

SUBLEADING POWER FACTORISATION OF AMPLITUDES

Robin van Bijleveld
arXiv:2503.10810 (JHEP07(2025)257)
r.vanbijleveld@nikhef.nl

in collaboration with
Eric Laenen, Leonardo Vernazza, Coenraad Marinissen and Guoxing Wang

Nikhef

LARGE LOGS EVERYWHERE

$$\hat{\sigma} \sim 1$$

$$\alpha_s \sim 0.1$$

$$+\alpha_s(L^2 + L + 1)$$

$$+\alpha_s^2(L^4 + L^3 + \dots + 1)$$

$$+\dots$$

$$+\alpha_s^n(L^{2n} + L^{2n-1} + \dots + 1)$$

$$+\dots$$

$$L \sim \log \left(\frac{\text{high scale}}{\text{low scale}} \right)$$

- At every order of your calculation there are logarithms that grow large in kinematic limits
- Actually, $\alpha_s L \sim 1$ is the expansion parameter. This is not small!

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- Factorisation theorem splits up amplitudes into single-scale objects
- Predict logarithmic structure up to some power in expansion parameter

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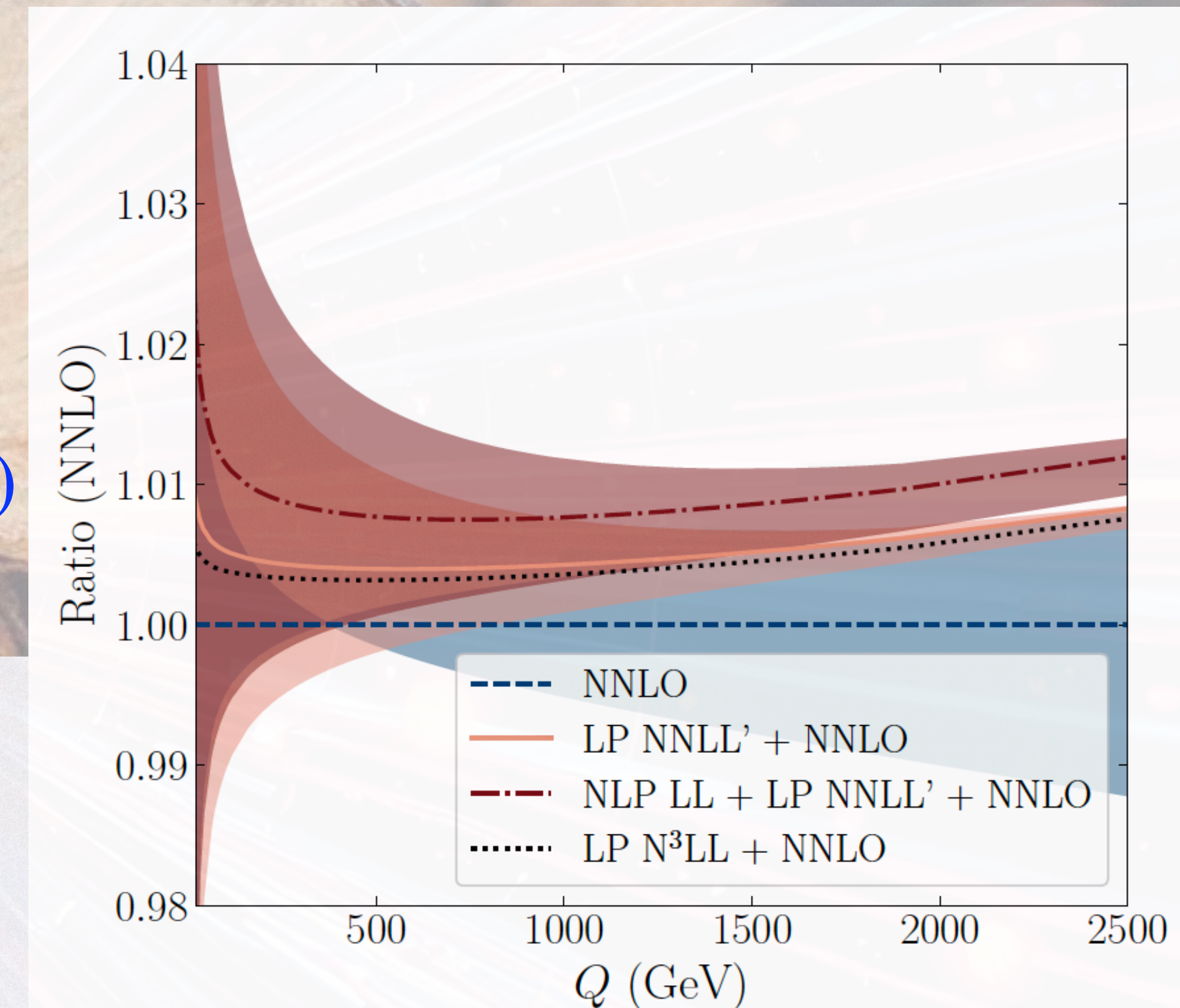
$$+\dots$$

$$+\alpha_s^n(L^{2n} + L^{2n-1} + \dots + 1)$$

$$+\dots$$

$$\alpha_s \sim 0.1$$

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(b) *Ratio with respect to NNLO*

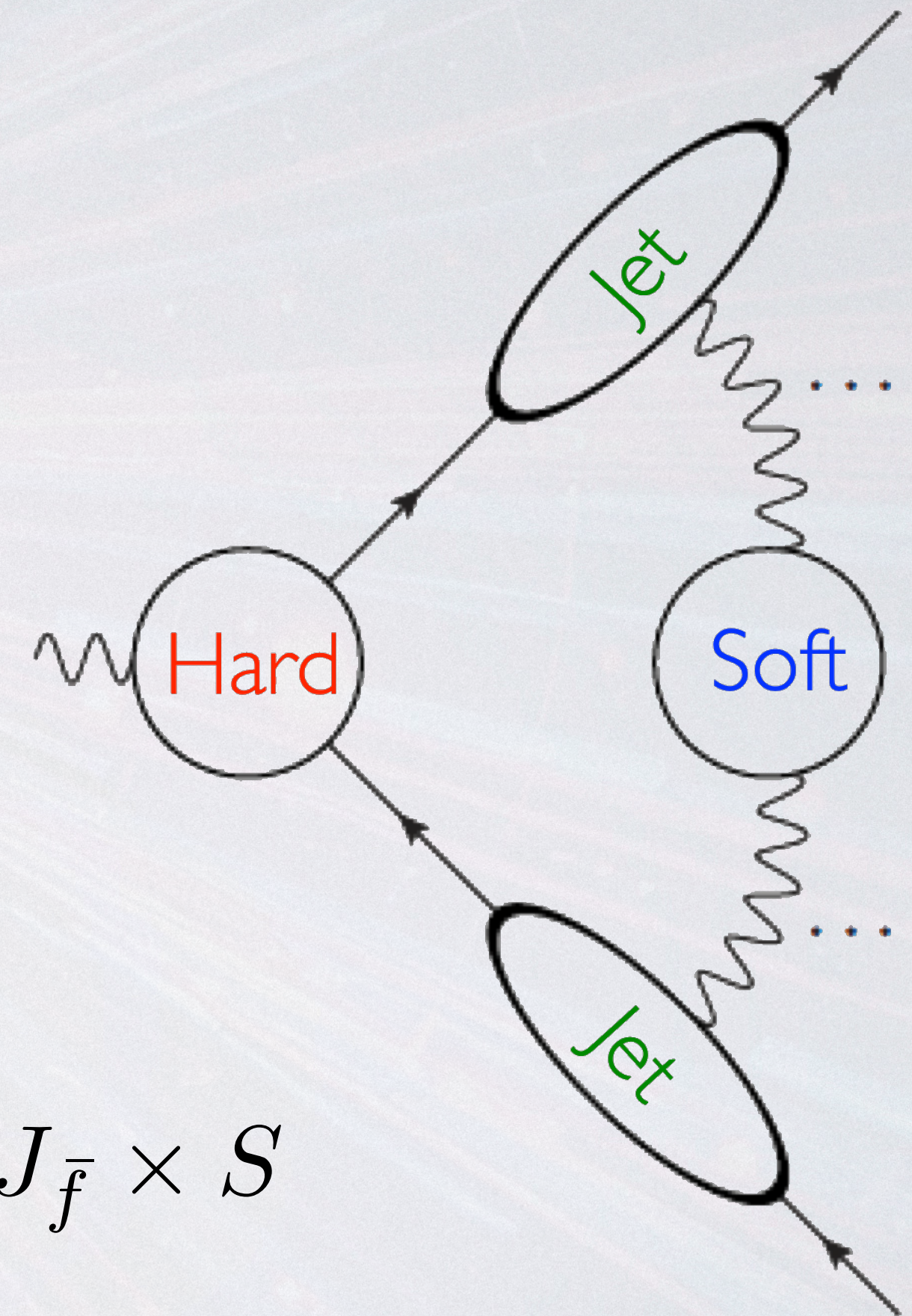
Van Beekveld, Laenen, Sinninghe Damsté, Vernazza (2021)

OUTLINE

- Factorisation theorem of amplitude
- Method of regions
- Jet functions up to subleading powers & check

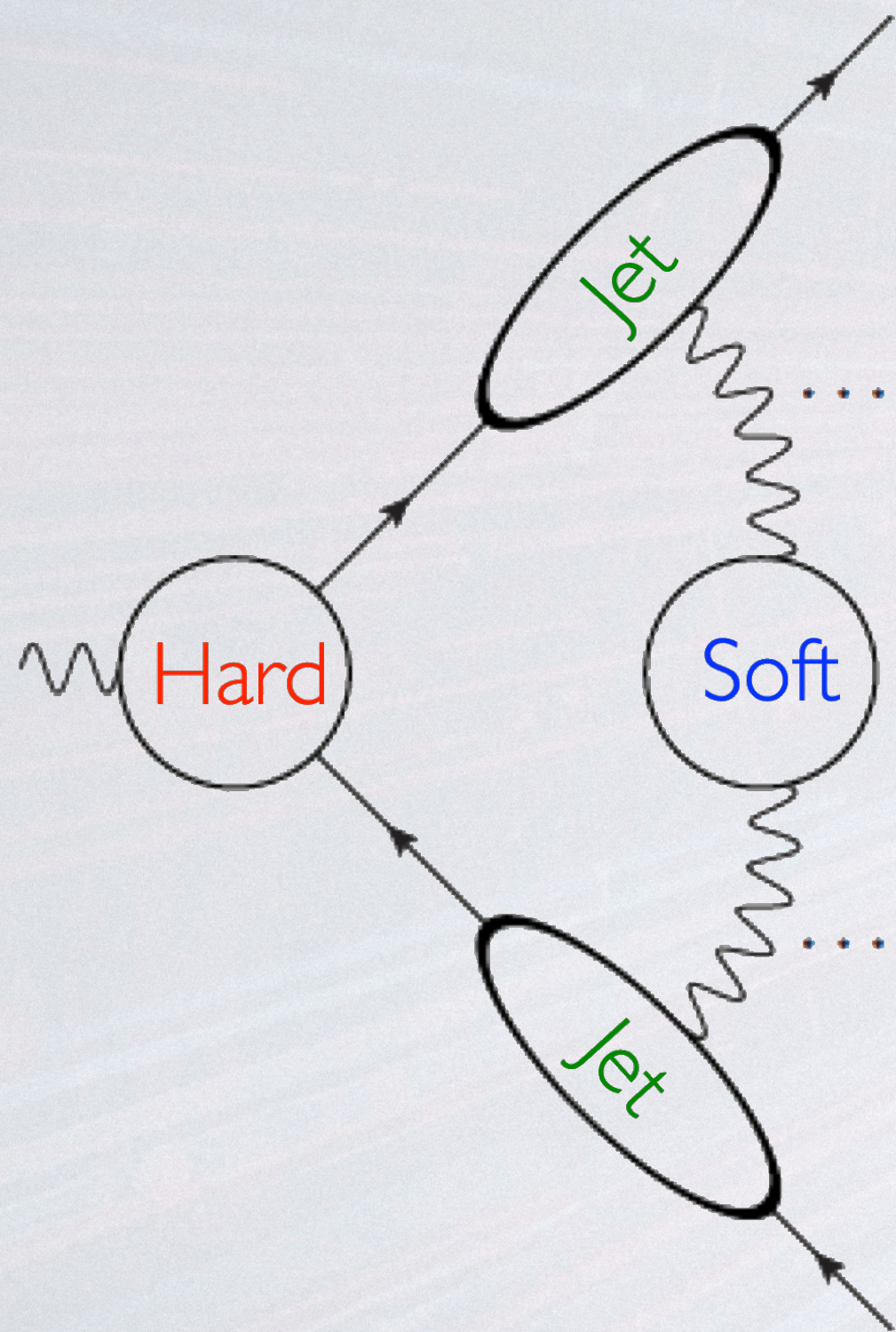
WHAT ARE THE INGREDIENTS AT LP?

