

Towards the complete NNLO BFKL Kernel

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One Central Emission at Two Loops

[arXiv:2412.20578](https://arxiv.org/abs/2412.20578) (10.1007/JHEP04(2025)161)

with S. Abreu, G. Falcioni, E. Gardi, C. Milloy, L. Vernazza

Two Central Emissions at One Loop

To Appear

with E. Byrne, V. Del Duca, E. Gardi, J. Smillie

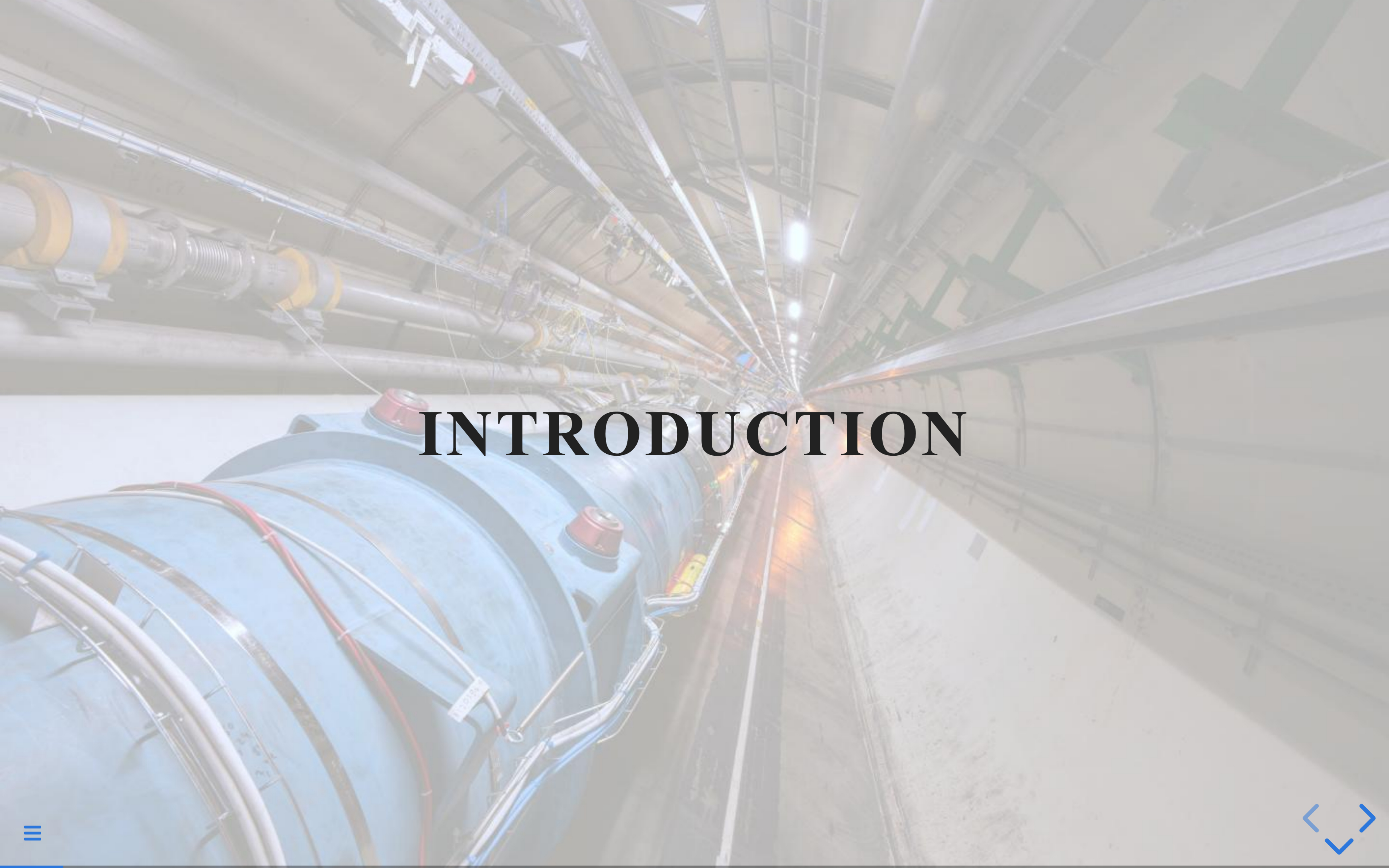
REF Conference

Milan, IT



Find these slides at gdelaurentis.github.io/slides/REFOct2025



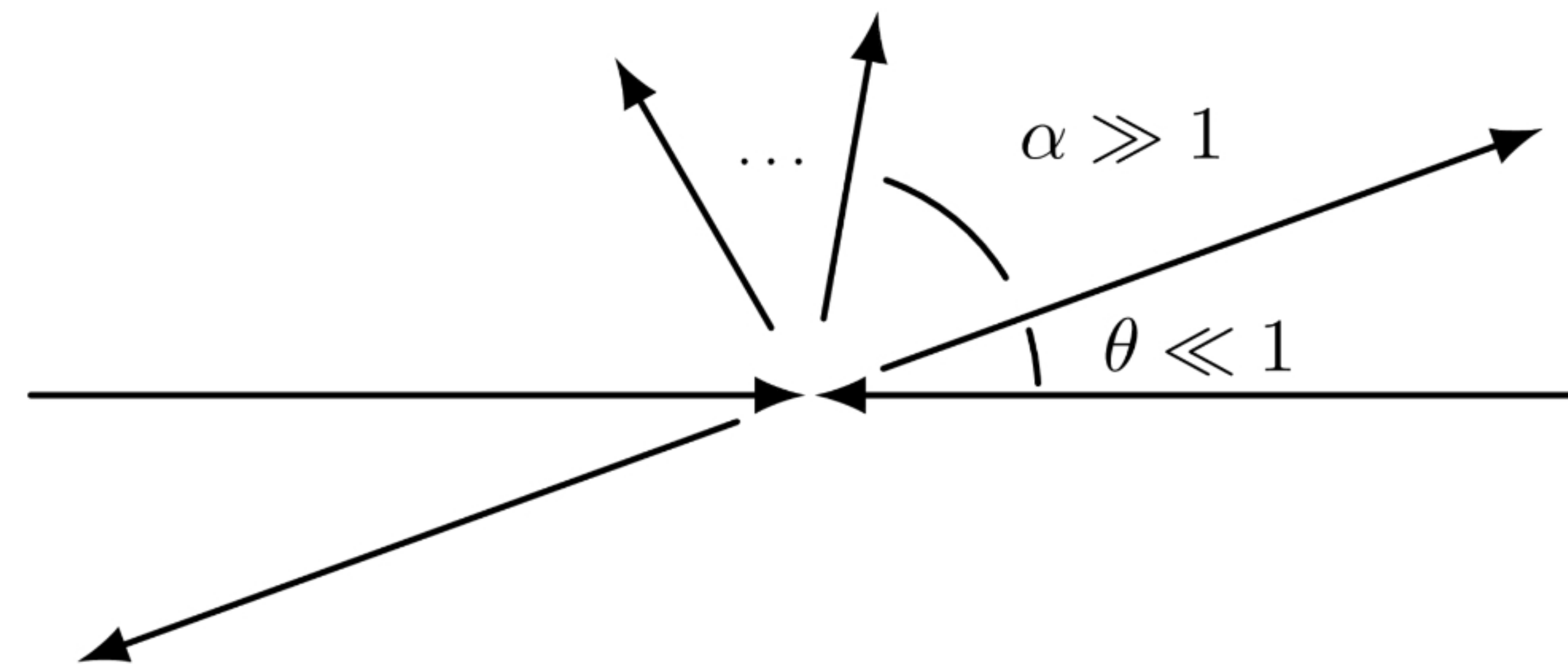


INTRODUCTION



LARGE LOGARITHMS FROM BIG RAPIDITY GAPS

- In the **forward limit** $s \gg |t|$, i.e. at large CoM energy vs. momentum transfer, when final state emissions develop **large rapidity gaps**



amplitudes are dominated by unphysically **large logarithms**

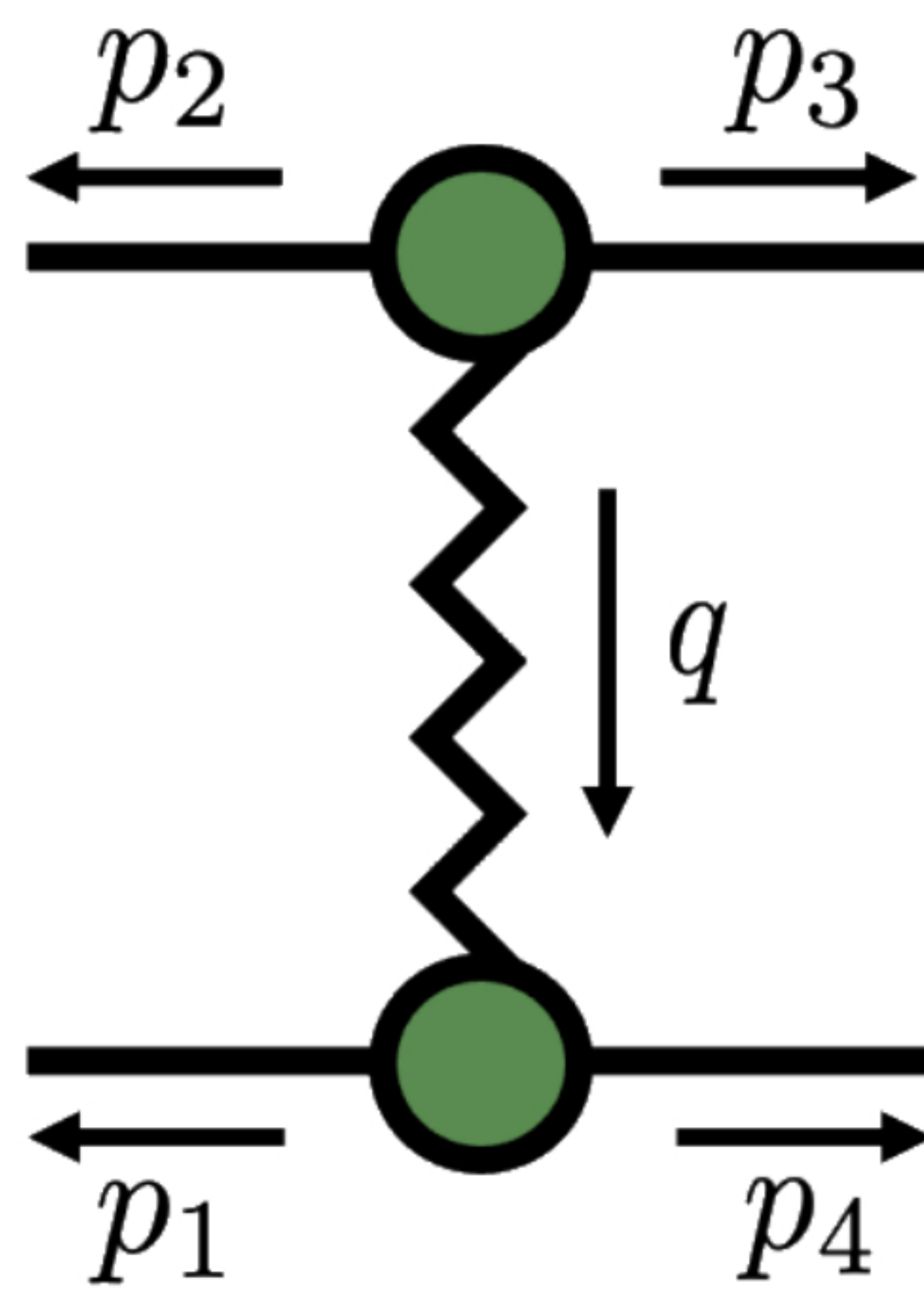
$$\mathcal{A} \approx \mathcal{O}(\alpha_s^n \log^n(s/|t|))$$

- The BFKL kernel captures the **exponentiation** of these large logarithms, allowing us to **resum** their contribution to the cross section. [Fadin, Kuraev, Lipatov '75](#); [Balitsky, Lipatov '78](#)
- In this kinematic limit, known as **Multi-Regge Kinematics** (MRK), an effective particle is exchanged in the t-channel, a Reggeon, from which more rapidity-gapped radiation can be emitted. Depending on whether the extra radiation is itself rapidity gapped we talk about next-to-MRK.

