Factorization violation in gluon amplitudes

Aniruddha Venkata
with Claude Duhr and Chi Zhang

Universität Bonn
Bethe Center for Theoretical Physics
avenkata@uni-bonn.de

October 16, 2025

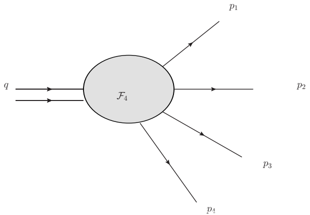
Questions and Plan

- Which S-matrices violate factorization? **All amplitudes with 2 incoming particles and $n-2$ outgoing particles, $n \geq 5$ particles**
- Is factorization violation a subleading color effect? **No!**
- Are splitting functions analytic in the their parameters? **No!**
- What is the domain of analyticity for the splitting functions?
- Is there an IR safe hard function? **Yes!**
- Are there (sufficiently inclusive) cross-section level observables that violate factorization? **No**
- How is factorization restored for inclusive cross-sections?

Setup–Kinematics

Denote the four particle form factor $\mathcal{F}_4^{(L)}\left(\{s_{ij}, s_{ijk}, q^2\}\right)$– Will be the four particle form factor in $\mathcal{N}=4$ SYM.

$$s_{ij} = (p_i + p_j)^2 \qquad\qquad s_{ijk} = (p_i + p_j + p_k)^2$$



$$-s_{12} + s_{34} + s_{412} + s_{123} = q^2 \qquad - s_{23} + s_{41} + s_{123} + s_{234} = q^2$$

$$-s_{34} + s_{12} + s_{234} + s_{341} = q^2$$

Setup-Dynamics (QCD)

In the $1 \to 4$ region, a Higgs decaying into 4 gluons,

$$H(q) \to g(p_1) + g(p_2) + g(p_3) + g(p_4)$$

In the $2 \to 3$ region, we consider Higgs production through gluon fusion.

$$g(p_1) + g(p_3) \to H(q) + g(p_2) + g(p_4)$$

The coupling of gluons to the Higgs is then captured through the effective coupling in the heavy top limit, which is given by

$$\begin{aligned} \mathcal{L}_{int} &= \frac{C}{2} H \mathrm{tr}(F_{\mu\nu} F^{\mu\nu}). \\ C &= \frac{\alpha_s}{6\pi v}\left(1 + \frac{11\alpha_s}{42\pi} + O(\alpha_s^2)\right). \end{aligned}$$

Glossary

- **Euclidean** $s_{ij}, s_{ijk}, q^2 \leq 0$. The natural region to carry our loop integrals (through Wick rotation).
- **Pseudo-Euclidean** The finite parts of the Euclidean result have the same functional form in the $s_{ij}, s_{ijk}, q^2 \geq 0$. This is the $1 \to 4$ kinematics. The difference is an overall phase.
- Admits a path of analytic continuation with no relative phase Eq.

$$\log\left(\frac{-s_{12}}{-s_{23}}\right) \to \log\left(\frac{-s_{12}e^{i\theta}}{-s_{23}e^{i\theta}}\right) = \log\left(\frac{-s_{12}}{-s_{23}}\right)$$

- **"Gluon fusion region"** $2 \to 3$ region $s_{ij} \leq 0$, $s_{123} = (q - p_4)^2 \geq 0$ etc.

Collinear limits

- Our interest here is in the collinear limit of scattering. We study the limit

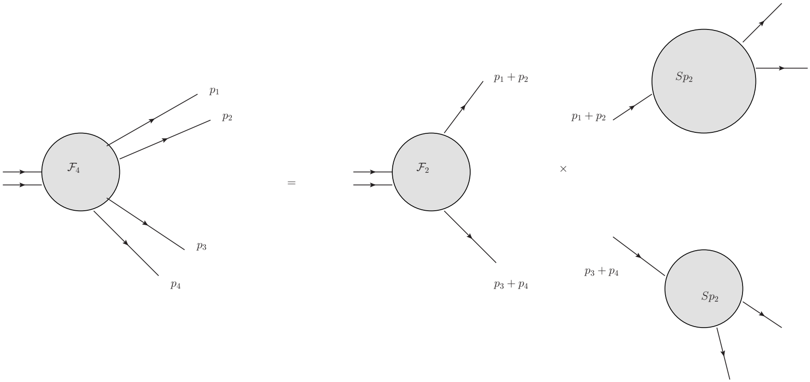
$$p_1 = \alpha p_2$$
$$p_3 = \beta p_4$$