- Hard function $\rightarrow$ Matching coefficient
- Jet function $\rightarrow$ Matrix element definition
- Soft function $\rightarrow$ Matrix element definition
- Leading power logarithms via $\mathcal{M}^\mu \sim J_f \times H_{f,\bar{f}}^\mu \times J_{\bar{f}} \times S$

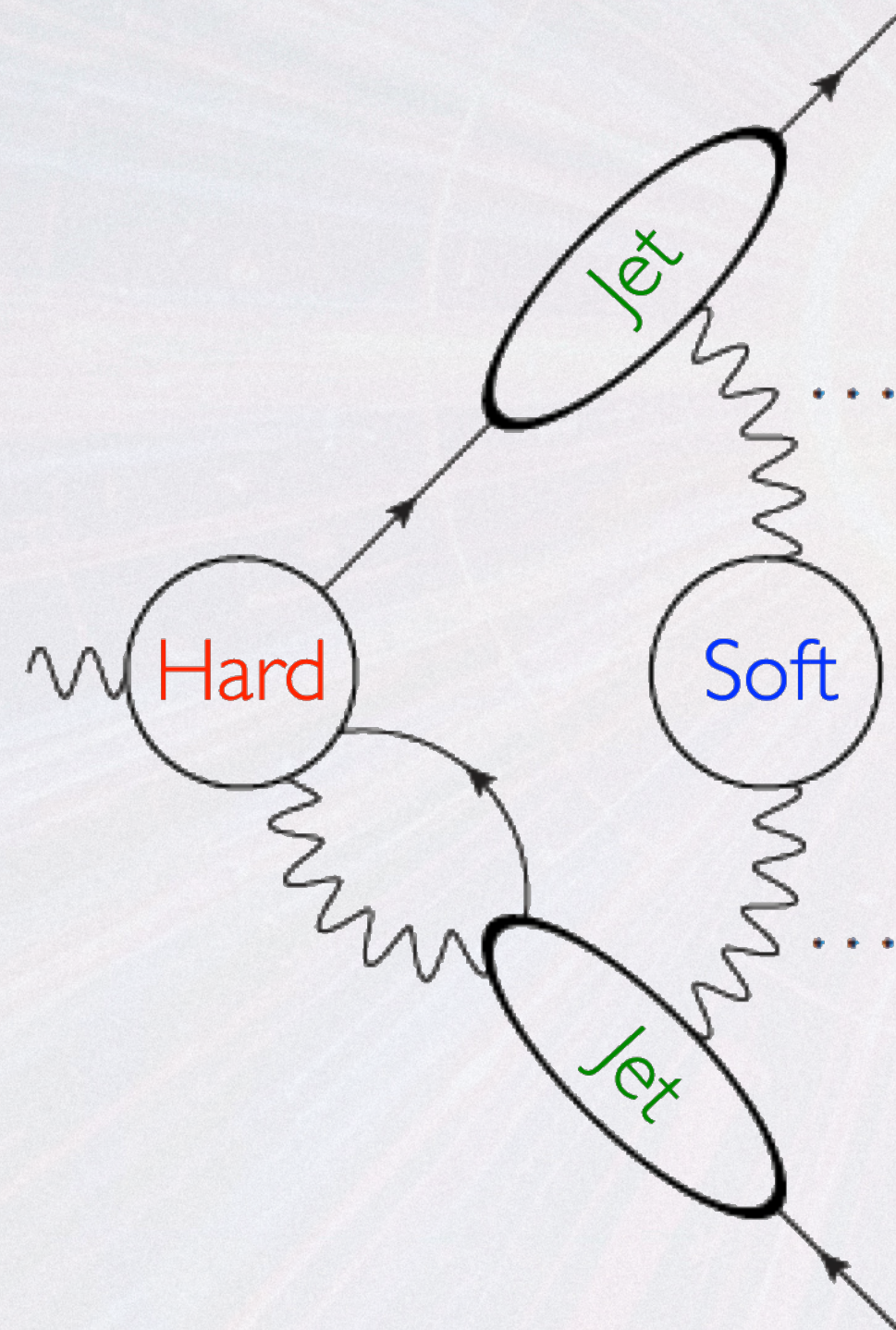


(SUB)LEADING POWER FACTORISATION

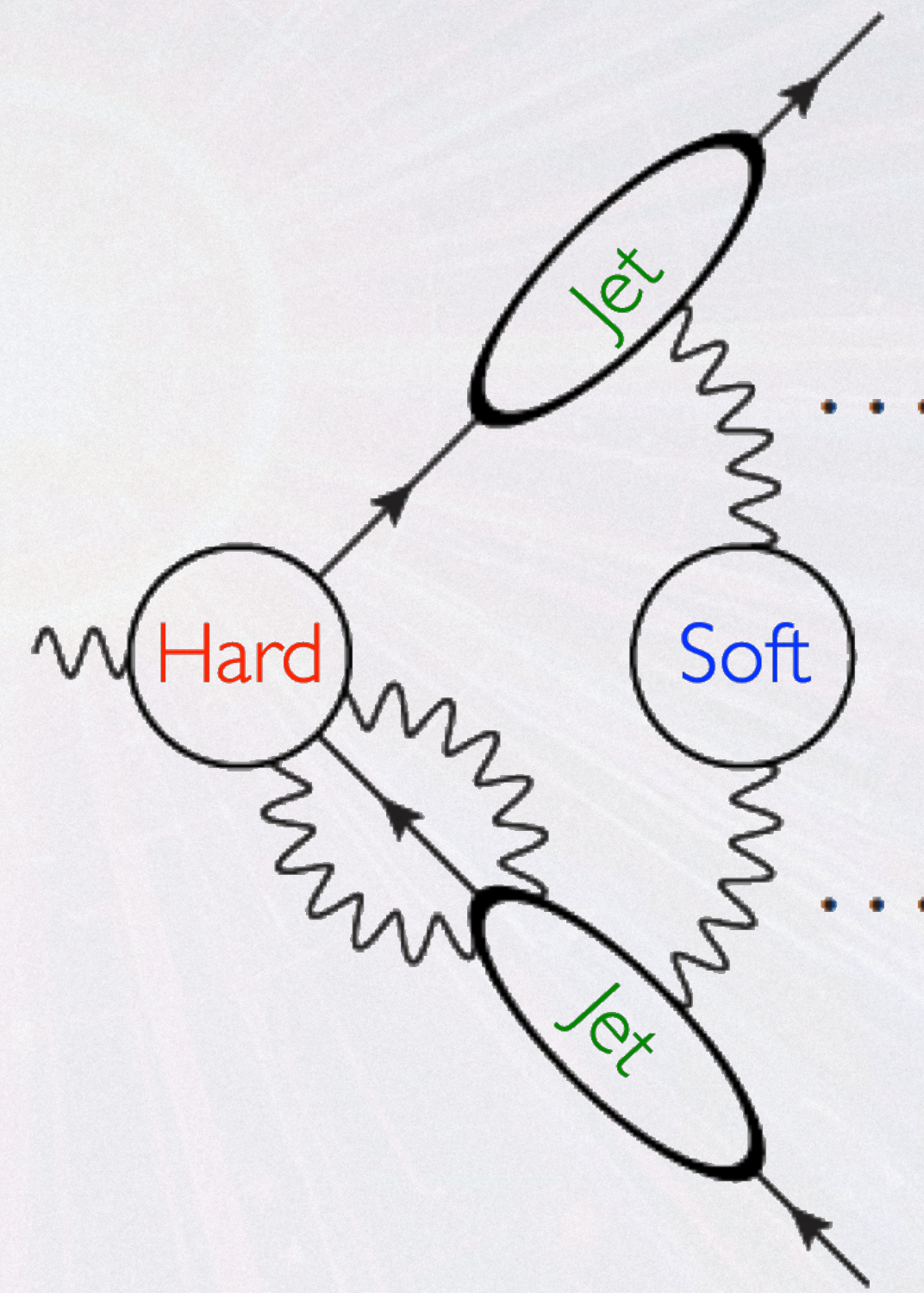
$\mathcal{O}(\lambda^0)$



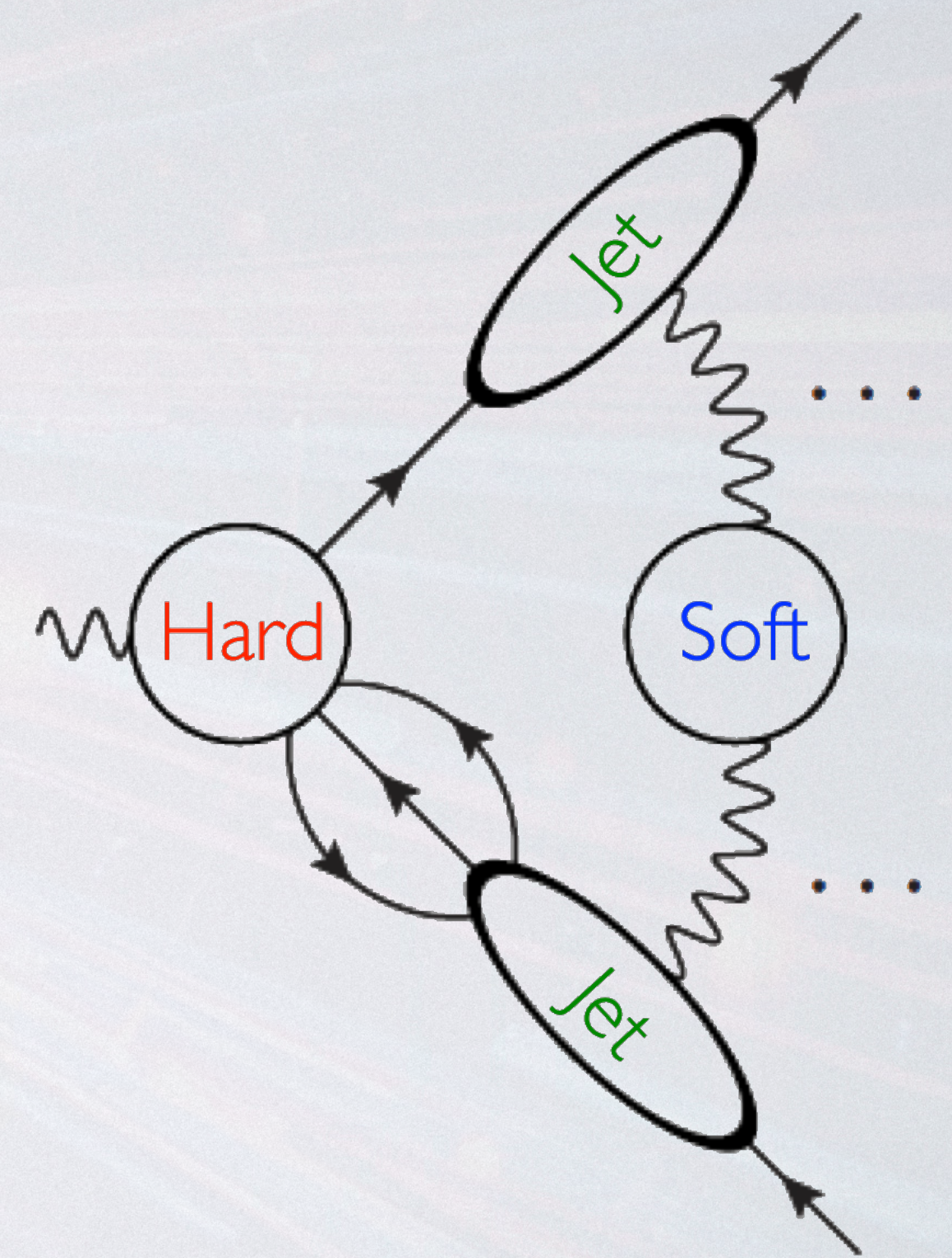
$\mathcal{O}(\lambda)$



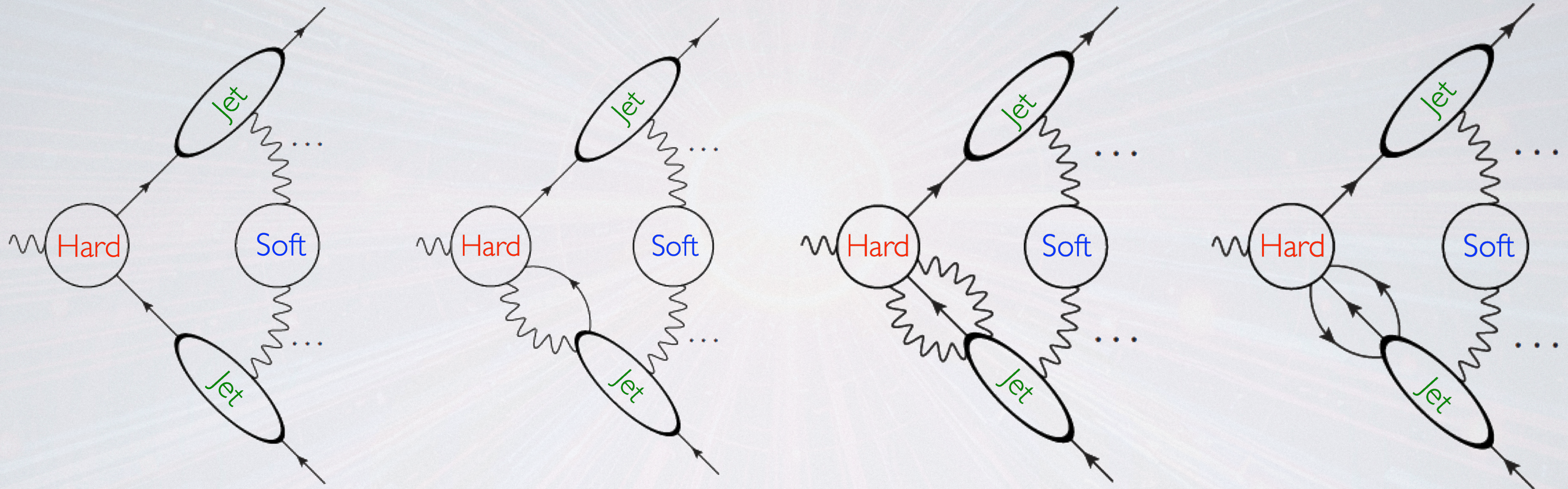
$\mathcal{O}(\lambda^2)$



$\mathcal{O}(\lambda^2)$



(SUB)LEADING POWER FACTORISATION



$$\mathcal{M}^\mu \sim J_f \times H_{f,\bar{f}}^\mu \times J_{\bar{f}} \times S + J_f \times H_{f,\bar{f}\gamma}^{\mu\rho} \otimes J_{\bar{f}\gamma\rho} \times S + (f\gamma\gamma) + (fff) + (f\gamma)(f\gamma) + \mathcal{O}(\lambda^3)$$