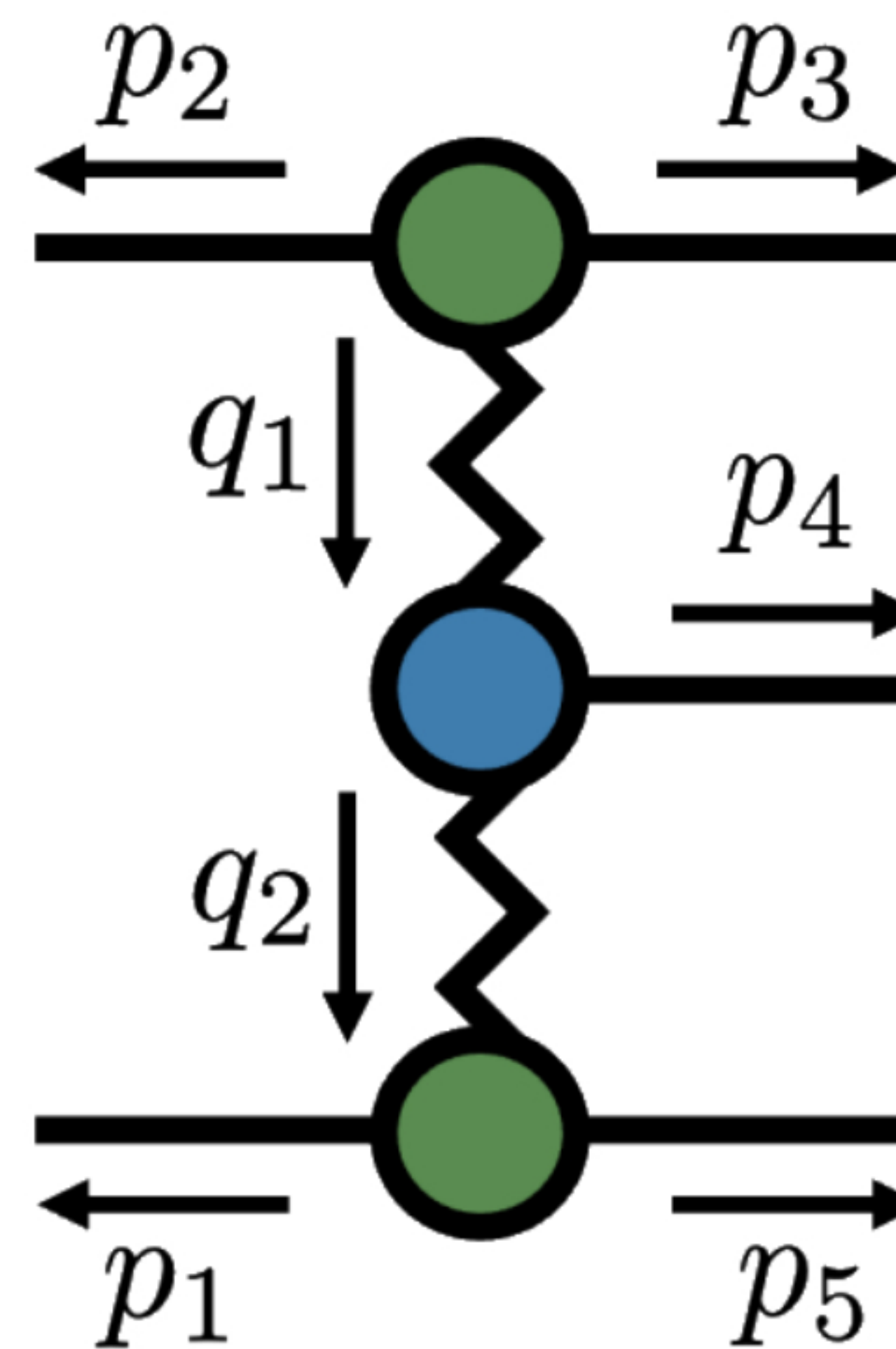


AMPLITUDE FACTORIZATION IN MRK AND NMRK

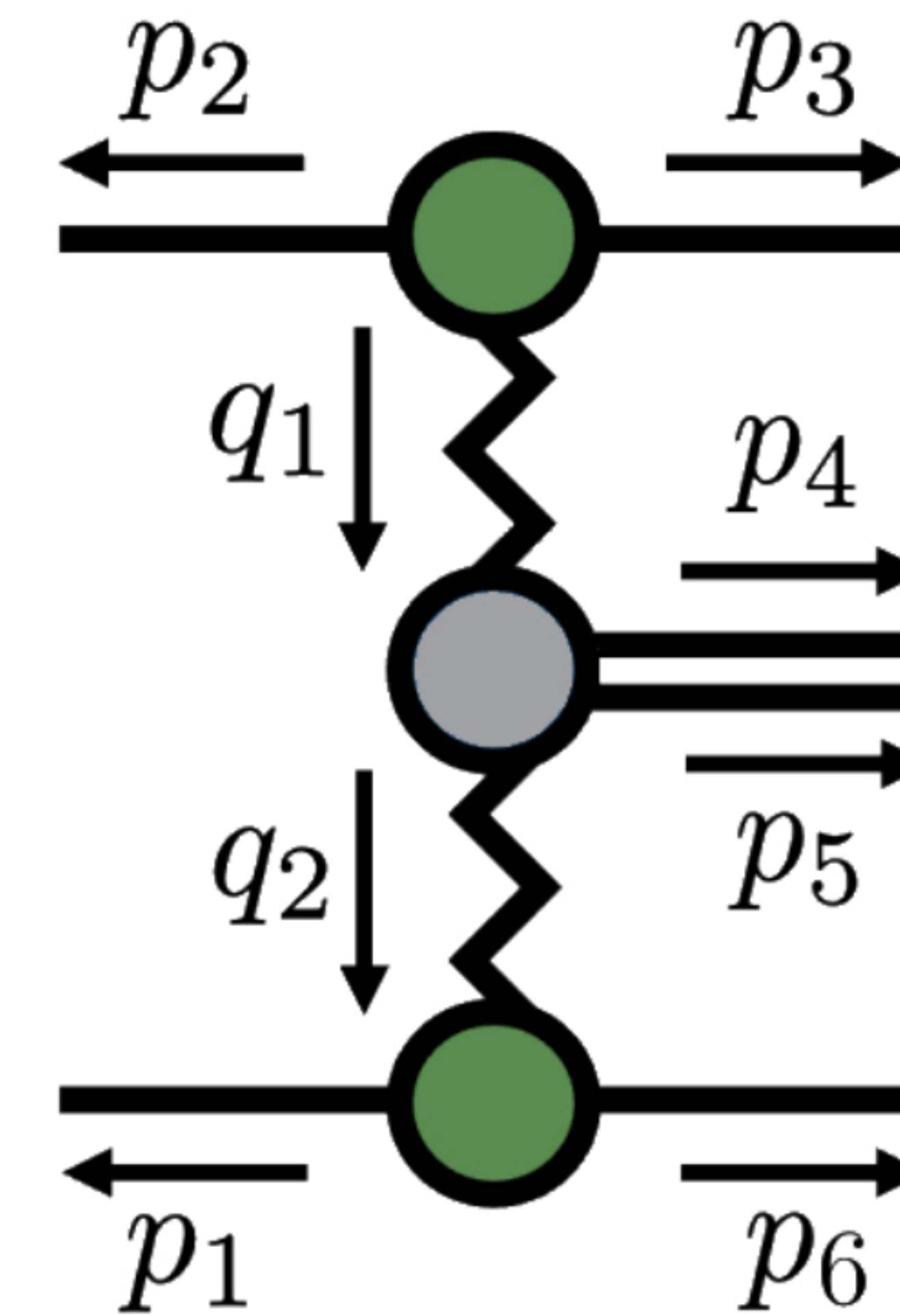
- In the (N)MRK we can picture the amplitude as follows



(a)



(b)



(c)

