (k^2)^{-1+\gamma}$ form a complete set.
- $\Rightarrow$ LO eigenvalues $\chi_0(\nu) = 2\psi(1) - \psi(\frac{1}{2} + i\nu) - \psi(\frac{1}{2} - i\nu)$ are real and sym. $\nu \leftrightarrow -\nu$
- LO BFKL is Conformal invariant.

$$G(k, k', Y) = \int \frac{d\nu}{2\pi^2 k k'} \left(\frac{k^2}{k'^2} \right)^{i\nu} e^{\bar{\alpha}_\mu \chi_0(\nu) Y}$$

BFKL equation in the $\mathcal{N}=4$ SYM case

- In $\mathcal{N} = 4$ SYM theory the coupling constant does not run.
- $\Rightarrow (k^2)^{-\frac{1}{2}+i\nu}$ are eigenfunctions at any order.

$$K(q, k) = \alpha_{\text{SYM}} K^{\text{LO}}(q, k) + \alpha_{\text{SYM}}^2 K^{\text{NLO}}(q, k) + \dots$$

BFKL equation in the $\mathcal{N}=4$ SYM case

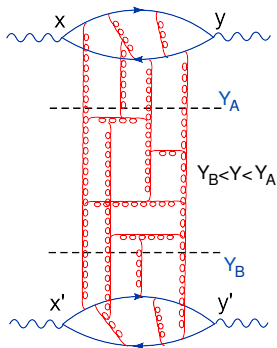
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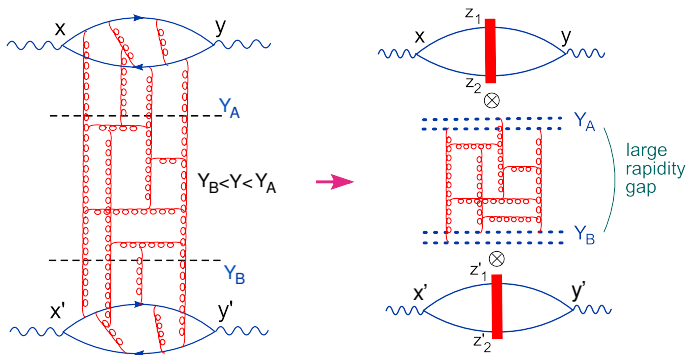
$$K(q, k) = \alpha_{\text{SYM}} K^{\text{LO}}(q, k) + \alpha_{\text{SYM}}^2 K^{\text{NLO}}(q, k) + \dots$$

$$\int d^2 q K(q, k) (q^2)^{-\frac{1}{2}+i\nu} = [\alpha_{\text{SYM}} \chi_0(\nu) + \alpha_{\text{SYM}}^2 \chi_1(\nu) \dots] (k^2)^{-\frac{1}{2}+i\nu}$$

$$G(k, k', Y) = \int \frac{d\nu}{2\pi^2 k k'} e^{[\alpha_{\text{SYM}} \chi_0(\nu) + \alpha_{\text{SYM}}^2 \chi_1(\nu) \dots]} \left(\frac{k^2}{k'^2} \right)^{i\nu}$$

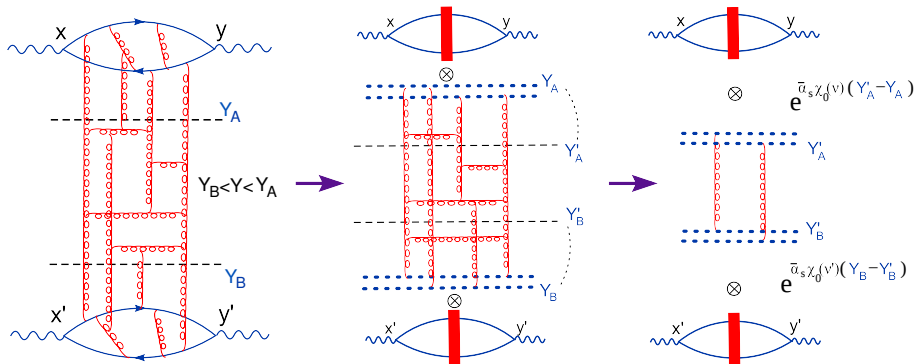
- The eigenvalues $\bar{\alpha}_\mu \chi_0(\nu) + \bar{\alpha}_\mu^2 \chi_1^{\text{SYM}}(\nu) + \dots$ are real and symmetric for $\nu \leftrightarrow -\nu$.





$$\langle j^\alpha(x) j^\beta(y) j^\rho(x') j^\lambda(y') \rangle \propto I_A^{\alpha\beta}(x, y; z_1, z_2) I_B^{\rho\lambda}(x', y'; z'_1, z'_2) \otimes \langle \text{tr}\{U_{z_1} U_{z_2}^\dagger\}^{Y_A} \text{tr}\{U_{z_3} U_{z_4}^\dagger\}^{Y_B} \rangle$$