METHOD OF REGIONS

- Philosophy: expand in λ before integrating

$$I = \int [dk] \mathcal{N}^\mu \frac{1}{(p_1 - k)^2 - m^2} \frac{1}{(p_2 + k)^2 - m^2} \frac{1}{k^2}$$

- E.g., if k is collinear: $(p_2 + k)^2 - m^2 \rightarrow 2p_2^- \cdot k + \mathcal{O}(\lambda^2)$

$$I_h = \left(\frac{\mu^2}{-\hat{s}} \right)^\epsilon \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \zeta_2 + \dots \right] \bar{u}(p_1) \gamma^\mu v(p_2) + \mathcal{O}(\lambda)$$

$$I_c = I_{\bar{c}} = \left(\frac{\mu^2}{m^2} \right)^\epsilon \left[\frac{1}{\epsilon^2} + \frac{2}{\epsilon} + 4 + \frac{\zeta_2}{2} + \dots \right] \bar{u}(p_1) \gamma^\mu v(p_2) + \mathcal{O}(\lambda)$$

METHOD OF REGIONS

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- Full result is sum of all regions:

$$I = I_h + I_c + I_{\bar{c}}$$
$$= \left(\frac{\mu^2}{-\hat{s}} \right)^\epsilon \left[\frac{1}{\epsilon} \left(1 - 2 \log \left(-\frac{m^2}{\hat{s}} \right) \right) - 4 \log \left(-\frac{m^2}{\hat{s}} \right) + \log^2 \left(-\frac{m^2}{\hat{s}} \right) + 2\zeta_2 \right] \bar{u}(p_1) \gamma^\mu v(p_2) + \mathcal{O}(\lambda, \epsilon)$$

METHOD OF REGIONS & FACTORISATION

- Method of regions: $I = I_c + I_{\bar{c}} + I_h$
- Factorisation at leading power:

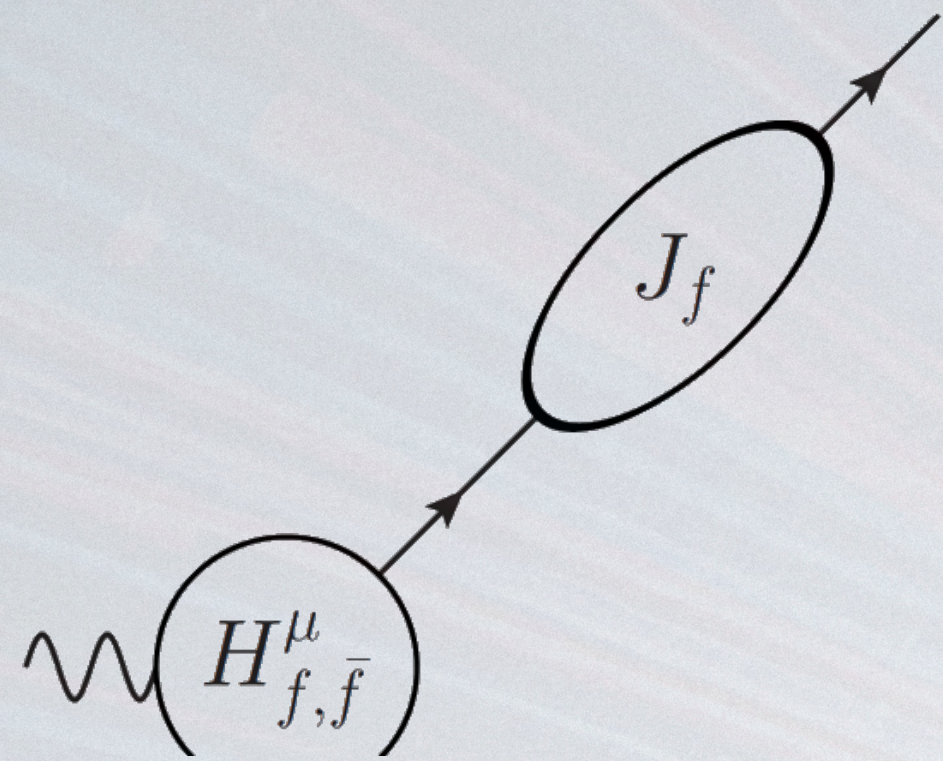
$$\mathcal{M}^\mu = J_f H_{f,\bar{f}}^\mu J_{\bar{f}} S \supset J_f^{(1)} H_{f,\bar{f}}^{(0)\mu} J_{\bar{f}}^{(0)} + J_f^{(0)} H_{f,\bar{f}}^{(1)\mu} J_{\bar{f}}^{(0)} + J_f^{(0)} H_{f,\bar{f}}^{(0)\mu} J_{\bar{f}}^{(1)}$$

$$I = \begin{array}{c} \text{red double arrow} \\ I_c \end{array} + \begin{array}{c} \text{red double arrow} \\ I_h \end{array} + \begin{array}{c} \text{red double arrow} \\ I_{\bar{c}} \end{array}$$

- Check with recent method of regions calculation for massive QED form factor

BUILDING BLOCKS

- Building blocks based on SCET
- Fermion: $\bar{\psi}(x)\Phi_i(x, \infty)$
- Gluon/photon: $\Phi_i(\infty, x)[iD^\mu\Phi_i(x, \infty)]$
- Combining them in products, and with $\partial_\perp$



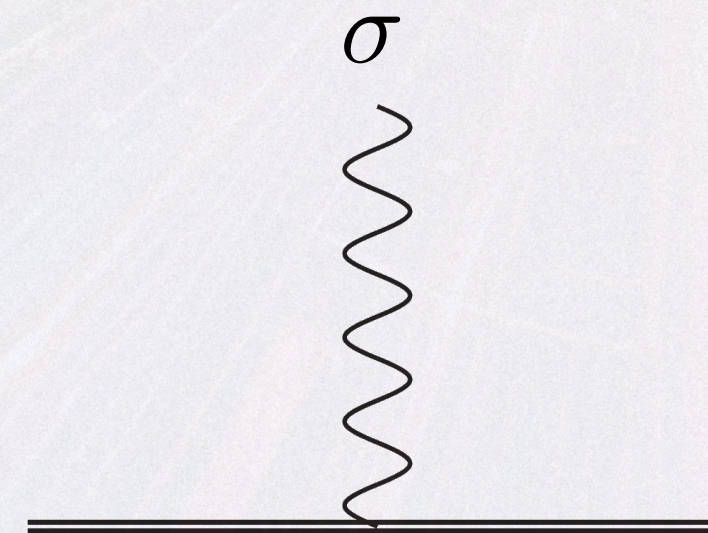
J_f JET FUNCTION

- Definition of the f-jet with a Wilson line:

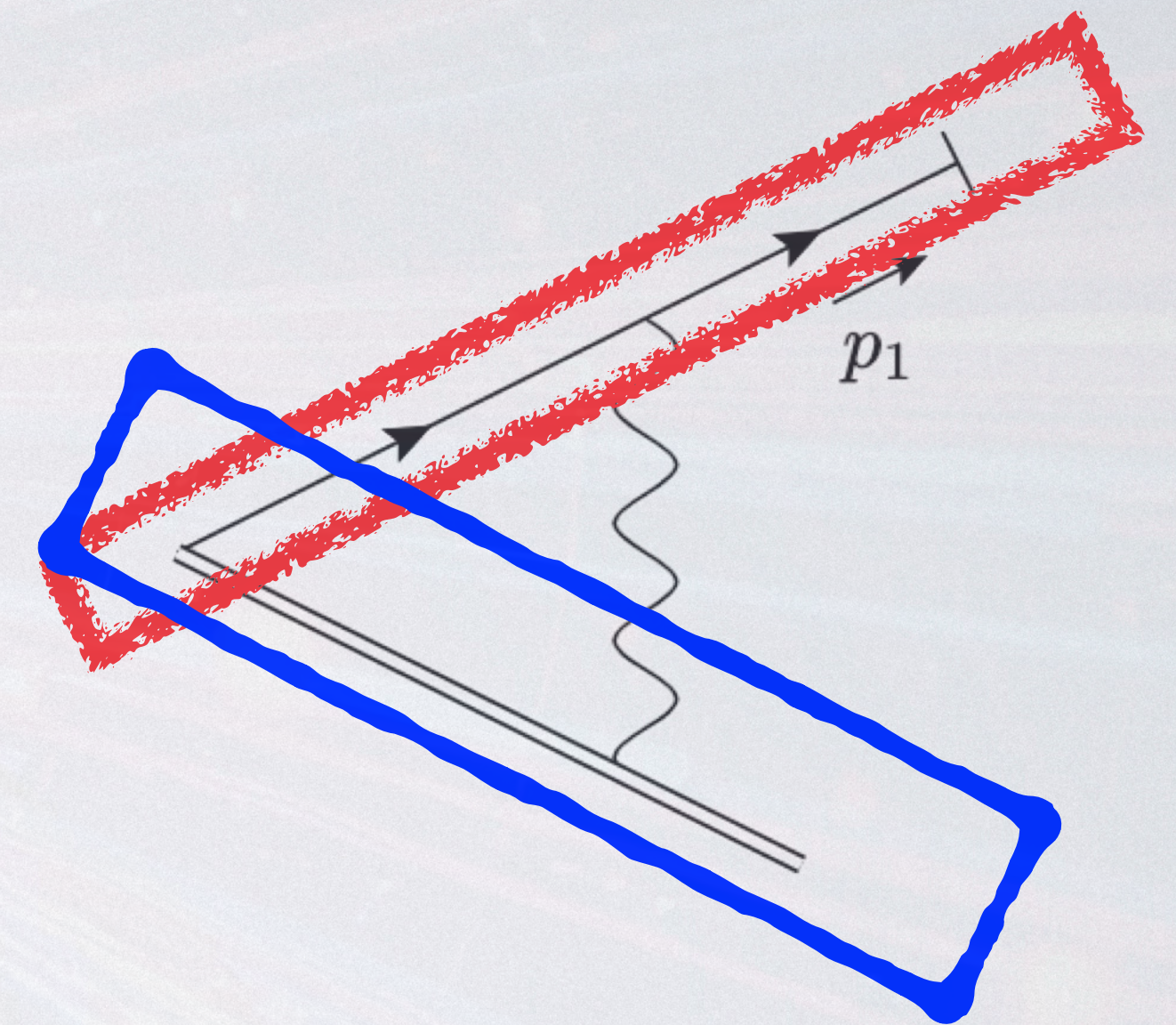
$$J_f(p_1, \bar{n}) = \langle p_1 | \bar{\psi}(0) \Phi_{\bar{n}}(0, \infty) | 0 \rangle$$

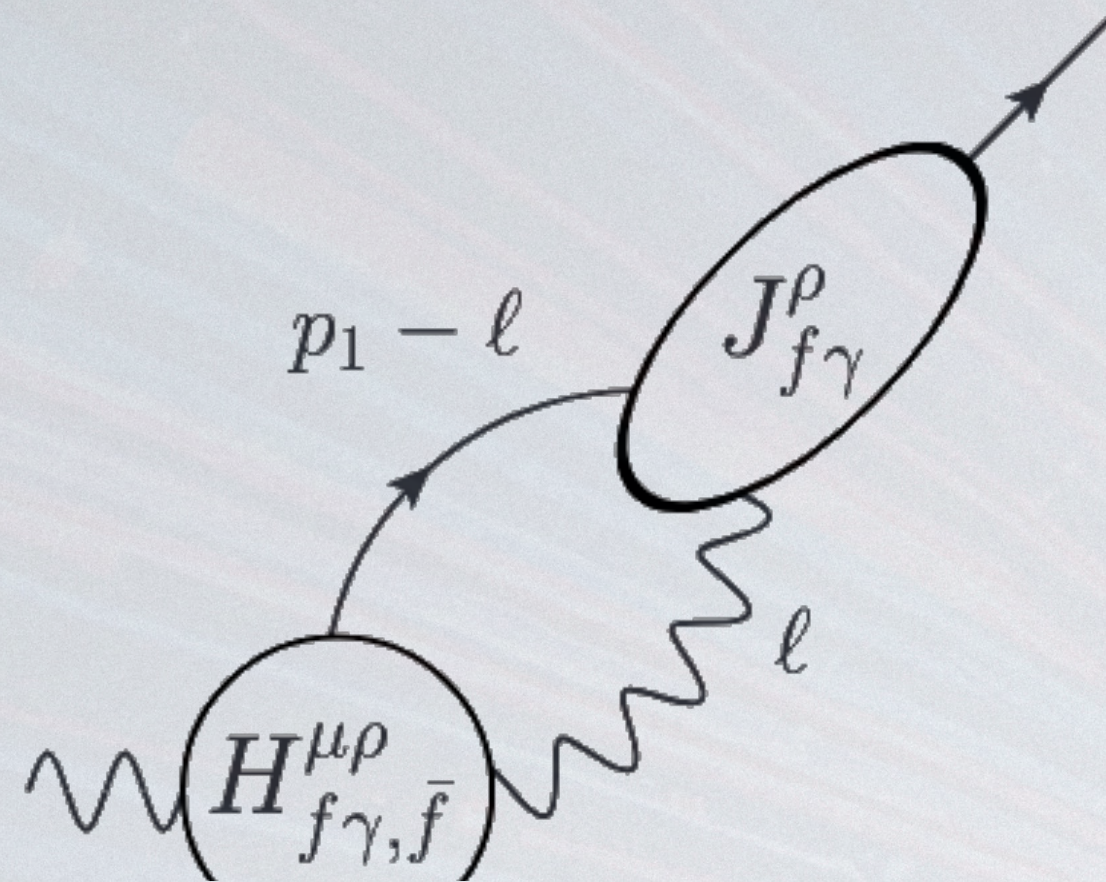
$$\Phi_{\bar{n}}(x, \infty) = \exp \left[-ie \int_{\infty}^0 d\lambda \bar{n} \cdot A(x + \lambda \bar{n}) \right]$$