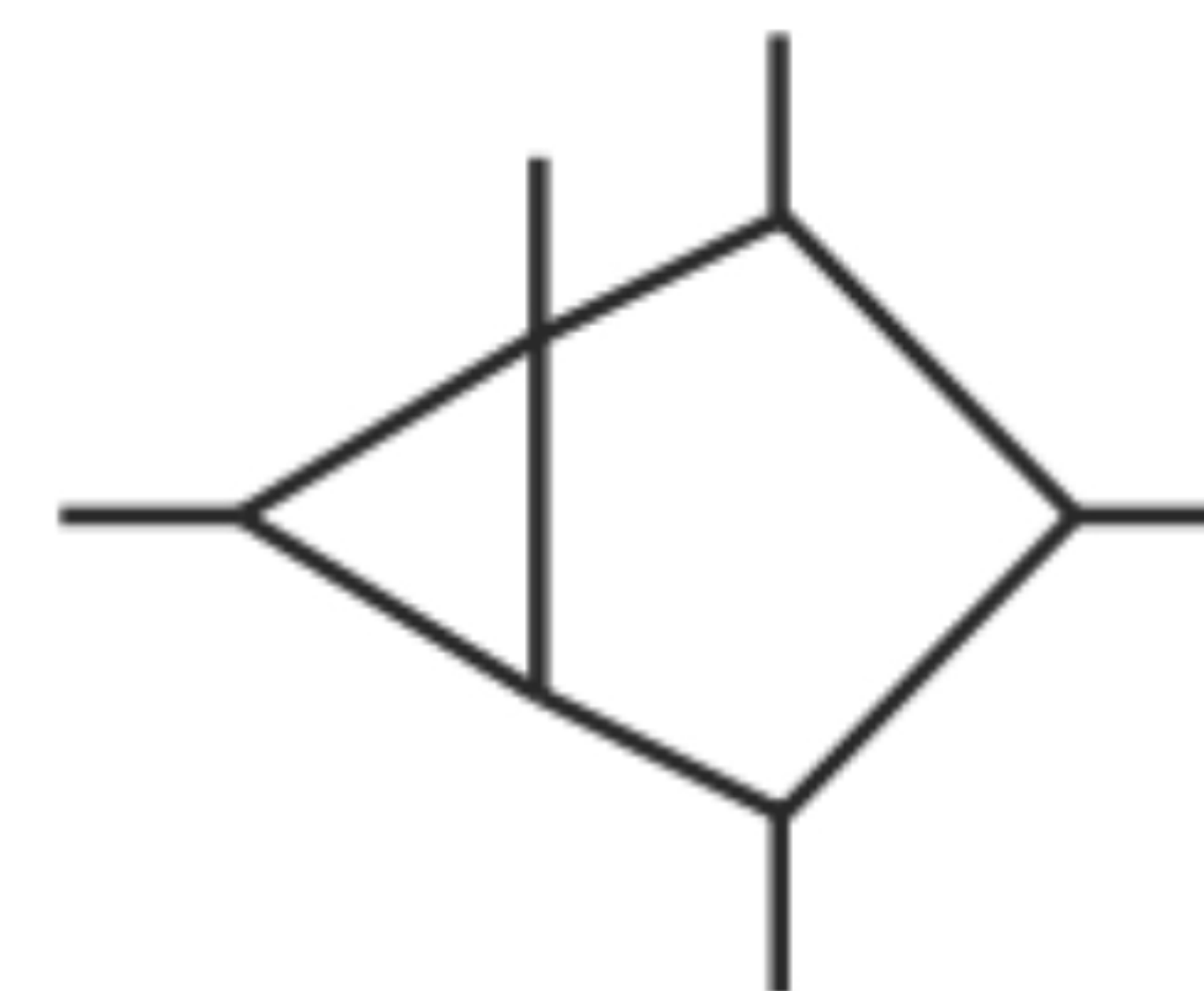
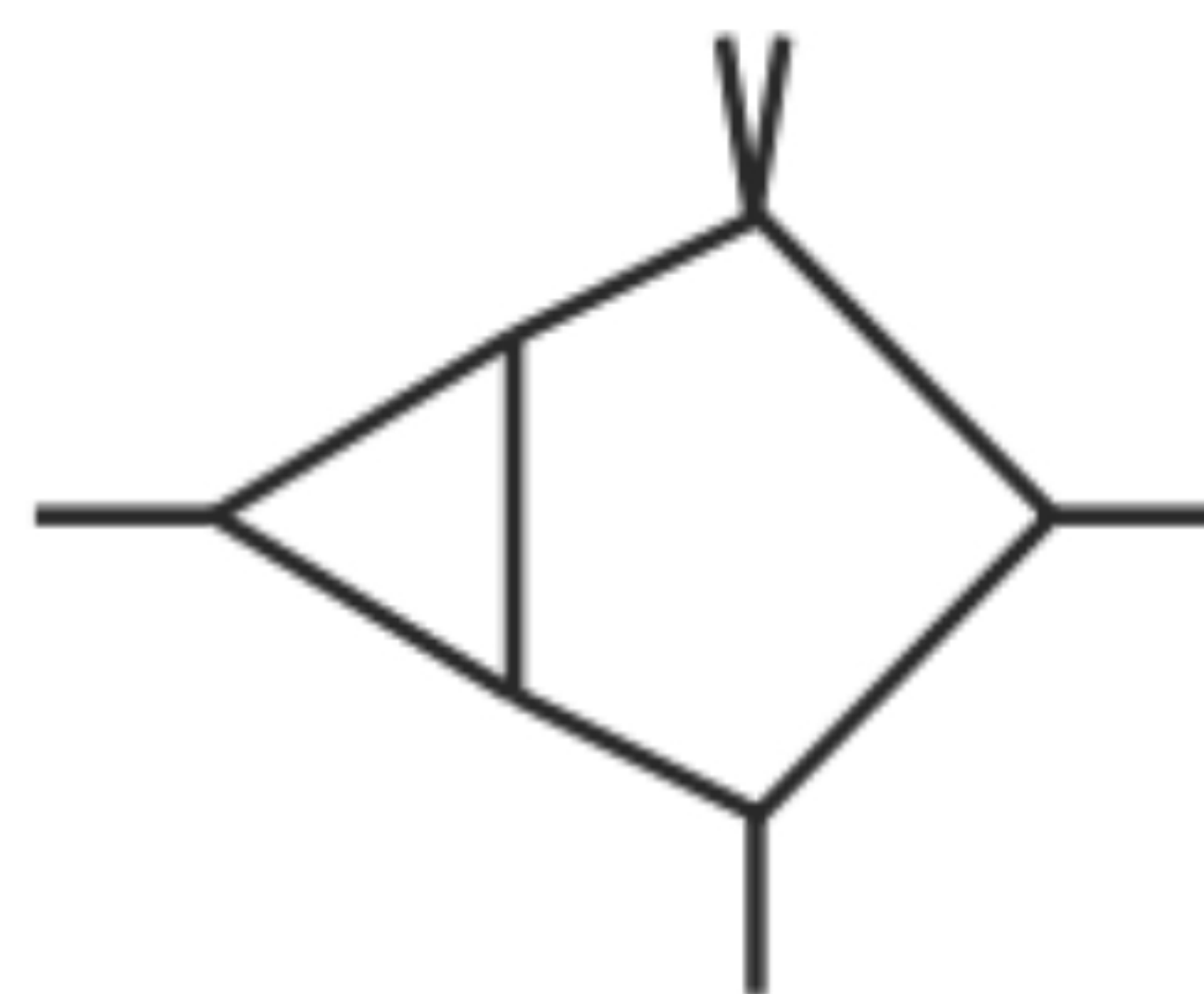
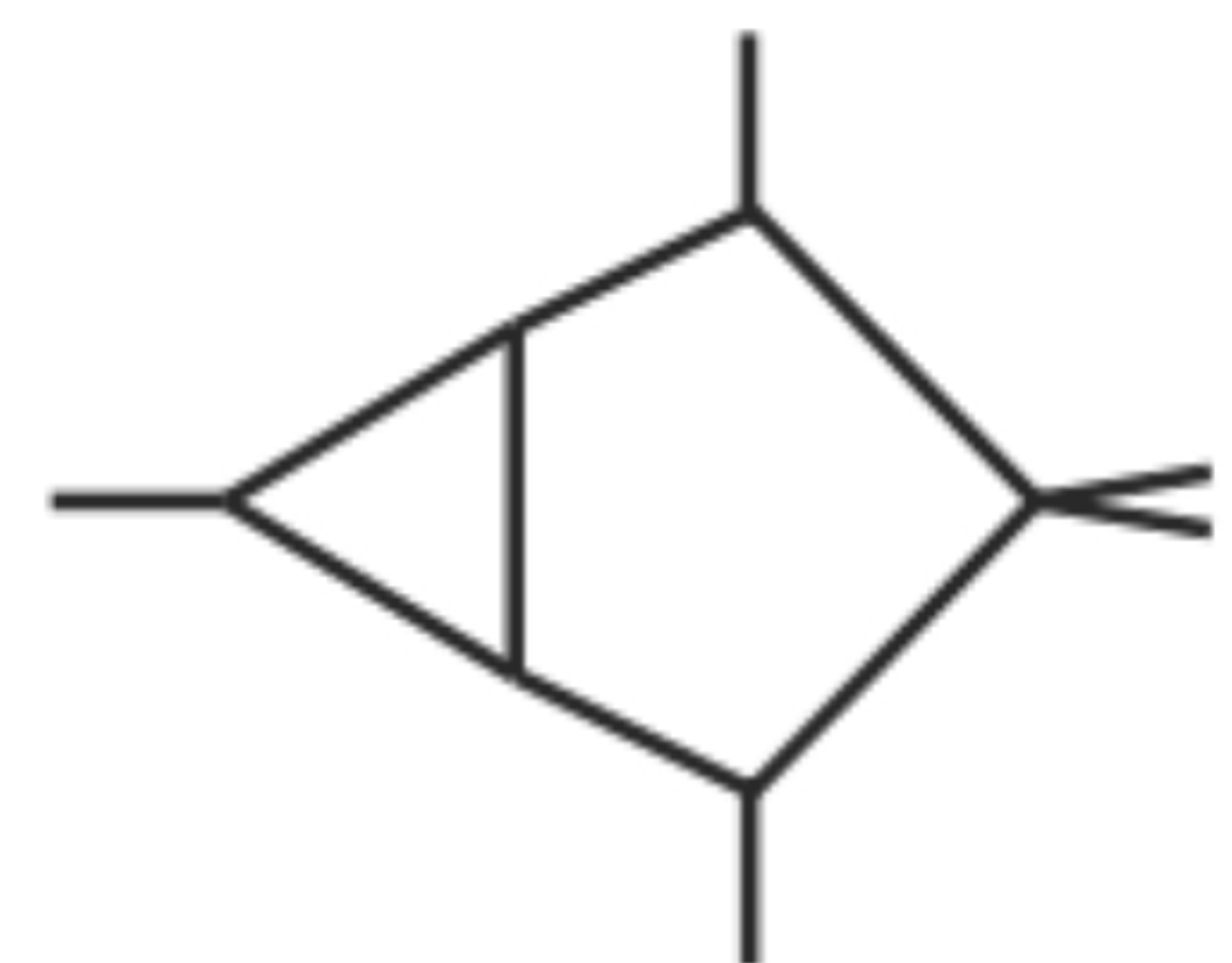
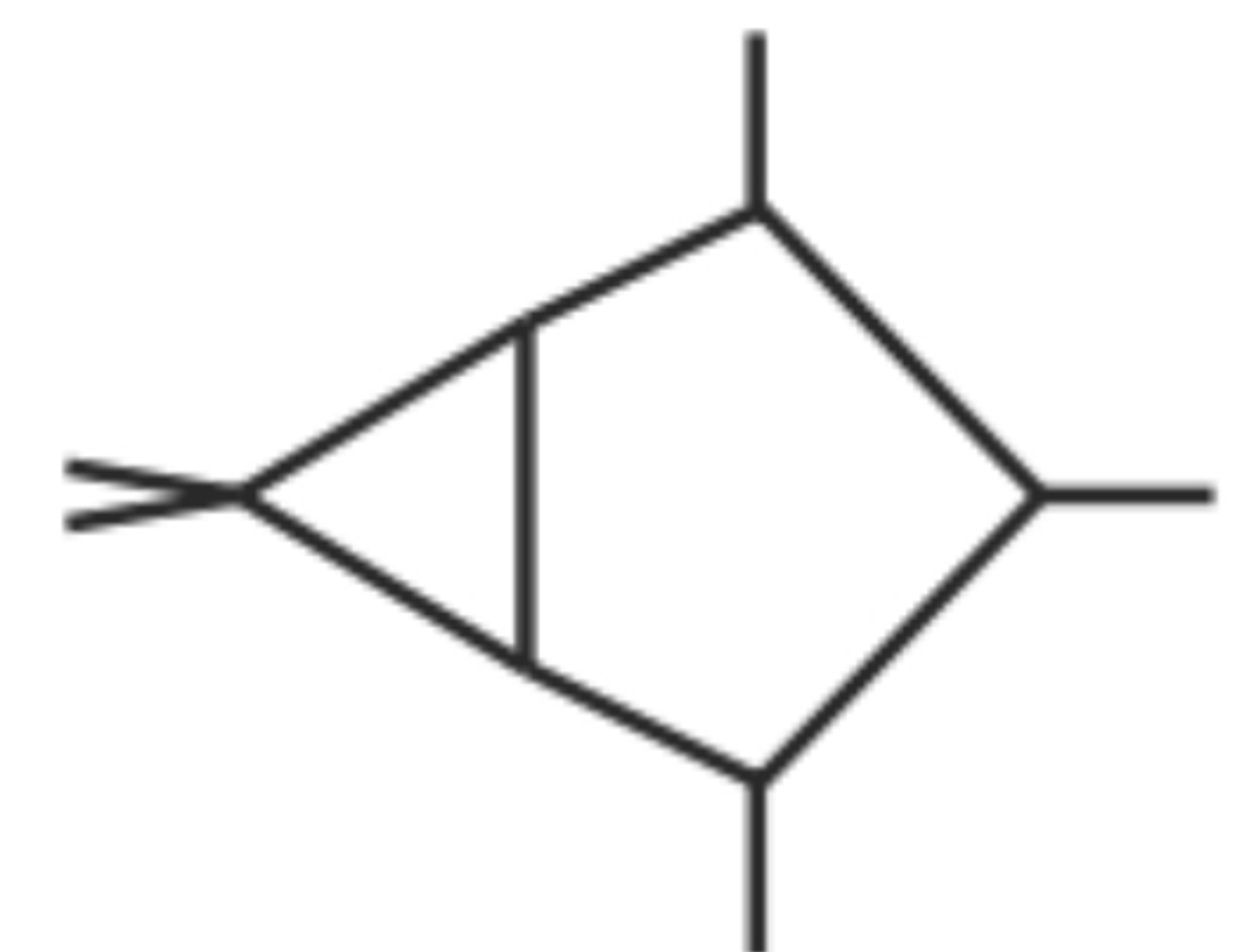