$\gamma^* \gamma^*$ scattering cross-section at LO



$$\langle j^\alpha(x) j^\beta(y) j^\rho(x') j^\lambda(y') \rangle \propto I_A^{\alpha\beta}(x, y; z_1, z_2) I_B^{\rho\lambda}(x', y'; z'_1, z'_2) \otimes \langle \text{tr}\{U_{z_1} U_{z_2}^\dagger\}^{Y_A} \rangle_A \langle \text{tr}\{U_{z_3} U_{z_4}^\dagger\}^{Y_B} \rangle_A$$

$$\mathcal{A}^{\alpha\beta\rho\lambda}(q_1, q_2) \propto i \frac{\alpha_s^2}{Q_1 Q_2} \int d\nu I_{\text{LO}}^{\alpha\beta}(\nu) I_{\text{LO}}^{\rho\lambda}(\nu) \left(\frac{Q_1^2}{Q_2^2} \right)^{i\nu} e^{\bar{\alpha}_\mu \chi_0(\nu) (Y_A - Y_B)}$$

$$Y_A = \frac{1}{2} \ln \frac{s}{Q_1^2}, \quad Y_B = -\frac{1}{2} \ln \frac{s}{Q_2^2}, \quad s = (q_1 + q_2)^2$$

$$\mathcal{A}^{\alpha\beta\rho\lambda}(q_1, q_2) \propto i \frac{\alpha_s^2}{Q_1 Q_2} \int d\nu I_{\text{LO}}^{\alpha\beta}(\nu) I_{\text{LO}}^{\rho\lambda}(\nu) \left(\frac{Q_1^2}{Q_2^2} \right)^{i\nu} e^{\bar{\alpha}_\mu \chi_0(\nu) \ln \frac{s}{Q_1 Q_2}}$$

$$\begin{aligned}
 (x-y)^4 T\{\bar{\psi}(x)\gamma^\mu\hat{\psi}(x)\bar{\psi}(y)\gamma^\nu\hat{\psi}(y)\} &= \int \frac{d^2z_1 d^2z_2}{z_{12}^4} \left\{ I_{\text{LO}}^{\mu\nu}(z_1, z_2) \left[1 + \frac{\alpha_s}{\pi}\right] [\text{tr}\{\hat{U}_{z_1}\hat{U}_{z_2}^\dagger\}]_{a_0} \right. \\
 &+ \int d^2z_3 \left[\frac{\alpha_s}{4\pi^2} \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left(\ln \frac{\kappa^2(\zeta_1 \cdot \zeta_3)(\zeta_1 \cdot \zeta_2)}{2(\kappa \cdot \zeta_3)^2(\zeta_1 \cdot \zeta_2)} - 2C \right) I_{\text{LO}}^{\mu\nu} + I_2^{\mu\nu} \right] \\
 &\times [\text{tr}\{\hat{U}_{z_1}\hat{U}_{z_3}^\dagger\}\text{tr}\{\hat{U}_{z_3}\hat{U}_{z_2}^\dagger\} - N_c \text{tr}\{\hat{U}_{z_1}\hat{U}_{z_2}^\dagger\}]_{a_0} \left. \right\}
 \end{aligned}$$

where

$$\begin{aligned}
 (I_2)_{\mu\nu}(z_1, z_2, z_3) &= \frac{\alpha_s}{16\pi^8} \frac{\mathcal{R}^2}{(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)} \left\{ \frac{(\kappa \cdot \zeta_2)}{(\kappa \cdot \zeta_3)} \frac{\partial^2}{\partial x^\mu \partial y^\nu} \left[-\frac{(\kappa \cdot \zeta_1)^2}{(\zeta_1 \cdot \zeta_3)} + \frac{(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)}{(\zeta_2 \cdot \zeta_3)} \right. \right. \\
 &+ \frac{(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_3)(\zeta_1 \cdot \zeta_2)}{(\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot \zeta_3)} - \frac{\kappa^2(\zeta_1 \cdot \zeta_2)}{(\zeta_2 \cdot \zeta_3)} \left. \right] + \frac{(\kappa \cdot \zeta_2)^2}{(\kappa \cdot \zeta_3)^2} \frac{\partial^2}{\partial x^\mu \partial y^\nu} \left[\frac{(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_3)}{(\zeta_2 \cdot \zeta_3)} - \frac{\kappa^2(\zeta_1 \cdot \zeta_3)}{2(\zeta_2 \cdot \zeta_3)} \right] \\
 &\left. + (\zeta_1 \leftrightarrow \zeta_2) \right\}
 \end{aligned}$$