- The kinematic space is spanned by α, β, s_{24}, a singular slice on the five dimensional kinematics.
- Additionally, the phases of α, β are related

$$(\alpha + 1)(\beta + 1)s_{24} = q^2$$

- We define $\alpha + 1 = z, \beta + 1 = z'$.

Factorization– Pseudo Euclidean region

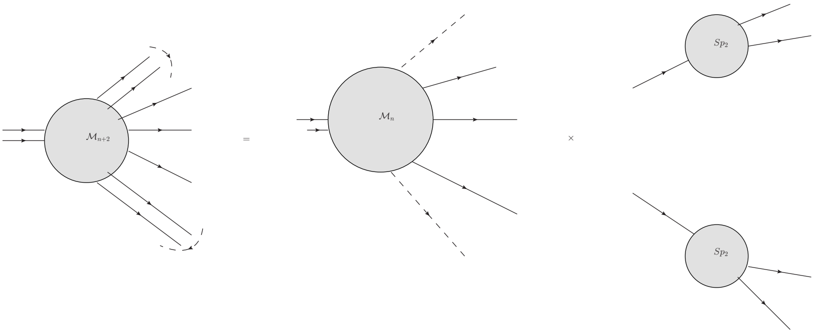


$$\mathcal{F}_{4,DC} = \mathcal{F}_2(q) \times \mathsf{Sp}(p_1, p_2) \times \mathsf{Sp}(p_3, p_4).$$

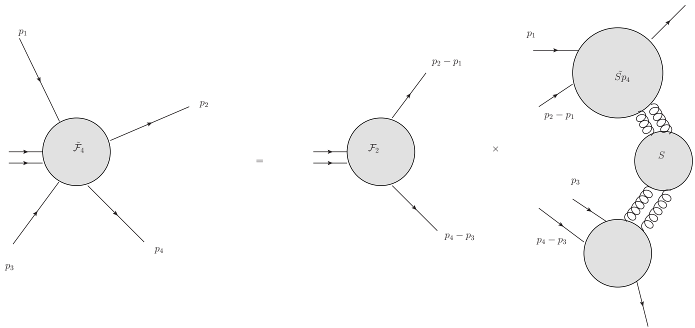
Pseudo Euclidean region

These splitting functions don't depend on the process – universality.
The hard sub-process can be a complicated higher point amplitude

$$\mathcal{M}_{\mathsf{n+4}}(p_1, p_2, p_3, p_4, q_1 \dots q_n) = \mathcal{M}_{\mathsf{n+4}}(q_1 \dots q_n)\mathsf{Sp}_2(p_1, p_2)\mathsf{Sp}_2(p_3, p_4)$$



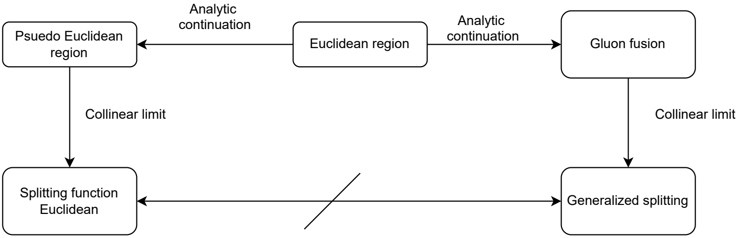
Generalized splitting functions–Higgs production region