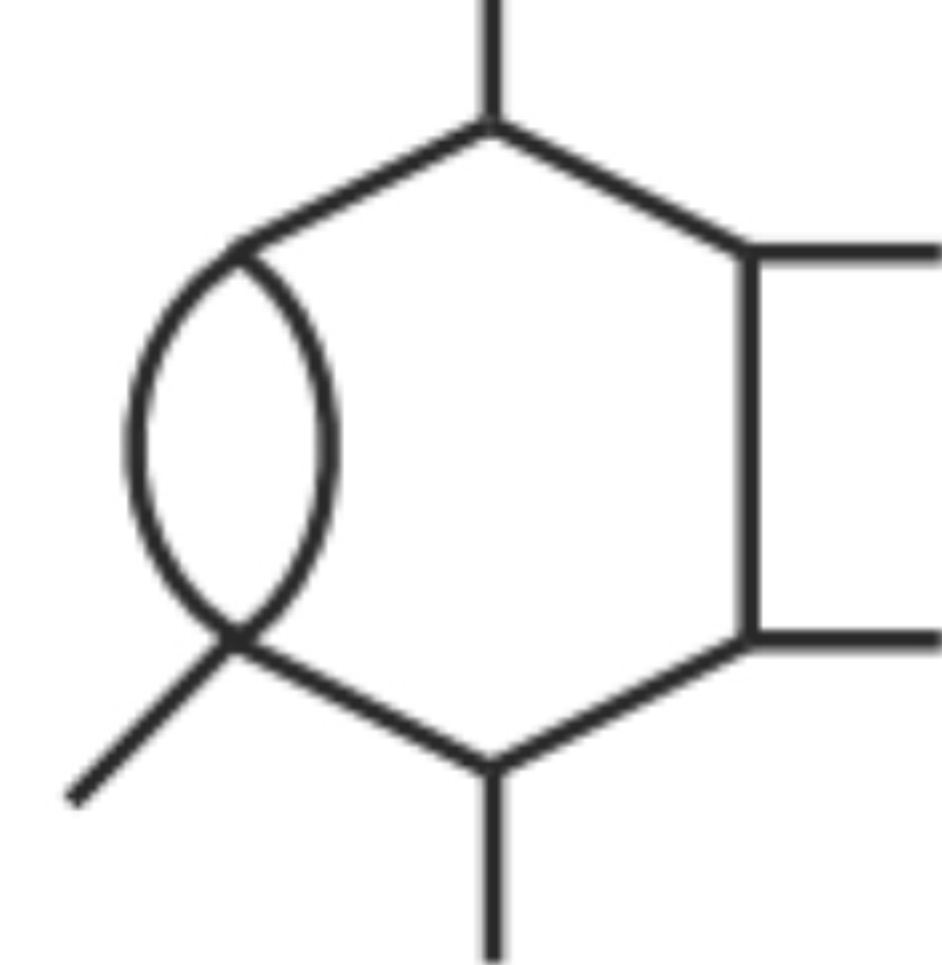
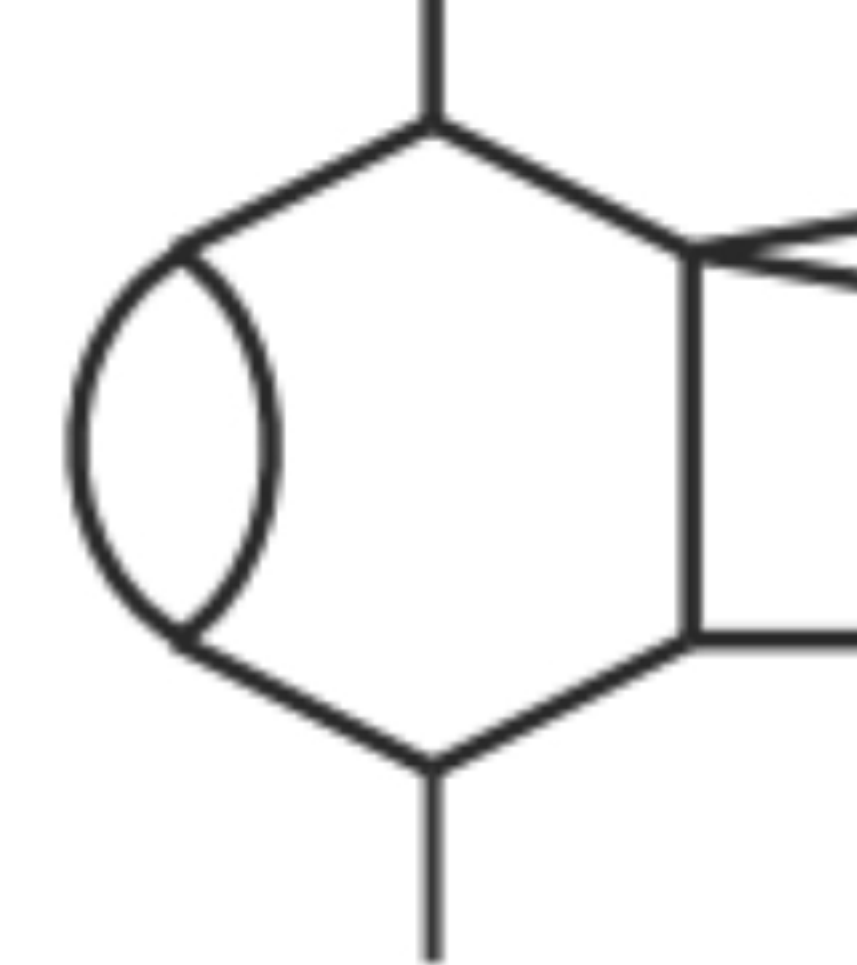
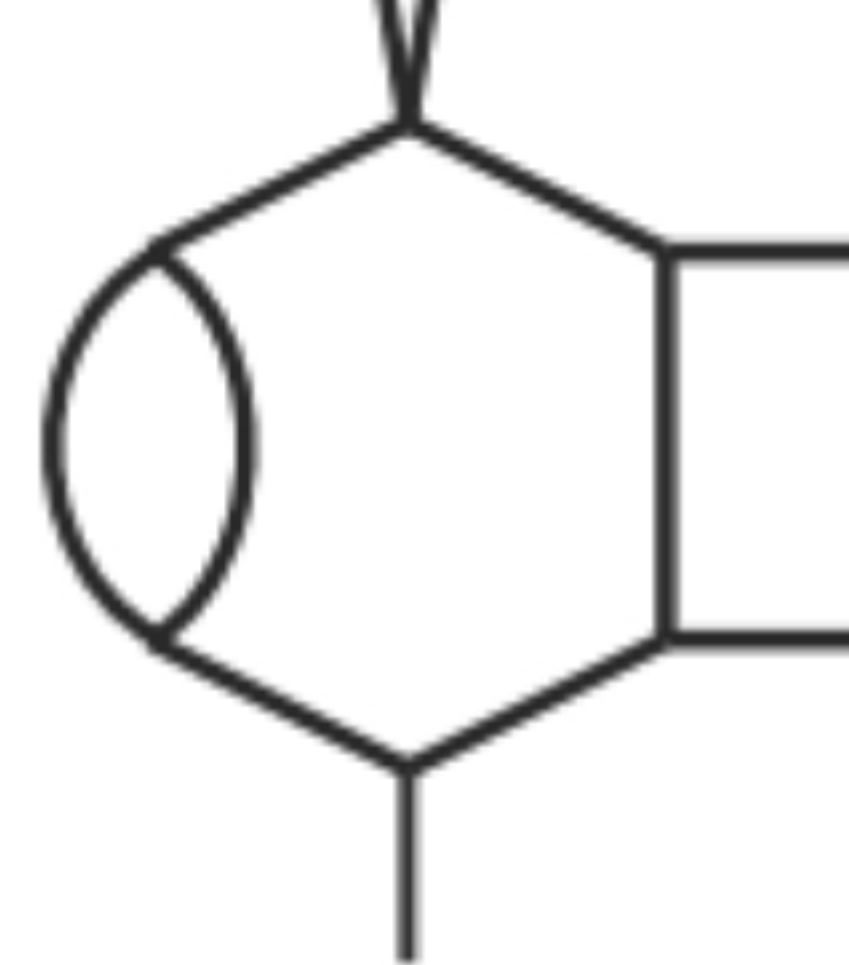
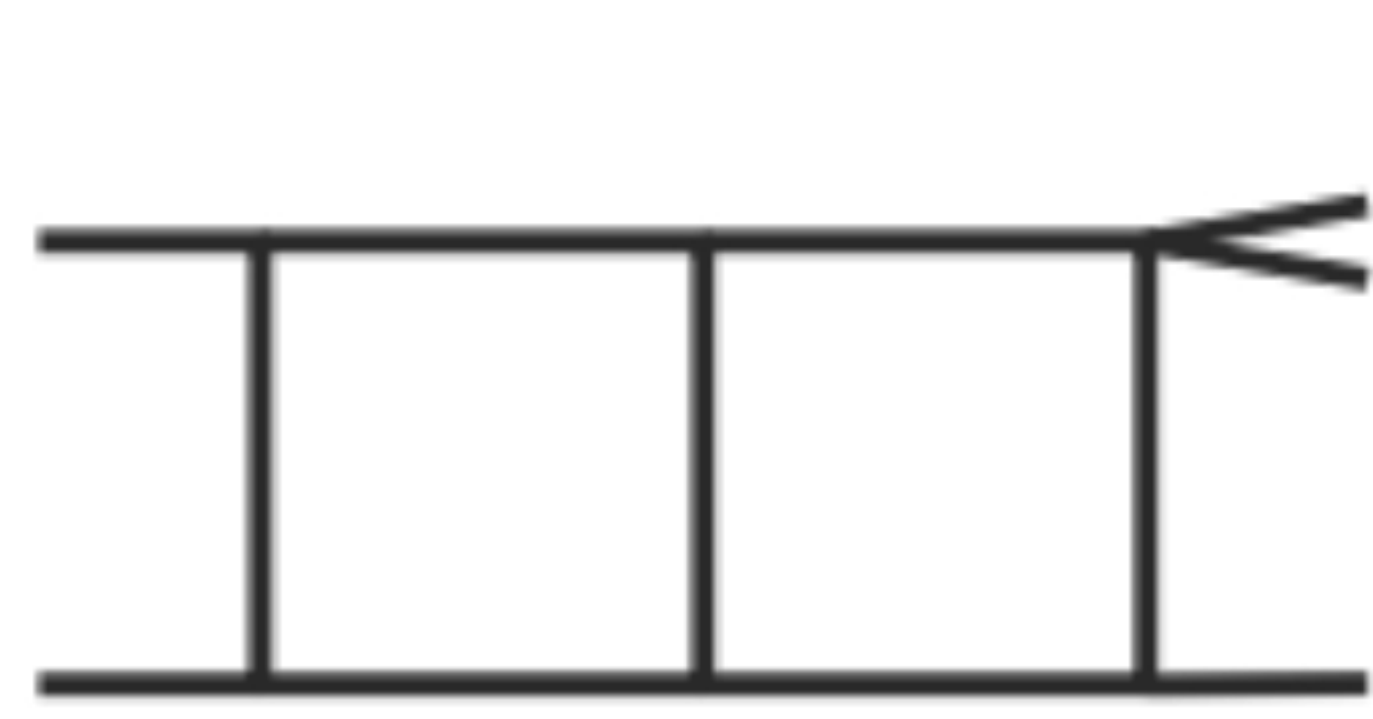
Images adapted from [Byrne '23](#)