- Eikonal Feynman rule



$$= \frac{\bar{n}^\sigma}{\bar{n} \cdot k} = \frac{p_2^{-\sigma}}{p_2^- \cdot k}$$





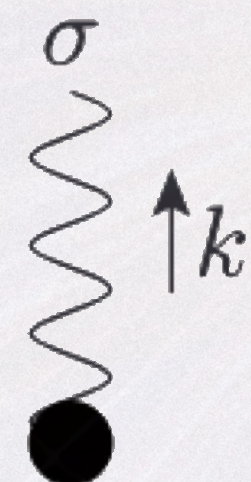
$J_{f\gamma}$ JET FUNCTION

fermion

photon

$$J_{f\gamma}^\rho(p_1, \bar{n}, \bar{n} \cdot \ell) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{-i\ell(\xi\bar{n})} \langle p_1 | \left[\bar{\psi}(0) \Phi_{\bar{n}}(0, \infty) \right] \left[\Phi_{\bar{n}}(\infty, \xi\bar{n}) (iD^\rho \Phi_{\bar{n}}(\xi\bar{n}, \infty)) \right] | 0 \rangle ,$$

- Effective Feynman rule

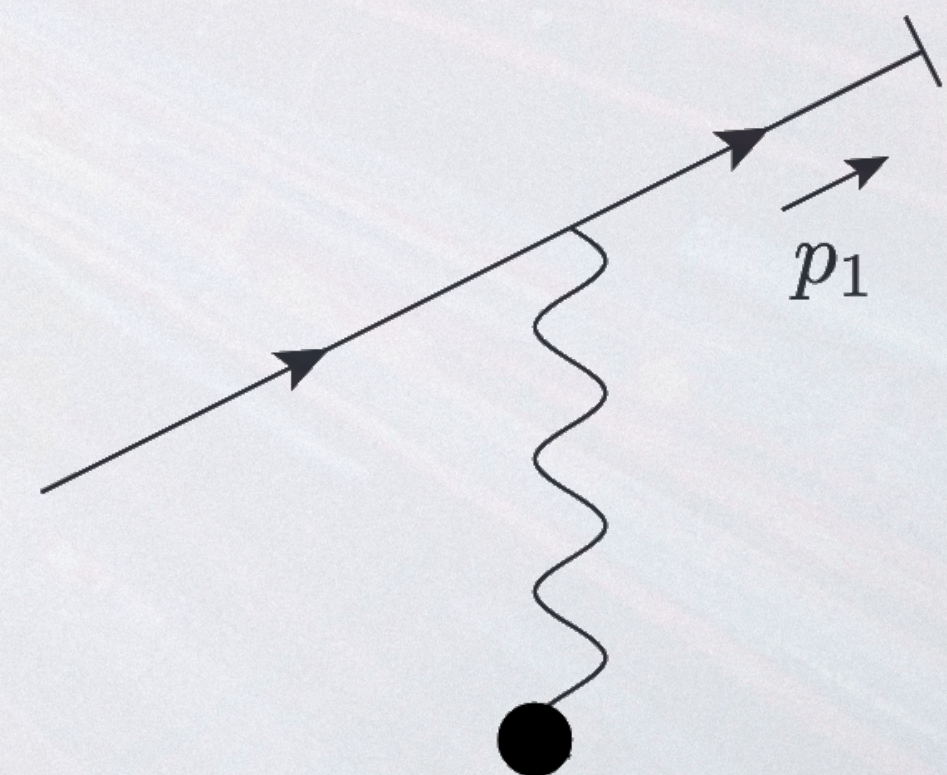


$$= \frac{-i}{k^2} \left(\eta^{\rho\sigma} - \frac{\bar{n}^\sigma k^\rho}{\bar{n} \cdot k} \right) \delta(\bar{n} \cdot k - \ell^+)$$

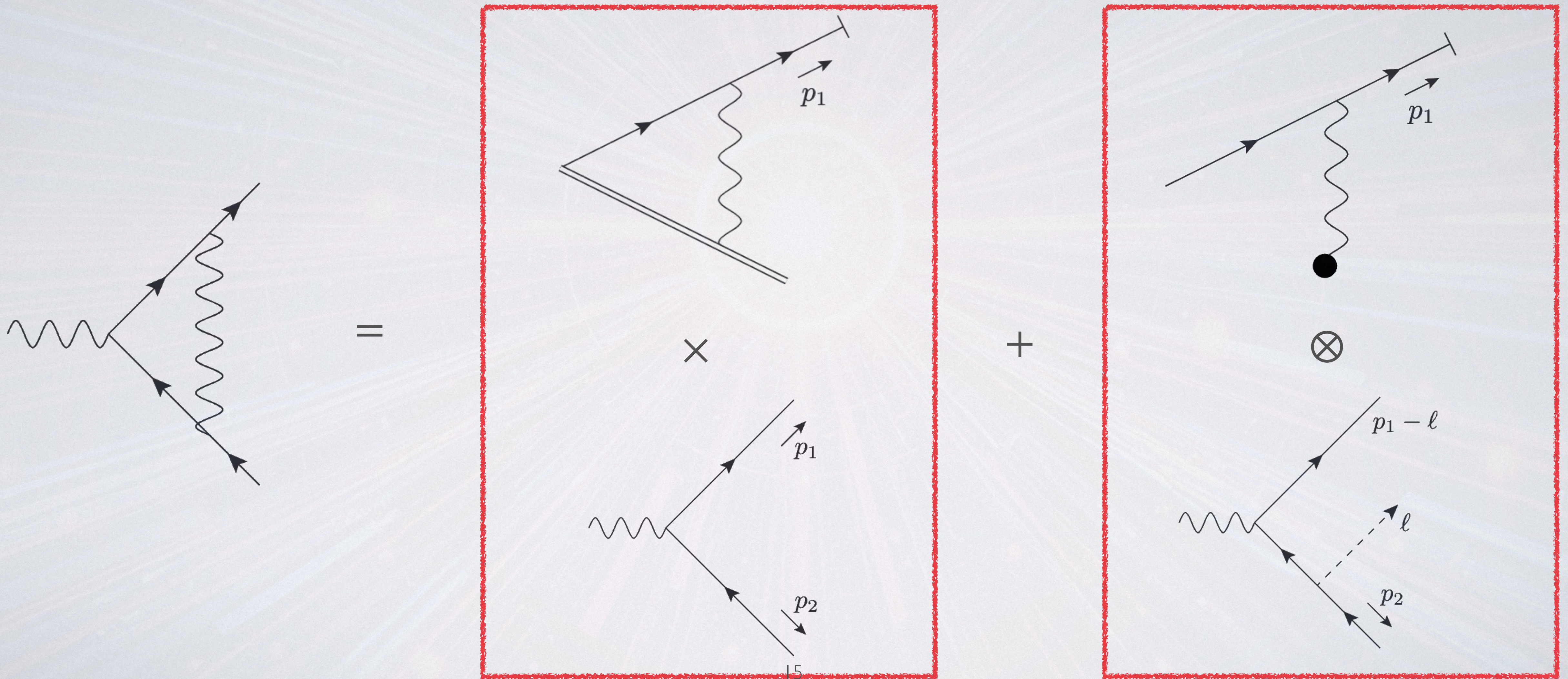
- Momentum fraction

$$x = \frac{\ell^+}{p_1^+}$$

- Convolution $J_{f\gamma} \otimes H_{f\gamma, \bar{f}}^\mu \equiv \int_0^1 dx J_{f\gamma}^\rho(x) H_{f\gamma, \bar{f} \rho}^\mu(x)$

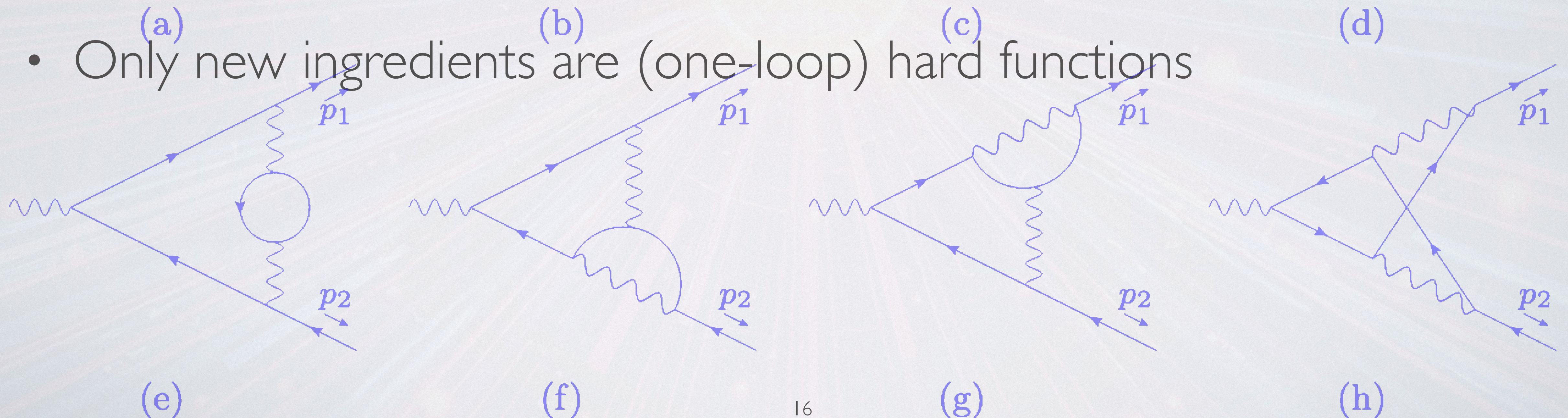


ONE-LOOP RESULTS



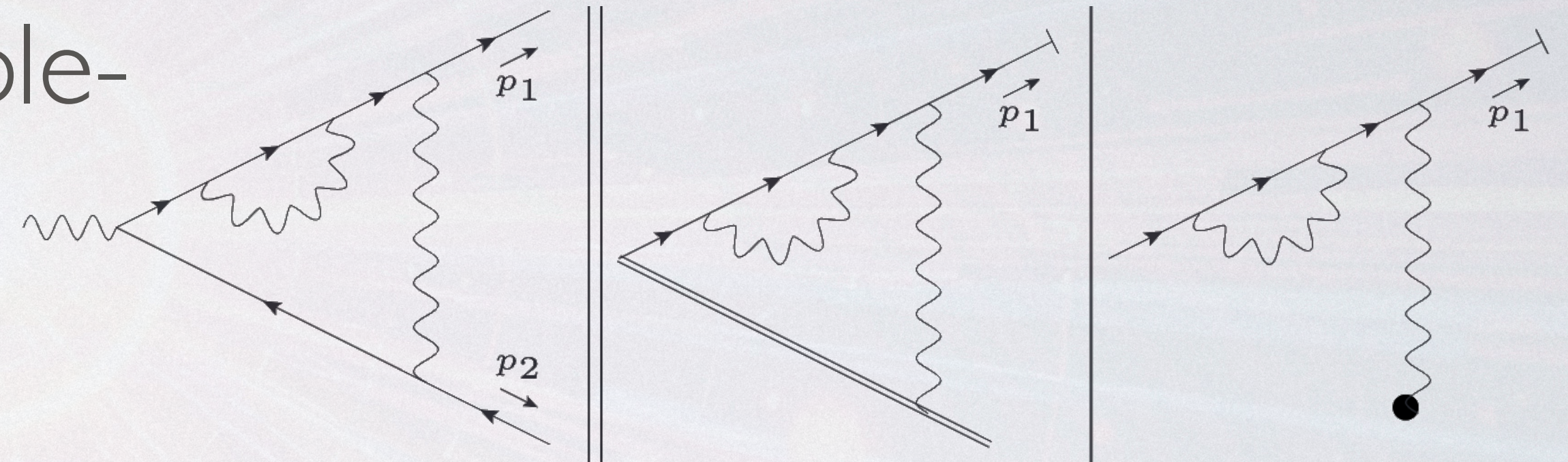
TWO LOOPS

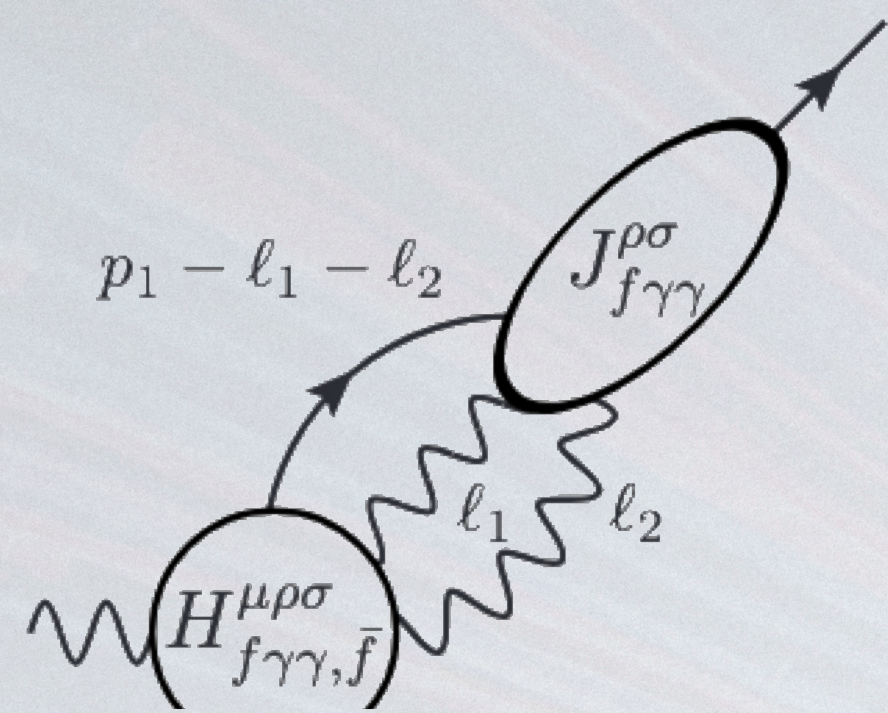
- One loop collinear, one loop anticollinear with existing jet functions
- One loop hard, one loop collinear with existing jet functions