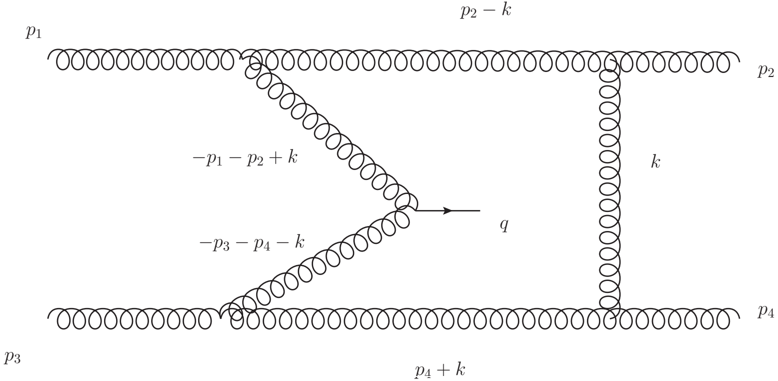
$$\tilde{\mathcal{F}}_{4,DC} = \mathcal{F}_2(q) \times \mathsf{Sp}_4(p_1, p_2, p_3, p_4).$$

Also Universal– **Planar Graphs are always going to factorize "Universally".**

Generalized splitting functions

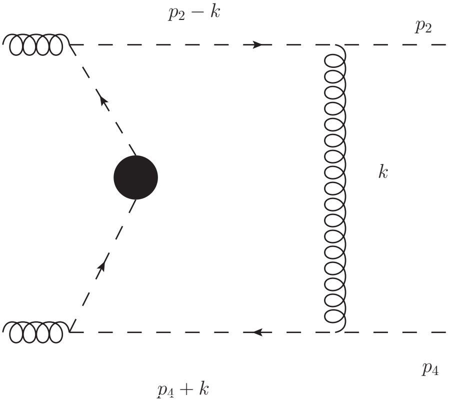


Glauber pinches



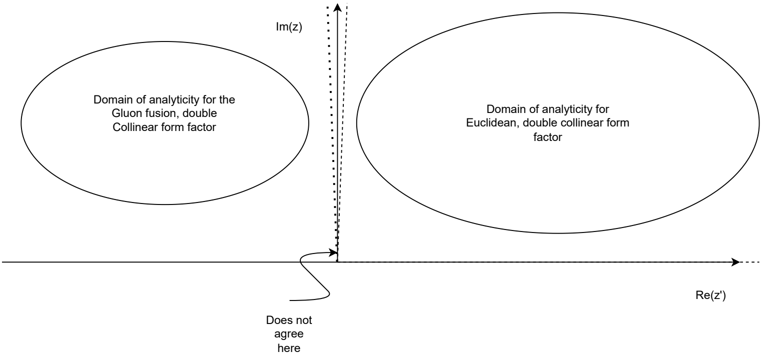
The region of interest is defined by $k \to 0$. The denominators $p_2 - k, -p_1 - p_2 + k$ are both onshell.

Glauber pinches in $\mathcal{N} = 4$ SYM



A non-planar graph with planar color flow. The graph also violates collinear factorization.

Analyticity of the form factors on the collinear slice



The domains of analyticity for the double collinear form factors. The edge of the wedge theorem cannot be applied. **Crossing symmetrty is violated on this slice.**

Explicit checks

- At tree level, the form factor is known

$$\mathcal{F}_4^{(0)} = \frac{\delta^4\left(q - \sum_{i=1}^{4} \lambda_i \tilde{\lambda}_i\right)\left(\langle 24\rangle\right)^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}.$$

- Here, the full form factor with color factors is generated from these color ordered ones by the equation

$$F_4 = \sum_{\text{perms}} T_1 T_2 T_3 T_4 \mathcal{F}_4(p_1, p_2, p_3, p_4)$$

- The form factor at higher loops are necessarily proportional to the tree level form factor (SUSY ward identity+R symmetry)

$$\begin{aligned} \mathcal{F}_4 &= \mathcal{F}_4^{(0)}\left(1 + \sum_{L=1}^{\infty} \bar{\mathcal{R}}^{(L)}\right) \\ &= \mathcal{F}_4^{(0)}\left(e^{\frac{\Gamma}{4\epsilon^2}(-(s_{12})^{-\epsilon} + cyclic) + \mathcal{E}_4^{(1)}}\right)\mathcal{R} \end{aligned}$$