where: the ziggly line is the **Regge trajectory** $\mathcal{R}$, the green blobs are **impact factors** $\mathcal{C}$, the blue blob is a one-emission **central vertex** $\mathcal{V}_g$, and the gray blob is a two-emission central vertex $\mathcal{V}_{gg}$.

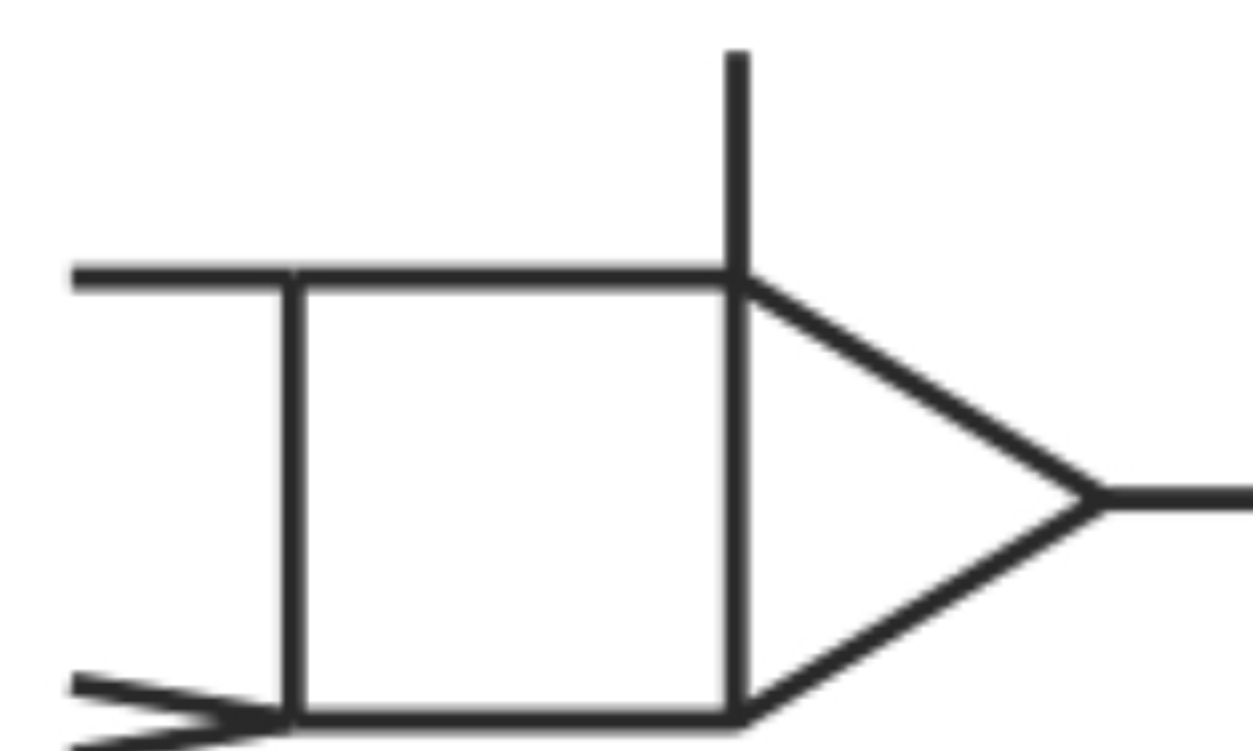
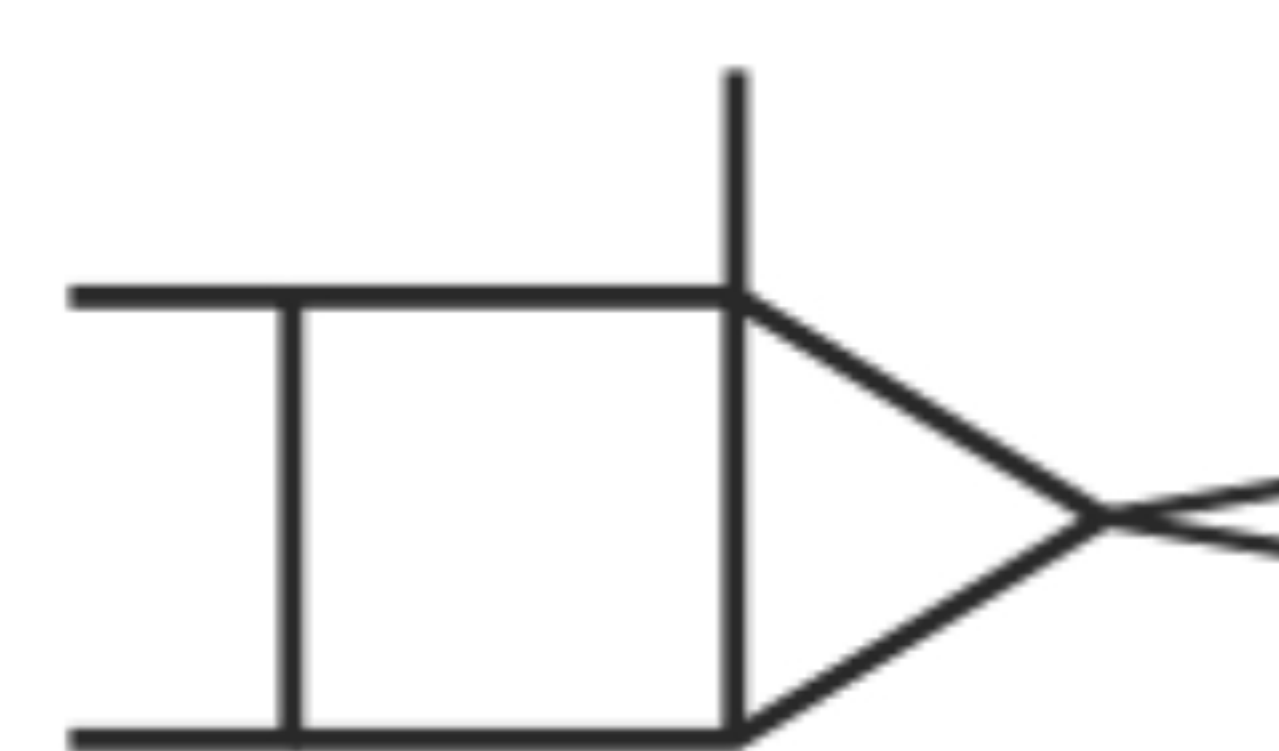