TWO LOOPS

- Two-loop jets are needed for double-collinear factorisation
- For most diagrams only need J_f and $J_{f\gamma}$
- Not sufficient for (crossed) ladder diagram





$J_{f\gamma\gamma}$ JET FUNCTION

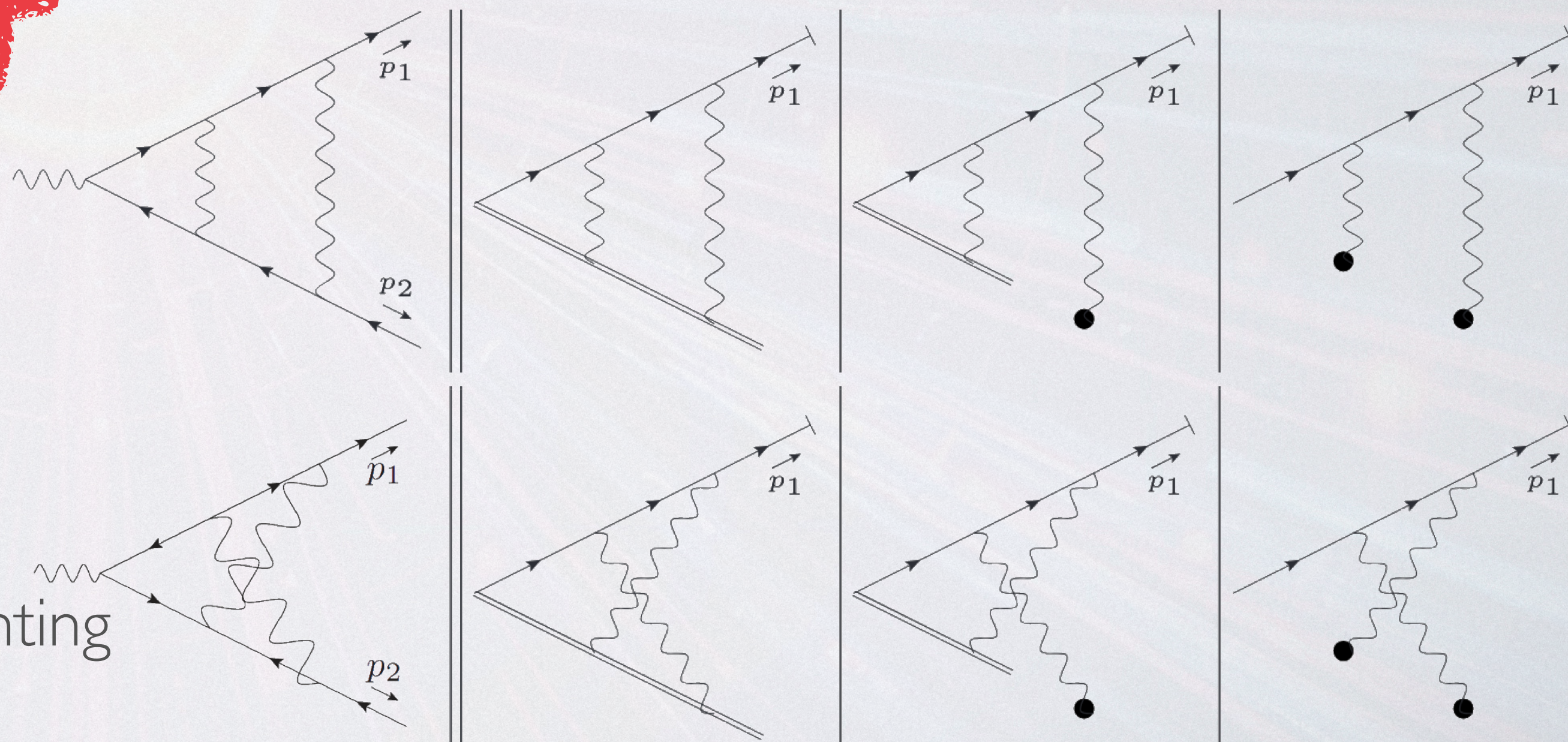
$$J_{f\gamma\gamma}^{\rho\sigma}(p_1, \bar{n}, \ell_1^+, \ell_2^+) = \int \frac{d\xi_1}{2\pi} \int \frac{d\xi_2}{2\pi} e^{-i\ell_1(\xi_1 \bar{n})} e^{-i\ell_2(\xi_2 \bar{n})} \langle p_1 | \left[\bar{\psi}(0) \Phi_{\bar{n}}(0, \infty) \right] \times \left[\Phi_{\bar{n}}(\infty, \xi_1 \bar{n}) (iD^\rho \Phi_{\bar{n}}(\xi_1 \bar{n}, \infty)) \right] \left[\Phi_{\bar{n}}(\infty, \xi_2 \bar{n}) (iD^\sigma \Phi_{\bar{n}}(\xi_2 \bar{n}, \infty)) \right] | 0 \rangle$$

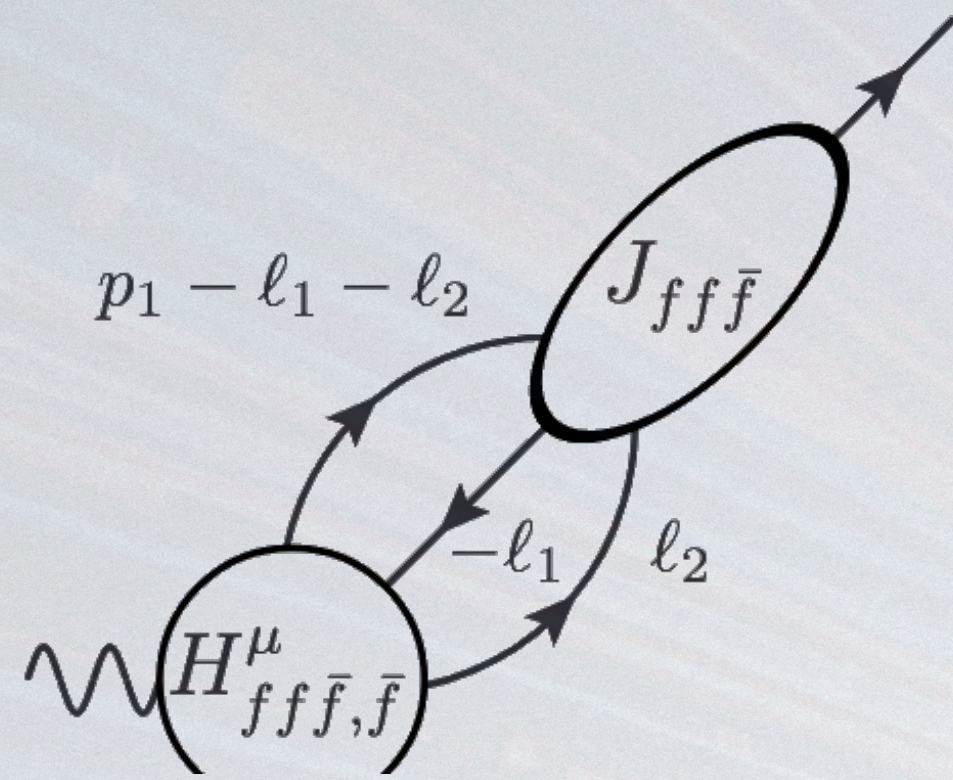
fermion

photon

photon

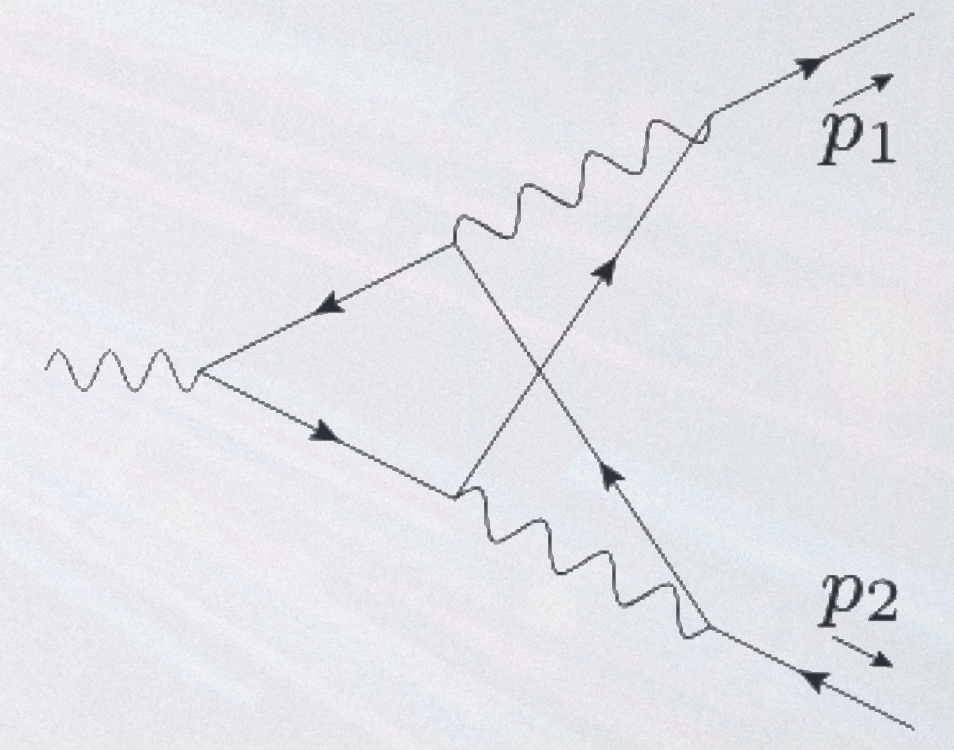
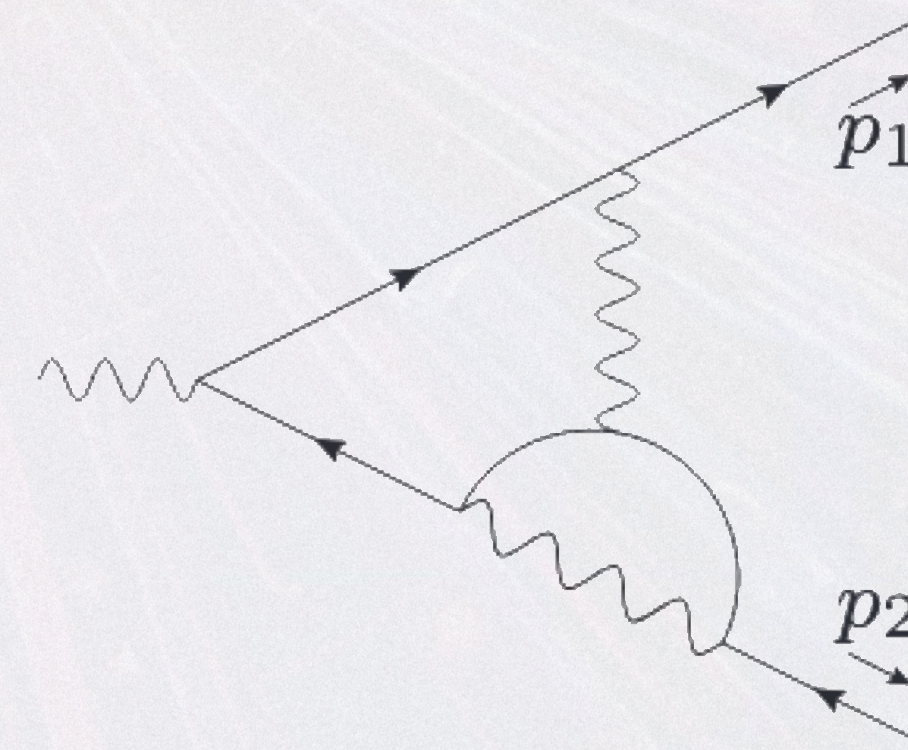
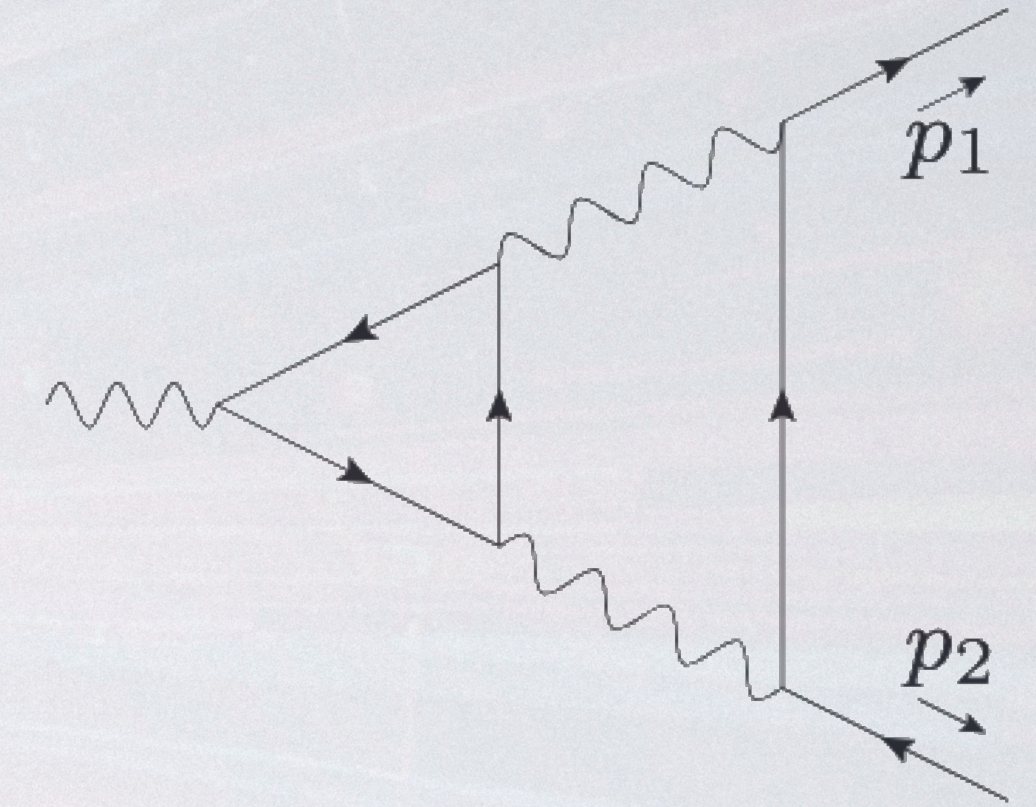
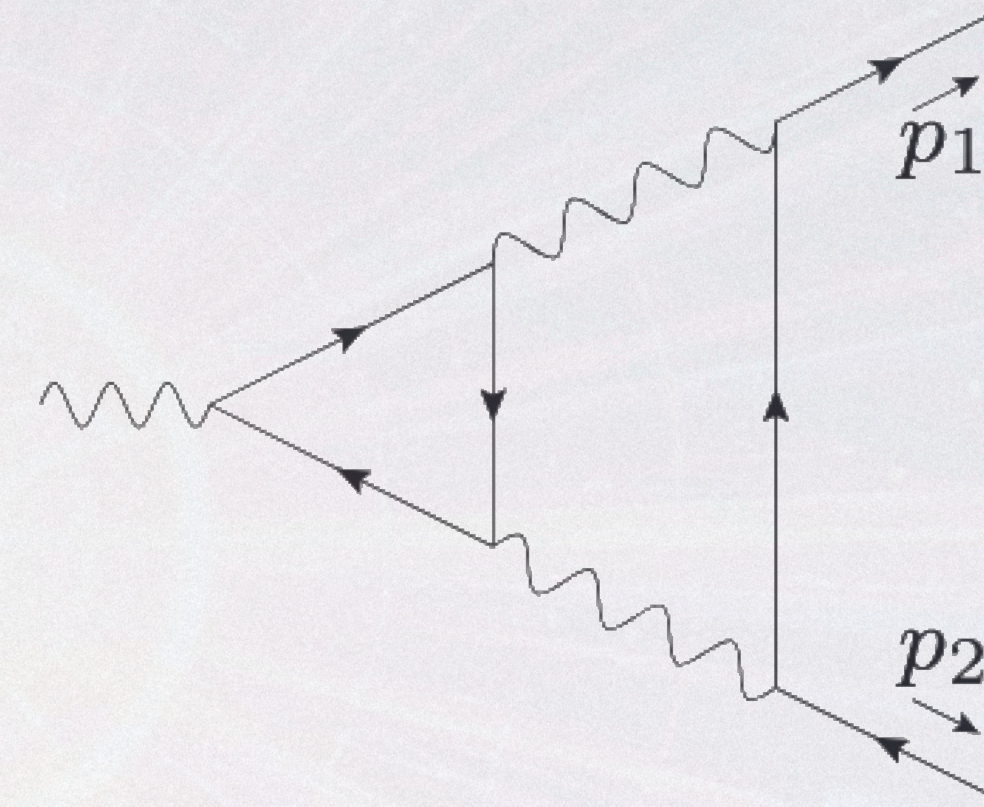
- (Crossed) Ladder needs a new jet function
- Starts at $\mathcal{O}(\lambda^2)$
- J_f , $J_{f\gamma}$ and $J_{f\gamma\gamma}$ do not overlap, so no double counting