One Loop result

$$
\begin{aligned}
\mathcal{F}_4^{(1)} &= \mathcal{F}_4^{(0)} \left(-\frac{g^2}{\epsilon^2} \left((-s_{12})^{-\epsilon} + \text{cyclic} \right) \right. \\
&+ g^2 \left(-2\mathrm{Li}_2 \left(1 - \frac{s_{123}}{q^2} \right) - \mathrm{Li}_2 \left(1 - \frac{q^2 s_{23}}{s_{123} s_{234}} \right) \right. \\
&- \log\left(\frac{s_{12}}{q^2}\right) \log\left(\frac{s_{23}}{q^2}\right) + \log\left(\frac{s_{123}}{q^2}\right) \log\left(\frac{s_{12} s_{23}}{s_{123} s_{234}}\right) + \frac{\pi^2}{6} \Bigg) \\
&+ \text{cyclic}
\end{aligned}
$$

In the double collinear limit

$$
-\log\left(\frac{s_{12}}{q^2}\right) \log\left(\frac{s_{23}}{q^2}\right) + \log\left(\frac{s_{123}}{q^2}\right) \log\left(\frac{s_{12} s_{23}}{s_{123} s_{234}}\right) \to
$$

$$
\log(s_{12}) \log\left(\frac{(\alpha+1)^2}{\alpha}\right) + \log(s_{34}) \log\left(\frac{(\beta+1)^2}{\beta}\right) + \text{finite} + O(s_{12}, s_{34})
$$

Appearance of a new leading power term

The Logarithms can be smoothly analytically continued into the gluon fusion region.
The Dilogarithm is curious

$$\mathrm{Li}_2\left(1-\frac{q^2 s_{23}}{s_{123}s_{234}}\right) \to \mathrm{Li}_2(0) = 0 + O(s_{12})$$

This relationship holds in the Euclidean region. However, The analytic continuation of the DiLog to the gluon fusion region reads

$$\mathrm{Li}_2\left(1-\frac{q^2 s_{23}}{s_{123}s_{234}}\right) \to \mathrm{Li}_2\left(1-\frac{q^2 s_{23}}{s_{123}s_{234}}\right) - i\pi\log\left(1-\frac{q^2 s_{23}}{s_{123}s_{234}}\right)$$

Correlated splitting function

The log in the gluon fusion region is

$$-i\pi \log\left(1 - \frac{q^2 s_{23}}{s_{123}s_{234}}\right)$$

- A new singular contribution in the double collinear limit.
- Correlates s_{12}, s_{34}.
- Violates factorization.
- Appears only when $Re(z') \leq 0$.
- Is this the only letter for five particle kinematics which can do this? Yes– Upto permutations.
- Clearly a planar effect.
- Checks upto 4 loops and 8 particles for MHV amplitude.

Connection to Regge physics

- Now repeat for amplitudes.
- I study the case of the six particle amplitude, and by universailty we have checked, all higher particles, can be computed.
- Just as before, we can factor out all the color and polarization factors.

$$A_6 = \sum_{\text{perms}} T_1 \ldots T_6 \mathcal{A}_6(p_1, \ldots p_6)$$

and

$$\mathcal{A}_6 = \frac{\langle 16 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle 61 \rangle} A_{BDS} \mathcal{R}(u_1, u_2, u_3)$$

- $\mathcal{R}$ is a remainder function which starts at two loops, a conformally invariant object. depends on three cross ratios $u_1 = \frac{s_{12} s_{45}}{s_{123} s_{345}}$

The double collinear limit

- We take $(u_1, u_2, u_3) = (0, 1, 0)$, holding hard scales finite which is equivalent to $\frac{u_1}{1-u_2}, \frac{u_3}{1-u_2}$ being held finite.
- In this limit we find

$$
\begin{aligned}
A_N \quad &\sim \textstyle\sum_{h,h'} \mathsf{Sp}_h^{(2)}(-p_1, p_2; \xi_1 + i\varepsilon)\, \mathsf{Sp}_{h'}^{(2)}(\pm p_3, \mp p_4; \xi_2 + i\varepsilon) \\
&\qquad\qquad \times \Delta_{\mathrm{I/II}}^{h,h'}\left(\tfrac{u_1}{1-u_2}, \tfrac{u_3}{1-u_2}\right) A_{N-2}(P^{h_P}, Q^{h_Q}, \ldots)
\end{aligned}
$$

This is the same limit as the MRK limit because of conformal invariance of the remainder function! These functions are known to all orders from integrability in the MRK context!