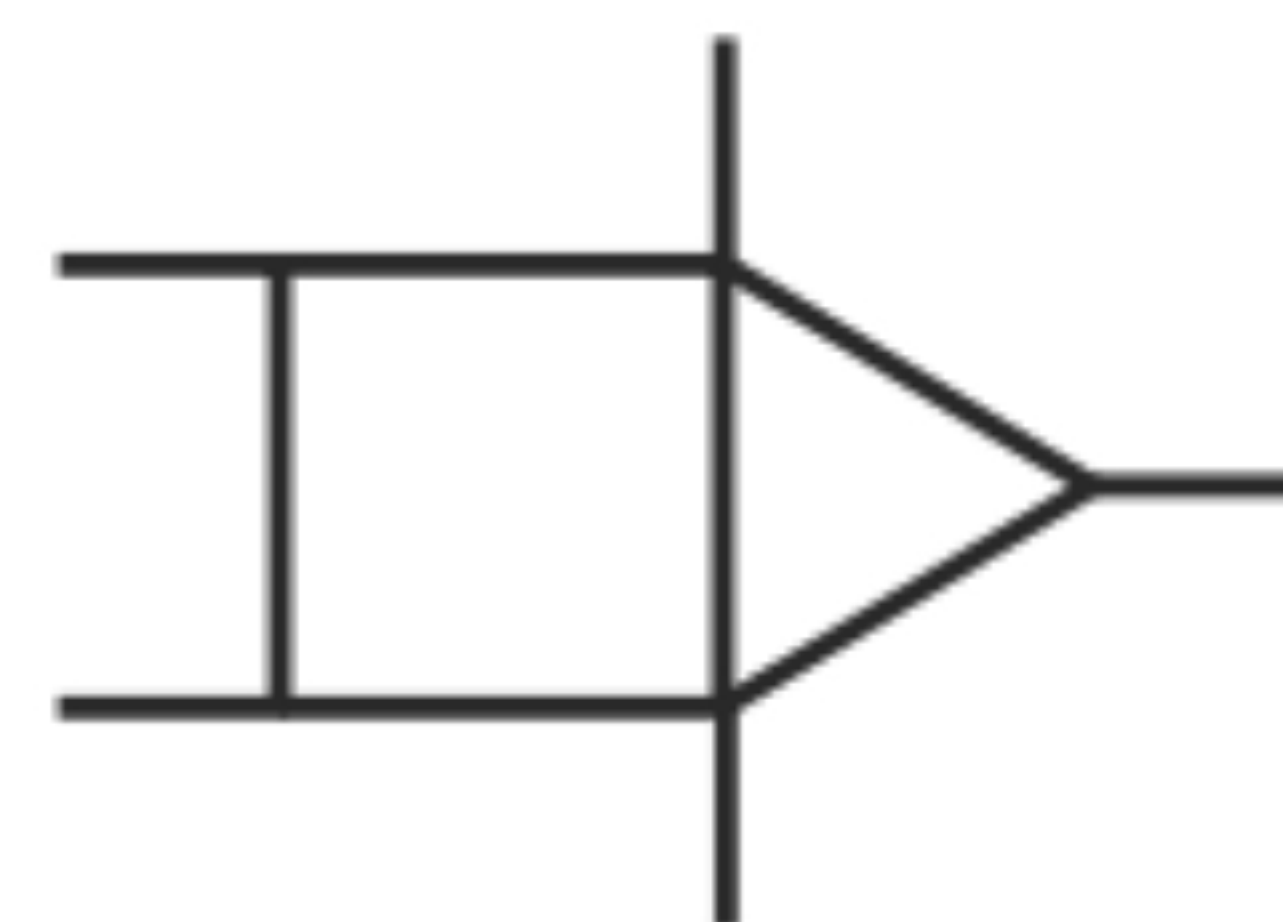
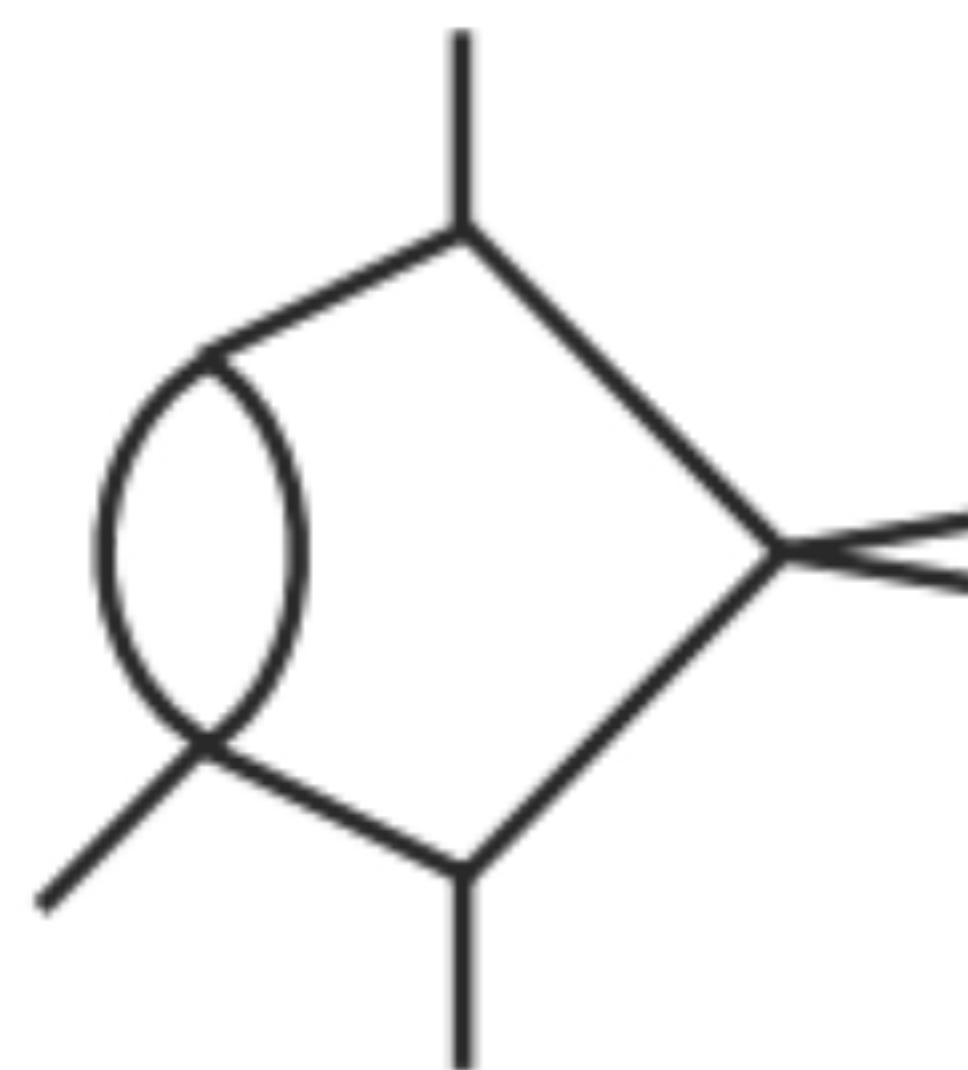
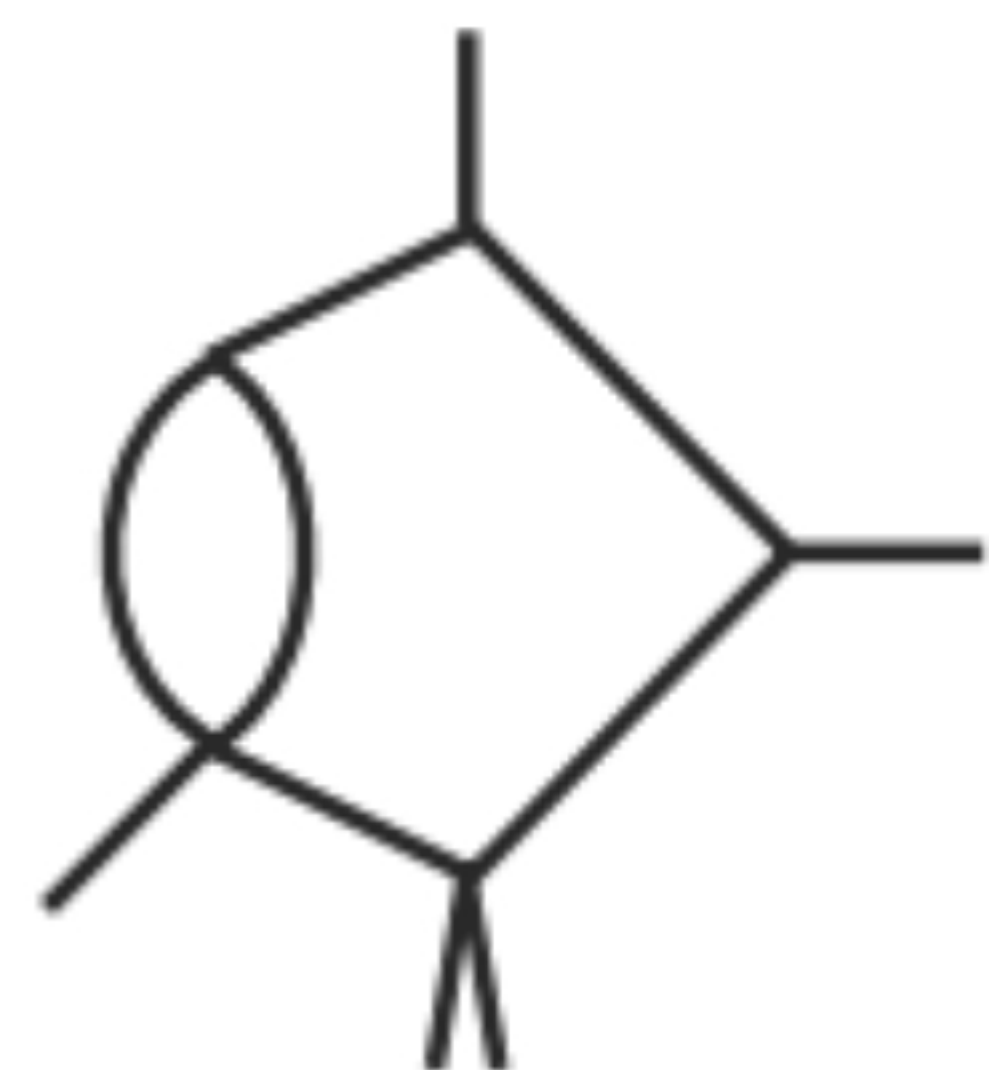
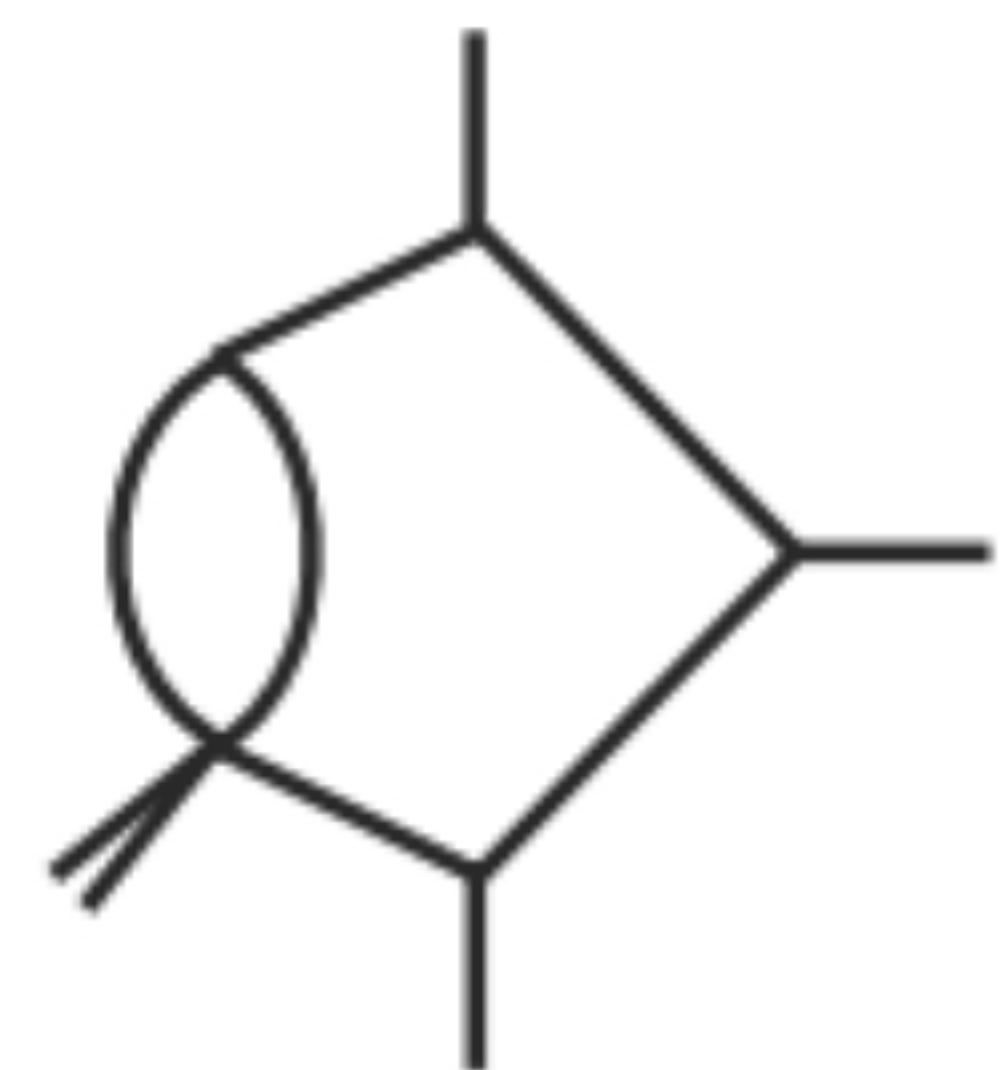
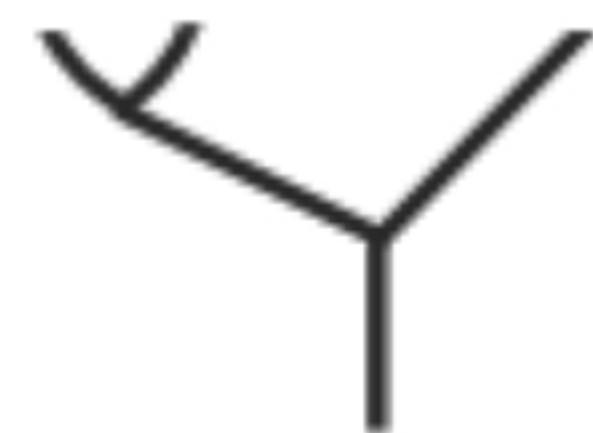
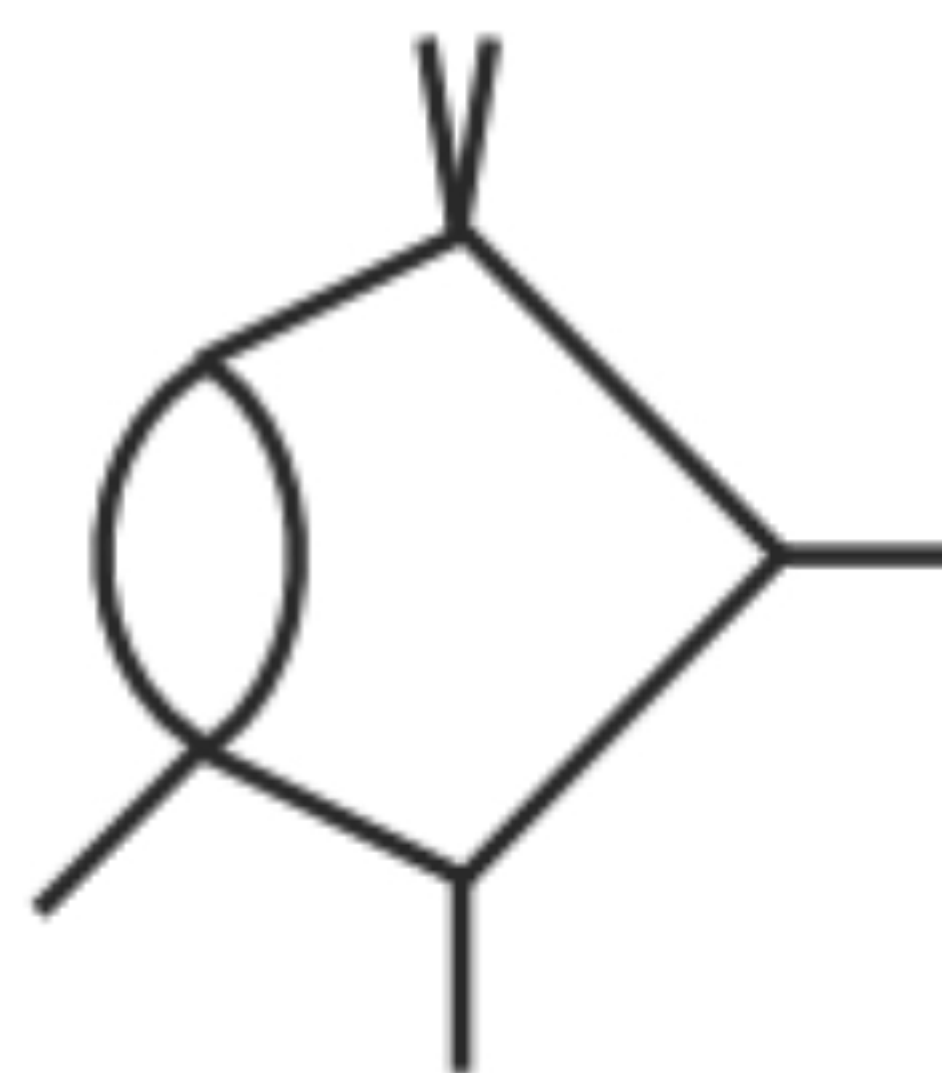
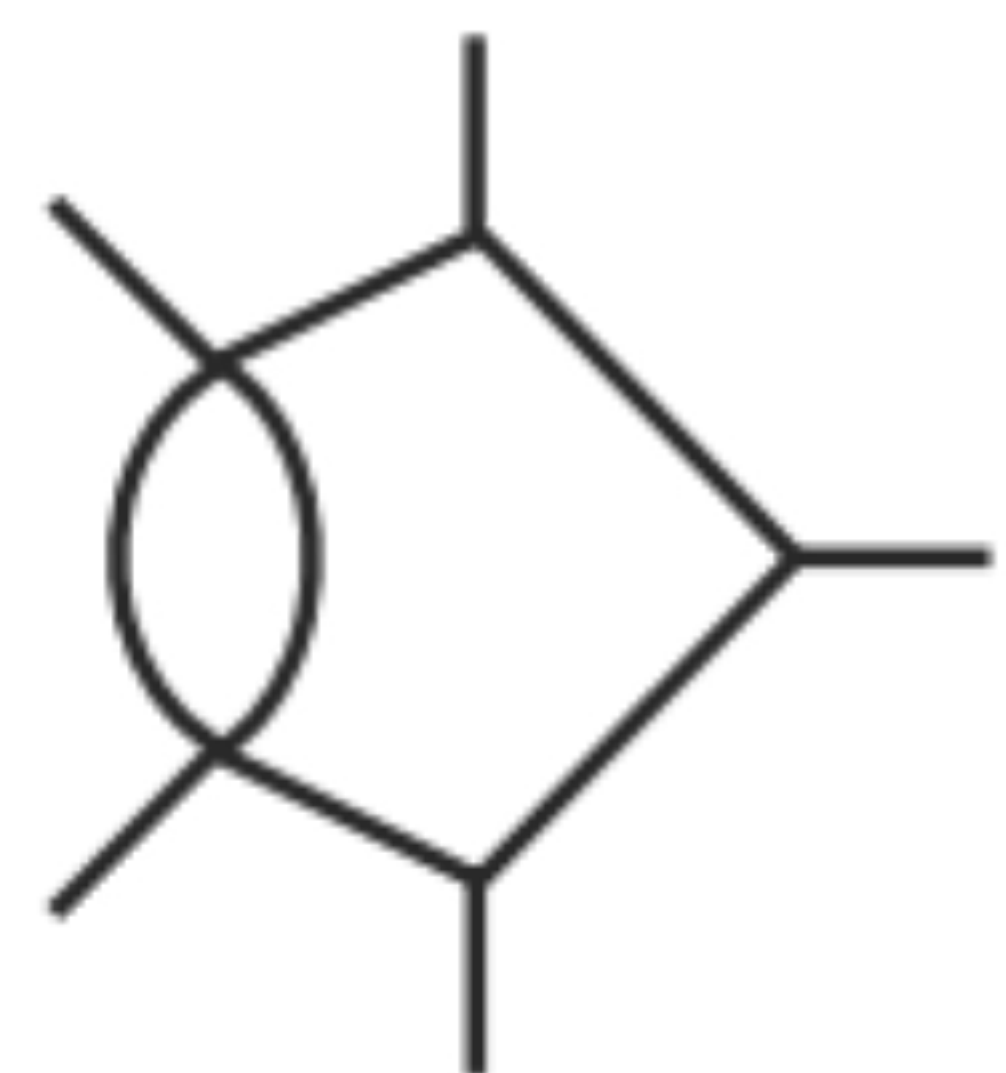
- Amplitudes factorise (very schematically, octet component only and up to Regge cuts)