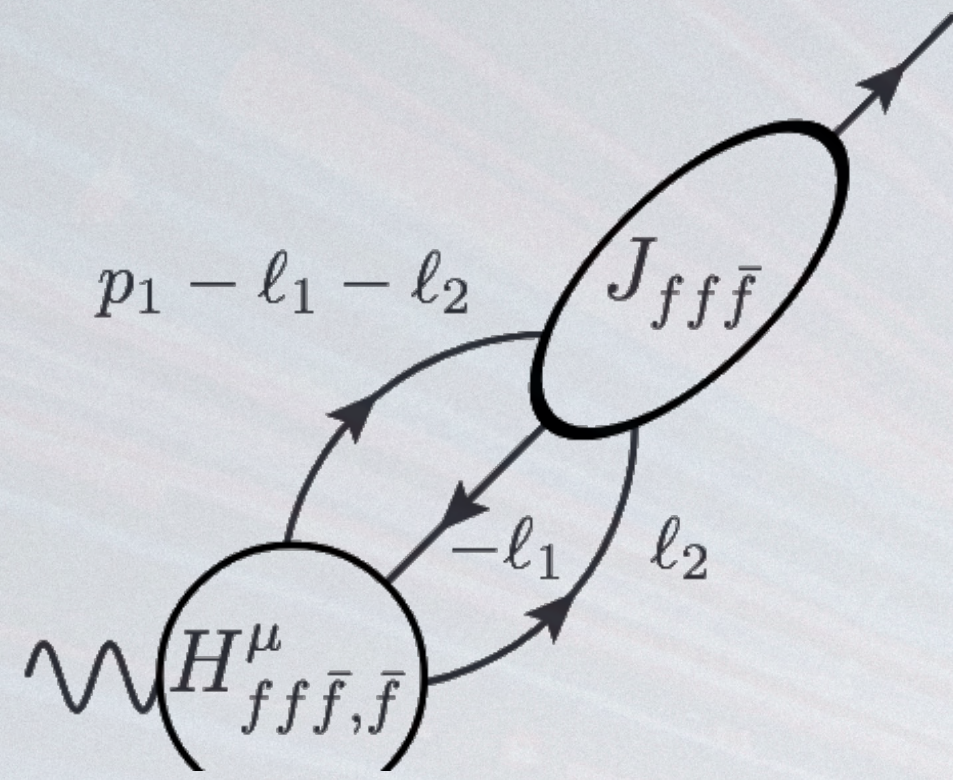




$J_{fff\bar{f}}$ JET FUNCTION

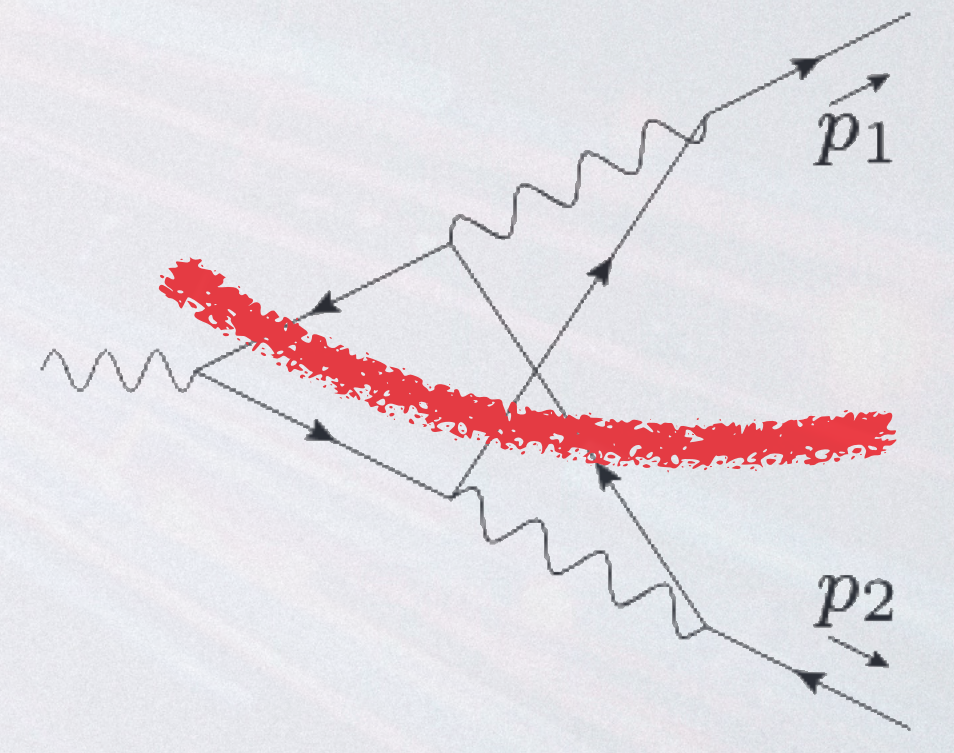
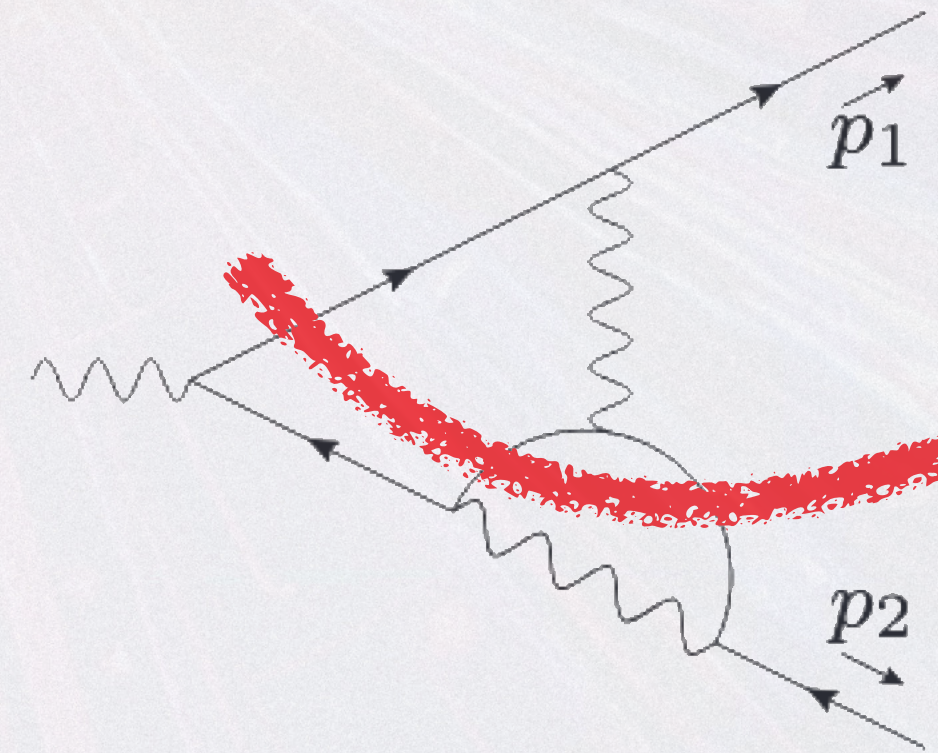
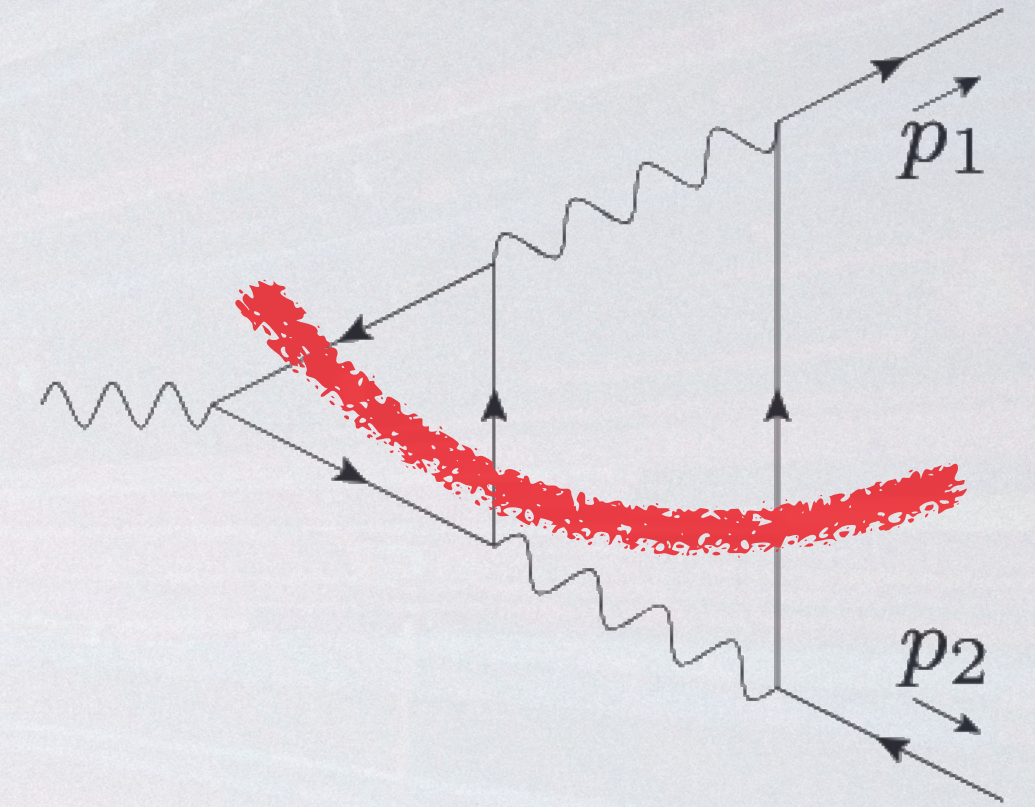
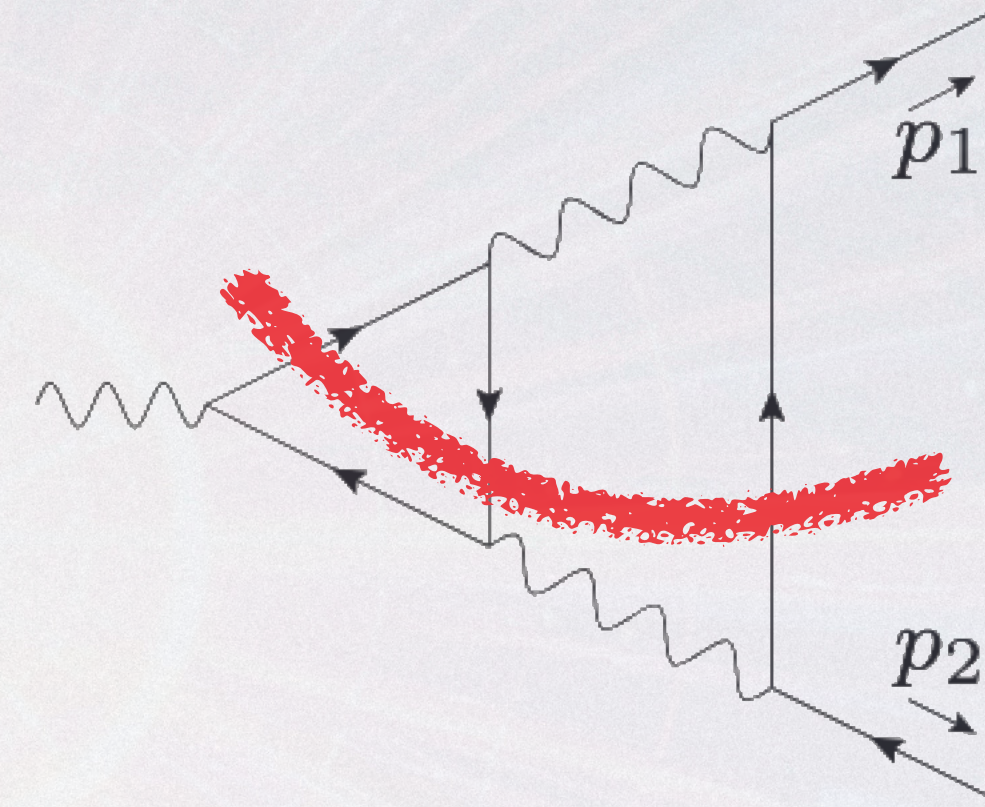
- Fermion attachments instead of photon attachments
- Furry diagrams
- Starts at $\mathcal{O}(\lambda^2)$