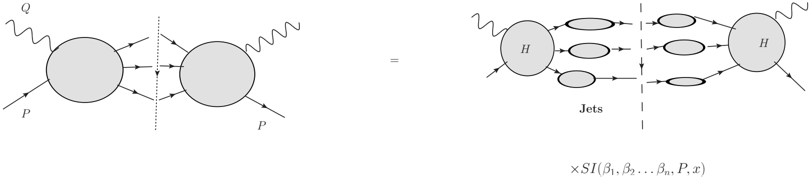
What about cross-sections?

Consider DIS– Similar but color dependant violation found by Catani-De Florian- Rodrigo

- Replace the hard, **final state** directions by Wilson lines.
- UV subtraction scheme: every soft parton, real or virtual has momentum less than Λ.
- Two components: The hard function+ final state jets which are like $e + e-$ final state jets, and soft+initial state function.
- Study the soft+inital state function.

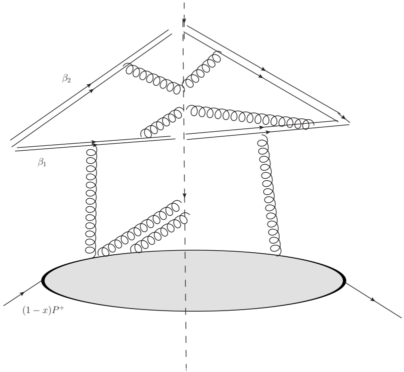
Time-like factorization

We will assume that all time like seperated final state partons are replaced by Wilson lines in the soft function $SI(\beta_1, \beta_2 \ldots \beta_n, P, x)$.



In addition, the incoming parton is also undergoing partonic splitting inside the color space matrix SI.

What is the soft +intial state function?



$$SI(\beta_1 \ldots \beta_n, x) = \sum_X \int dy e^{iyxP^+} \langle P|\psi^\dagger(y) \prod_{i=1}^{n} e^{-ig\int_0^\infty A(\beta_i\lambda)\cdot\beta_i d\lambda} |X\rangle$$
$$\times \text{C.C}$$

This matrix element is written in light cone gauge $A^+ = 0$.

Optical Theorem to the rescue

$$\begin{aligned} & \sum_X \int dy e^{iyxP^+} \langle P|\psi^\dagger(y) \prod_{i=1}^{n} e^{-ig\int_0^\infty A(\beta_i\lambda)\cdot\beta_i d\lambda} |X\rangle \\ & \times \int dz e^{-izxP^+} \langle X|\psi(z) \prod_{i=1}^{n} e^{ig\int_0^\infty A(\beta_i\lambda)\cdot\beta_i d\lambda} |P\rangle \\ = \;& 2\mathrm{Im} \int \frac{dz}{2\pi} e^{-izxP^+} \langle P|\psi^\dagger(z)\psi(0)|P\rangle \end{aligned}$$

This is the PDF. The optical theorem is true in the presence of a UV regulator Λ. Factorization requires only being fully inclusive in *soft* radiation, for the cancellation of Wilson lines.

Conclusions and Outlook

- Factorization violation at amplitude level.
- Limits don't commute in general because crossing symmetry is broken on a collinear slice.
- In $\mathcal{N} = 4$, we find the factorization violation to all orders.
- Splitting function = Regge Kernel.
- This contribution cancels upon summing over the cuts of the hard part at the cross section level.
- Are non global observables sensitive to this correlated splitting?
- There is a need to study these soft+initial state functions more carefully to relax the inclusivity in soft radiation condition.