$$\mathcal{A}_4 \approx \mathcal{C} \mathcal{R} \mathcal{C}, \quad \mathcal{A}_5 \approx \mathcal{C} \mathcal{R} \mathcal{V}_g \mathcal{R} \mathcal{C}, \quad \mathcal{A}_6 \approx \mathcal{C} \mathcal{R} \mathcal{V}_{gg} \mathcal{R} \mathcal{C}$$

where each component admits an expansion in powers of α_s , thus e.g. $\mathcal{A}_4^{(1)}$ gives us $\mathcal{C}^{(1)}$ and $\mathcal{R}^{(1)}$



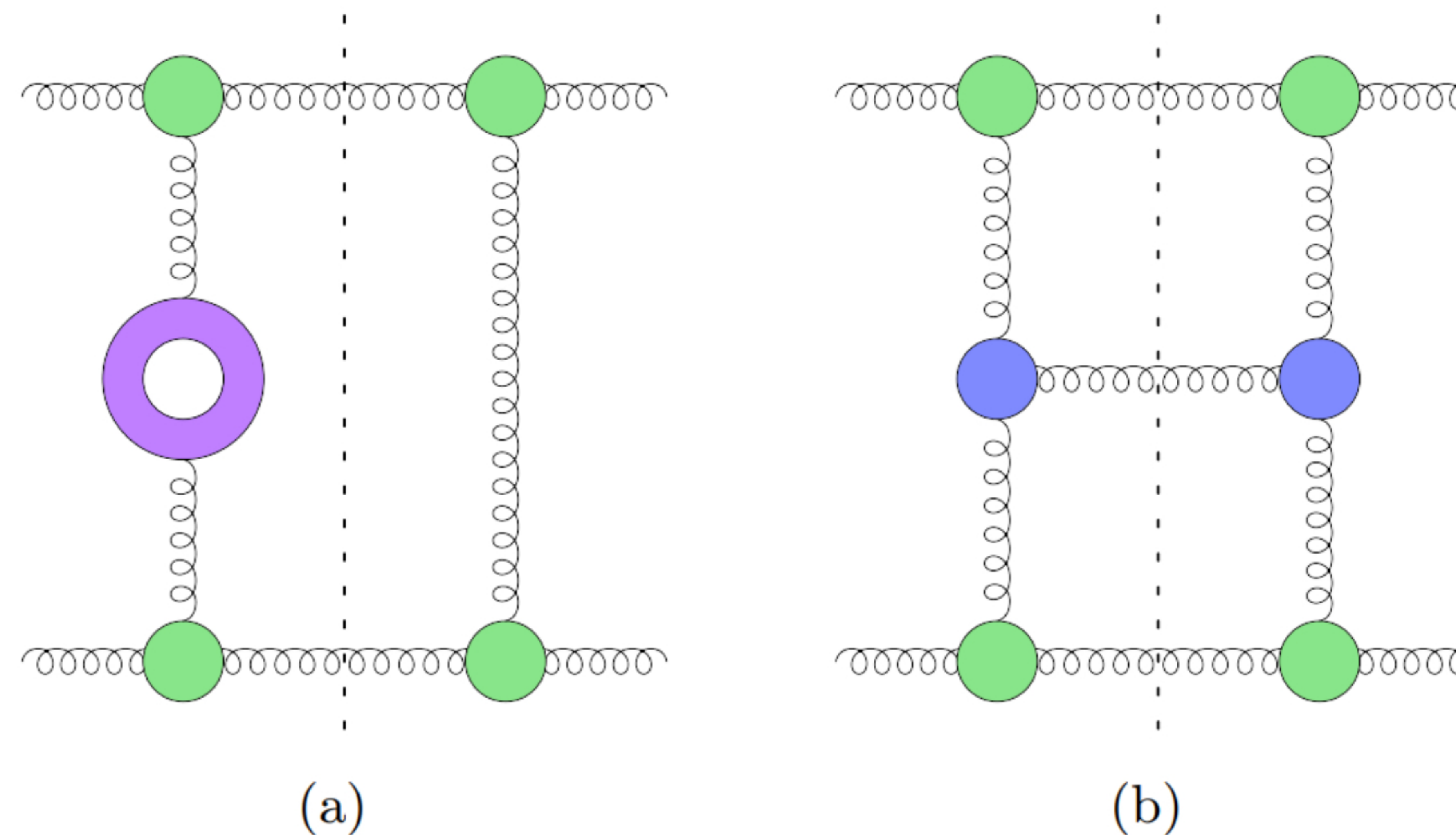
KERNEL COMPONENTS



LEADING ORDER KERNEL COMPONENTS

Leading-Log (LL) Resummation: $\mathcal{O}(\alpha_s^n \log^n(s/|t|))$

- The two components of the leading order (LO) BFKL kernel, required for resummation of leading logarithms (LL), are