$J_{fff\bar{f}}$ JET FUNCTION

- Fermion attachments instead of photon attachments
- Furry diagrams
- Starts at $\mathcal{O}(\lambda^2)$



OUTLOOK

- Include radiation in jets
- Move towards cross sections
- Derive resummation formulas from factorisation theorem

SUMMARY

- Definition of jet functions and calculate them up to two loops in QED
- Check results with method of regions results and it holds exactly

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- Definition of jet functions and calculate them up to two loops in QED
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Thanks for your attention!

The background is a complex, abstract composition. At the center is a bright, glowing yellow circle with a textured, sun-like surface. From this center, a dense network of thin, radiating lines extends outwards, filling the entire frame. These lines are primarily blue and red, with some yellow and white highlights. The lines vary in thickness and direction, creating a sense of dynamic movement and depth. Scattered throughout the background are small, glowing particles in blue, red, and yellow, adding to the overall vibrant and energetic feel. The text "BACK-UP SLIDES" is centered horizontally and vertically, overlaid on the central yellow circle and the radiating lines.

BACK-UP SLIDES

WHERE DO THE DIVERGENCES COME FROM?

- Deduce from generic Feynman integral where the IR singularities are

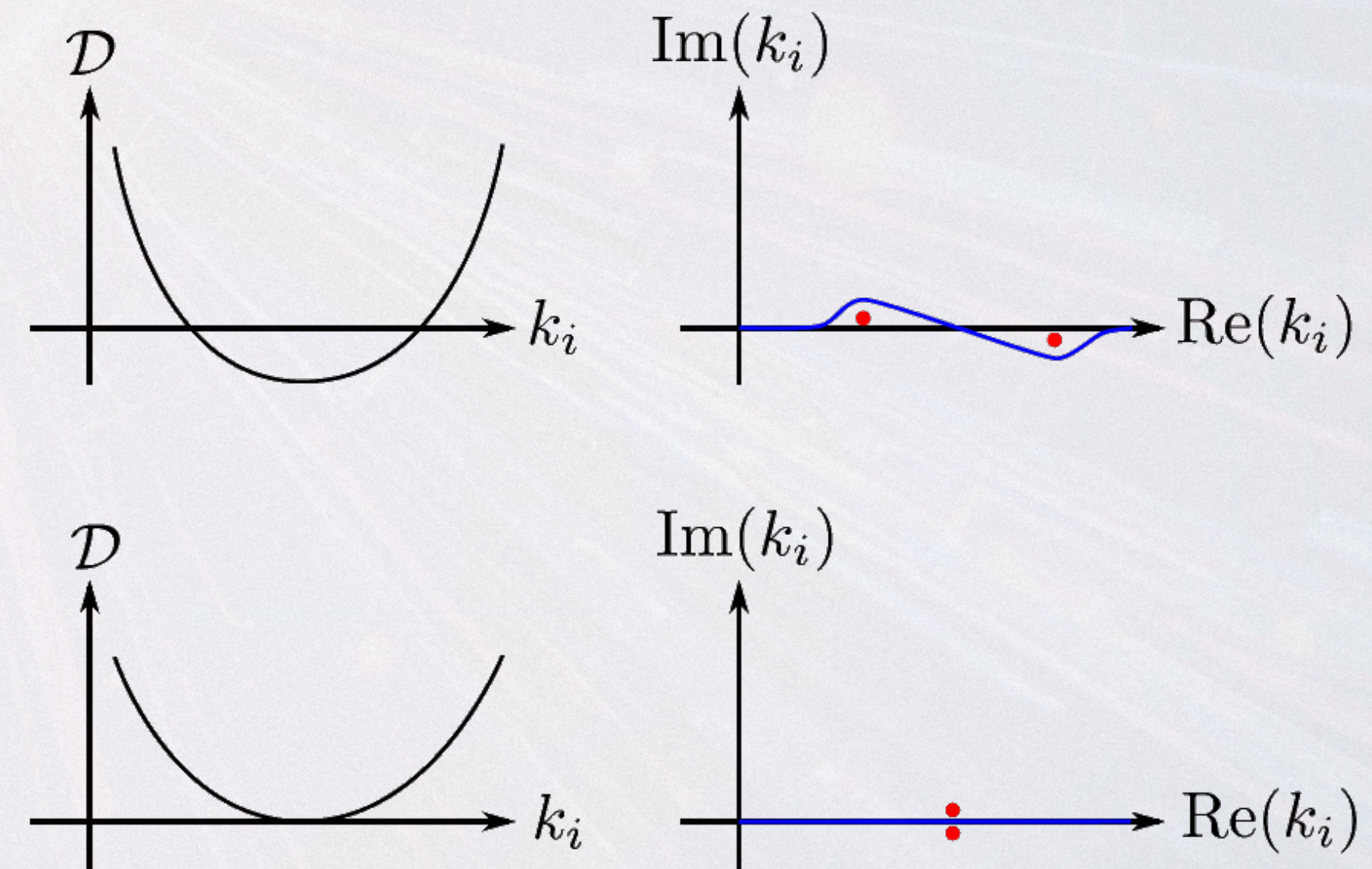
- Generic denominator: $D(p_r, k_j, \alpha_i) = \sum_{i=1}^I \alpha_i (\ell_i^2 - m_i^2) + i\eta$

- Landau equations

- Pinched surface

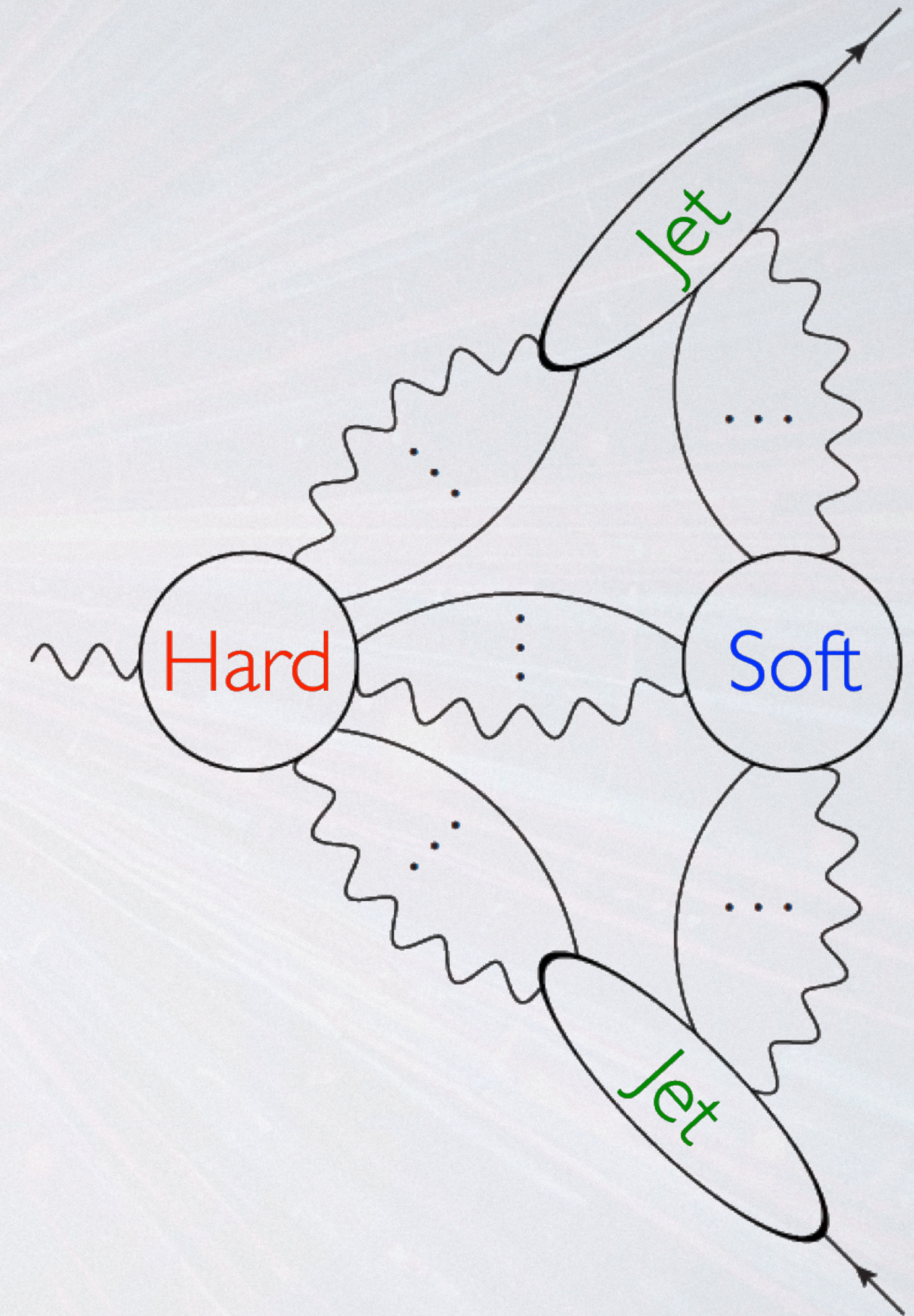
$$\ell_i^2 - m_i^2 = 0 \quad \text{or} \quad \alpha_i = 0$$

$$\frac{\partial D}{\partial \ell_i^\mu} = 0, \quad \forall i, \mu$$



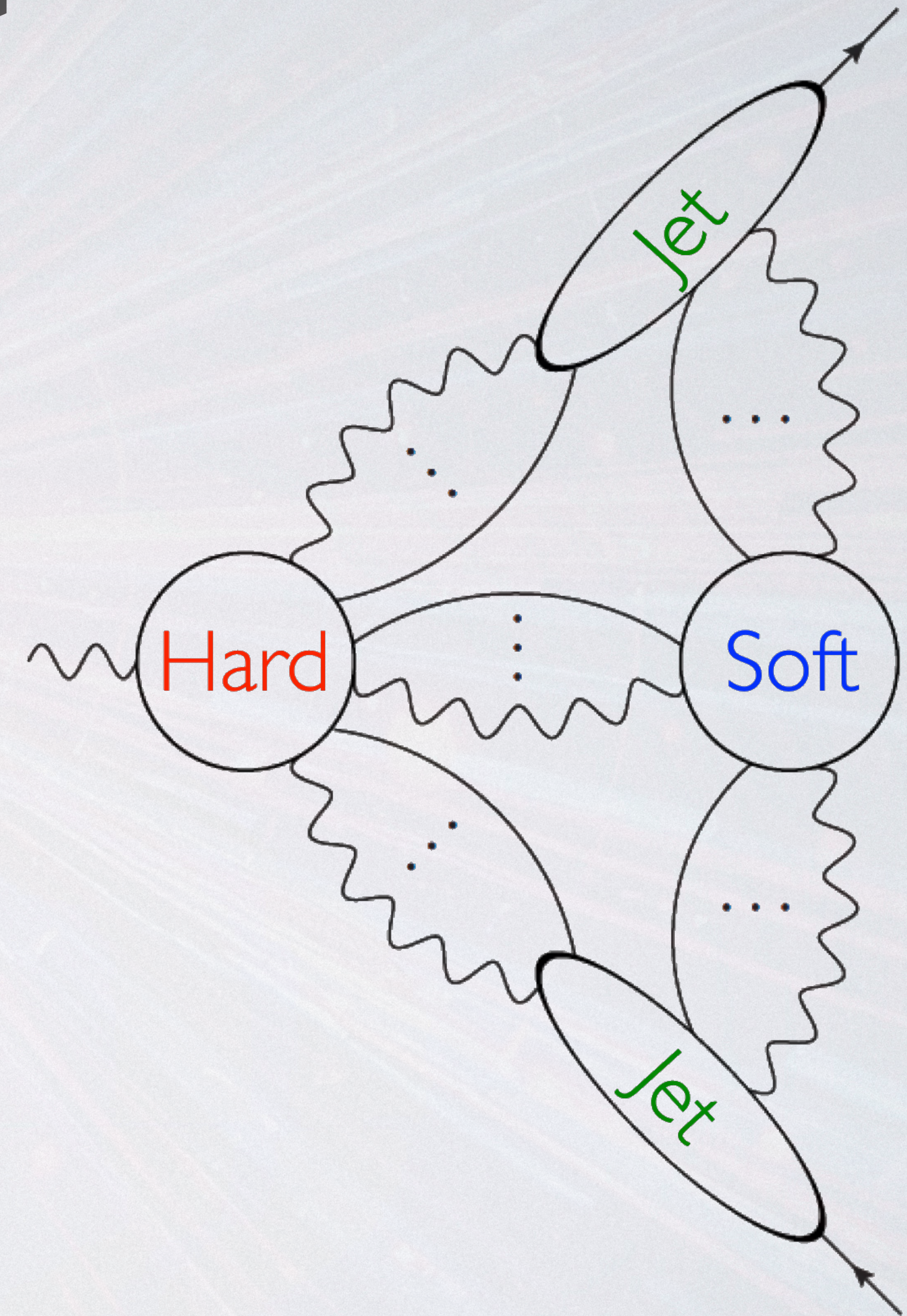
REDUCED DIAGRAMS

- Solve the Landau equations
- General reduced diagram for (QED) process for $\gamma^* \rightarrow f_1 + \cdots + f_n$
- Collinear and soft singularities
- Note: this is just a necessary condition



FROM REDUCED DIAGRAMS TO POWER COUNTING

- When is it a sufficient condition?
- Power counting parameter λ to quantify singularities
- All lines and connections have some scaling in terms of λ
- Derive which kind of diagrams can exist at a certain order in λ
- Degree of divergence $\gamma_G \geq 0$

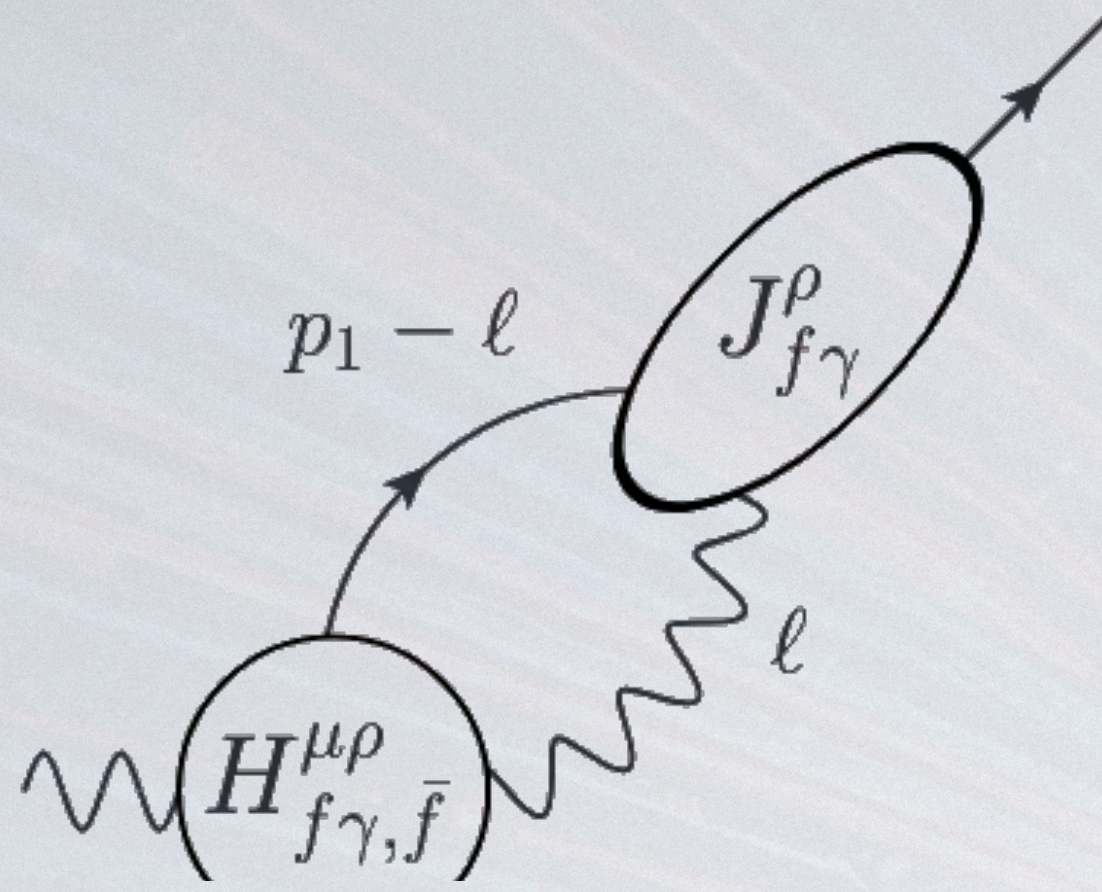


FROM POWER COUNTING TO FACTORISATION

- Collinear: $k^\mu \sim Q(1, \lambda^2, \lambda)$
- Anticollinear: $k^\mu \sim Q(\lambda^2, 1, \lambda)$
- Soft: $k^\mu \sim Q(\lambda^2, \lambda^2, \lambda^2)$
- Hard: $k^\mu \sim Q(1, 1, 1)$

- Collinear fermion:
$$\frac{i(\not{k} + m)}{k^2 - m^2} \sim \frac{\gamma^- + \lambda^2 \gamma^+ + \lambda \gamma^\perp + \lambda}{\lambda^2} \sim \frac{1}{\lambda^2}$$

	$m = 0$	$m \sim \lambda Q$
Collinear fermion	λ^{-2}	
Soft fermion	λ^{-2}	λ^{-1}
Collinear photon	λ^{-2}	
Soft photon	λ^{-4}	
Collinear loop	λ^4	
Soft loop	λ^8	



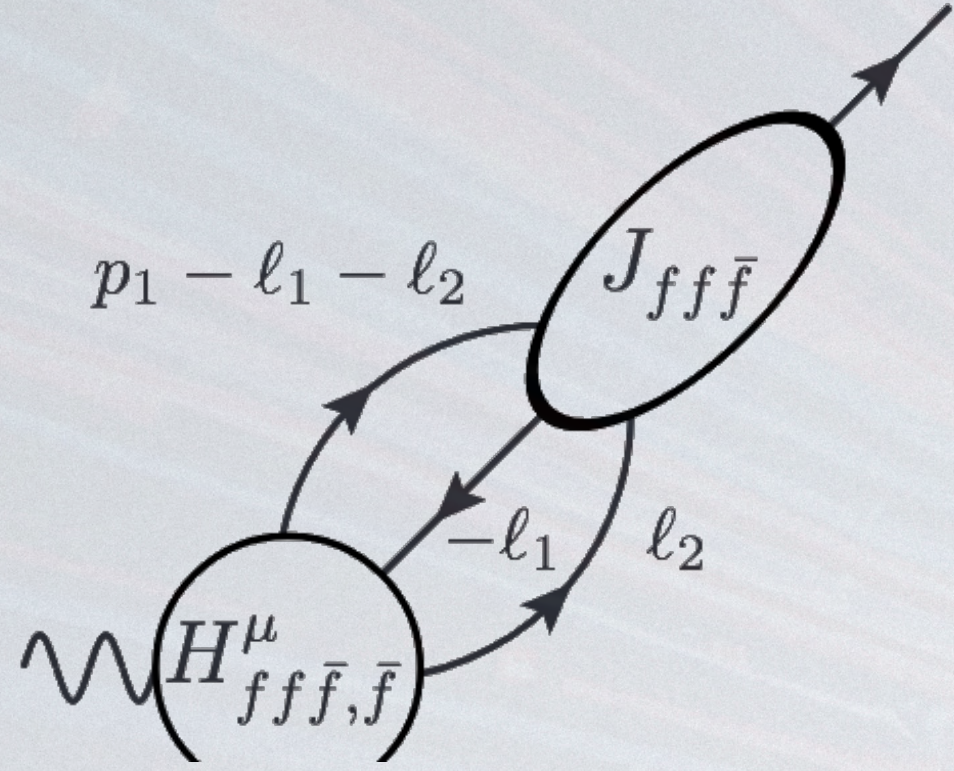
$J_{f\partial\gamma}$ JET FUNCTION

- Expand hard function such that it depends only on the large momentum component

$$H_{f\gamma,\bar{f}}^{\mu\rho}(p_1 - \ell, \ell; p_2) = H_{f\gamma,\bar{f}}^{\mu\rho}(p_1 - \hat{\ell}, \hat{\ell}; p_2) + \ell_{\perp\sigma} H_{f\partial\gamma,\bar{f}}^{\mu\rho\sigma}(\hat{p}_1 - \hat{\ell}, \hat{\ell}; \hat{p}_2) + \mathcal{O}(\lambda^2)$$

$$J_{f\partial\gamma}^{\rho\sigma}(\hat{p}_1 - \hat{\ell}, \hat{\ell}, \bar{n}) = J_{f\gamma}^{\rho}(\hat{p}_1 - \hat{\ell}, \hat{\ell}, \bar{n}) \ell_{\perp}^{\sigma}$$

$$J_{f\partial\gamma}^{\rho\sigma}(p_1, \bar{n}, \bar{n} \cdot \ell) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{-i\ell(\xi\bar{n})} \langle p_1 | \left[\bar{\psi}(0) \Phi_{\bar{n}}(0, \infty) \right] i\partial_{\perp}^{\sigma} \left[\Phi_{\bar{n}}(\infty, \xi\bar{n}) (iD^{\rho} \Phi_{\bar{n}}(\xi\bar{n}, \infty)) \right] | 0 \rangle$$



$J_{fff\bar{f}}$ JET FUNCTION

$$J_{fff\bar{f}}^{(I)}(p_1, \bar{n}, \ell_1^+, \ell_2^+) = \int_{-\infty}^{\infty} \frac{d\xi_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\xi_2}{2\pi} e^{-i\ell_1(\xi_1 \bar{n})} e^{-i\ell_2(\xi_2 \bar{n})} \\ \times \langle p_1 | \left[\bar{\psi}(0) \Phi_{\bar{n}}(0, \infty) \right] \left[\Phi_{\bar{n}}(\infty, \xi_1 \bar{n}) \psi(\xi_1 \bar{n}) \right] \left[\bar{\psi}(\xi_2 \bar{n}) \Phi_{\bar{n}}(\xi_2 \bar{n}, \infty) \right] | 0 \rangle$$

$$J_{fff\bar{f}}^{(II)}(p_1, \bar{n}, \ell_1^+, \ell_2^+) = \int_{-\infty}^{\infty} \frac{d\xi_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\xi_2}{2\pi} e^{-i\ell_1(\xi_1 \bar{n})} e^{-i\ell_2(\xi_2 \bar{n})} \\ \times \langle p_1 | \left[\Phi_{\bar{n}}(\infty, 0) \psi(0) \right] \left[\bar{\psi}(\xi_1 \bar{n}) \Phi_{\bar{n}}(\xi_1 \bar{n}, \infty) \right] \left[\bar{\psi}(\xi_2 \bar{n}) \Phi_{\bar{n}}(\xi_2 \bar{n}, \infty) \right] | 0 \rangle$$

NO OVERLAP BETWEEN JETS

- At leading power, collinear photon propagator becomes

$$\frac{-i\eta^{\rho\sigma}}{k^2} \rightarrow \frac{-i}{k^2} \frac{\bar{n}^\rho k^\sigma}{\bar{n} \cdot k}$$

- Effective Feynman rule for photon propagator in $J_{f\gamma}$ and $J_{f\gamma\gamma}$

$$\frac{-i}{k^2} \left(\eta^{\rho\sigma} - \frac{\bar{n}^\sigma k^\rho}{\bar{n} \cdot k} \right) \delta(\bar{n} \cdot k - \ell^+)$$

MOMENTUM ROUTING FOR CROSSED LADDER

- cc region: contributions from J_f , $J_{f\gamma}$ and $J_{f\gamma\gamma}$
 - Due to $\delta(\vec{n} \cdot k_i - \ell_i^+)$,
loop momentum k_i^μ through photon
- cc' region: contribution from $J_{ff\bar{f}}$
 - Due to $\delta(\vec{n} \cdot k_i - \ell_i^+)$,
loop momentum k_i^μ through fermions