Images from [Byrne, Del Duca, Dixon, Gardi, Smillie '22](#)

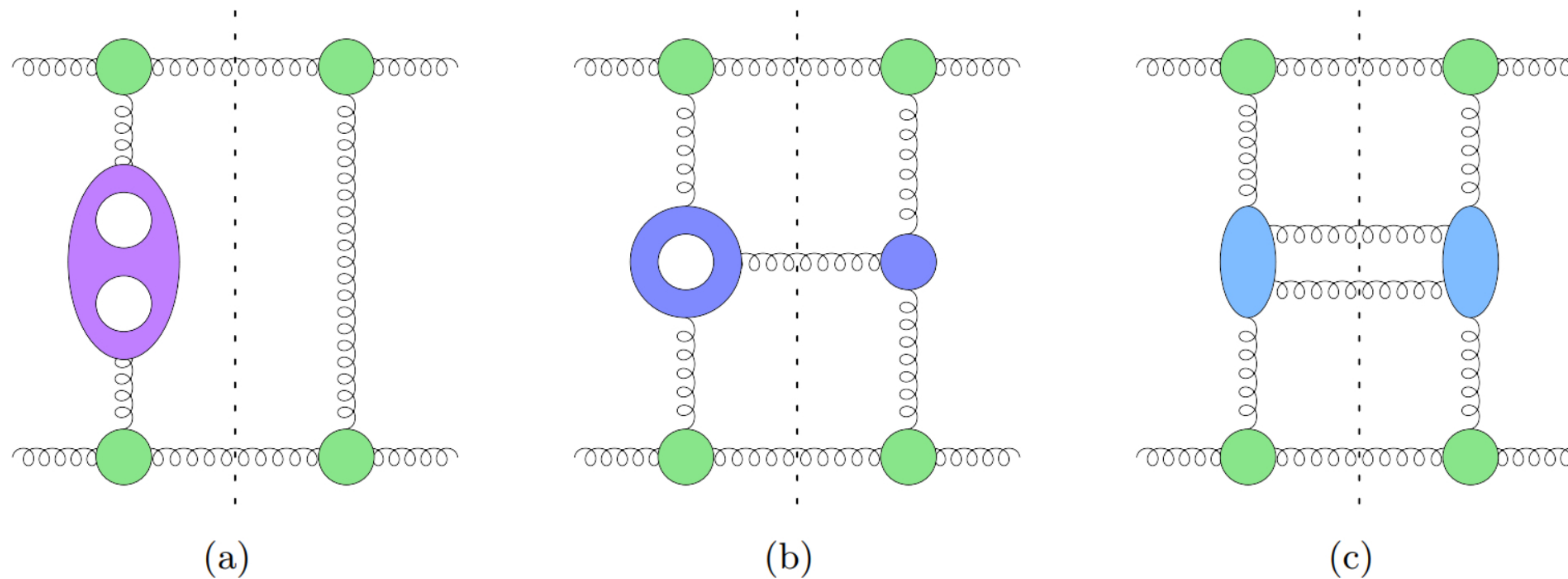
where pictured is a forward squared amplitude with a final-state cut.

(a) is a correction to the Regge trajectory $\mathcal{R}^{(1)}$

(b) is the leading order central emission vertex (CEV) $\mathcal{V}_g^{(0)}$ in MRK

NLO KERNEL

Next-To-Leading-Log (NLL) Resummation: $\mathcal{O}(\alpha_s^n \log^{n-1}(s/|t|))$



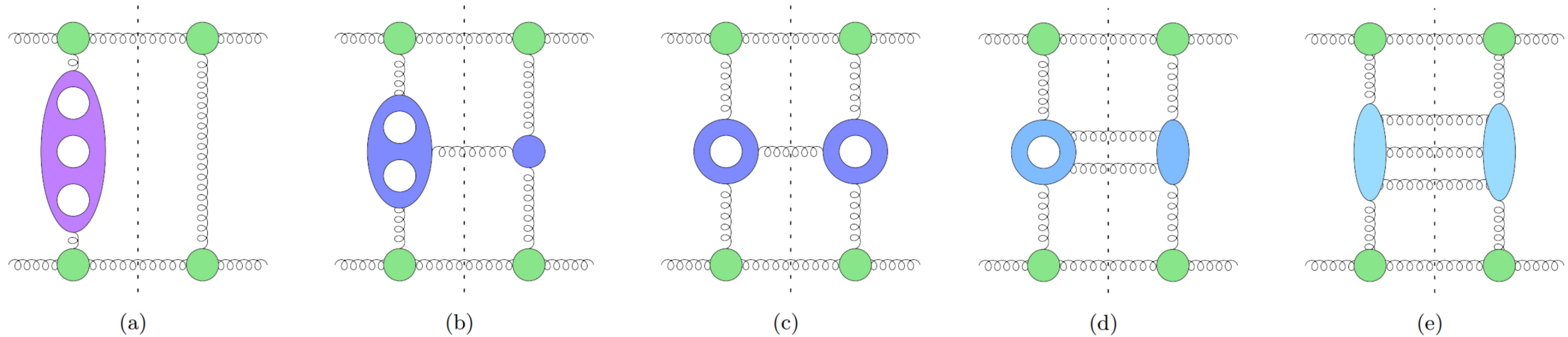
(a) two-loop correction to the Regge trajectory, $\mathcal{R}^{(2)}$

(b) one-loop correction to the one-emission CEV $\mathcal{V}_g^{(1)}$ in MRK

(c) leading two-emission CEV $\mathcal{V}_{gg}^{(0)}$, this requires an next-to-MRK (NMRK) tree computation:
the two central gluons are not rapidity gapped

NNLO KERNEL

NNLL Resummation: $\mathcal{O}(\alpha_s^n \log^{n-2}(s/|t|))$



- (a) Three loop $2 \rightarrow 2$ MRK, from three Reggeons to three-loop correction to the trajectory, $\mathcal{R}^{(3)}$
[Falcioni, Gardi, Maher, Milloy, Vernazza '21](#); [Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi '21](#)
- (b) Two-loop correction to the central emission vertex $\mathcal{V}_g^{(2)}$ for one gluon
[Abreu, GDL, Falcioni, Gardi, Milloy, Vernazza '24](#); [Buccioni, Caola, Devoto, Gambuti '24](#)
 by expanding in the MRK limit the recently available two-loop five-parton amplitudes
[GDL, Ita, Klinkert, Sotnikov '23](#); [GDL, Ita, Sotnikov '23](#); [Agarwal, Buccioni, Devoto, Gambuti, von Manteuffel, Tancredi '23](#)
- (c) The CEV for one emission $\mathcal{V}_{gg}^{(0)}$ (at higher orders in epsilon) [Fadin, Fucilla, Papa '23](#)
- (d) The last missing component is the next-to-maximally-helicity-violating (NMHV) one-loop two-gluon CEV $\mathcal{V}_{g^+g^-}^{(0)}$, this requires expanding in NMRK the one-loop six-gluon amplitude
[Byrne, GDL, Del Duca, Gardi, Smillie - in progress](#); [GDL, Maitre '19](#)
- (e) The leading CEV for three emissions $\mathcal{V}_{ggg}^{(0)}$ from an NNMRK limit at tree level

[Byrne, Del Duca, Gardi, Mo, Smillie '25](#)

NMRK NUMERICAL EXPANSION

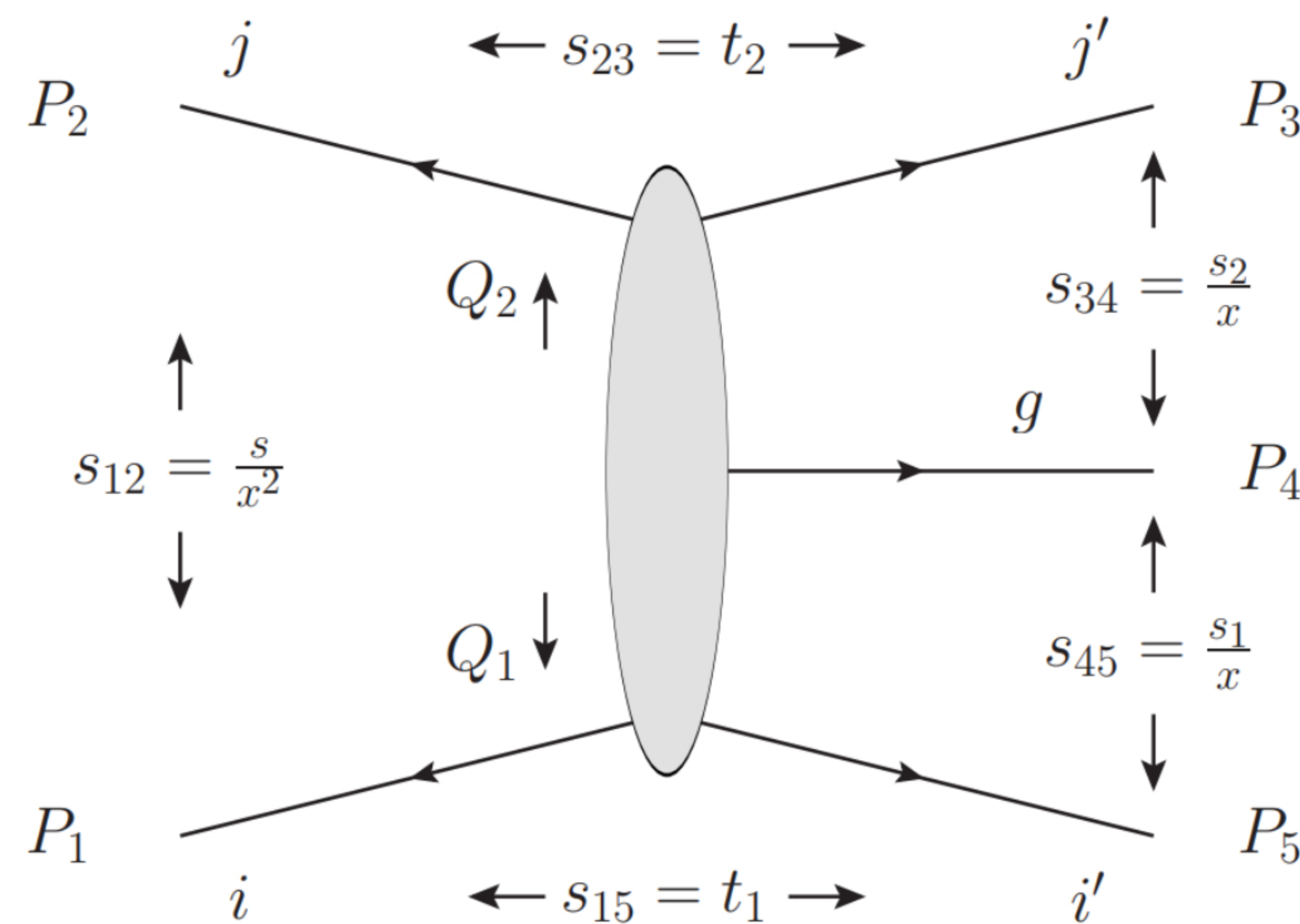


MINIMAL VARIABLES FOR (N)MRK

- The problem is most easily formulated in terms of lightcone momenta

$$\begin{aligned} p &= (p^+, \quad p^-, \quad p_\perp, \quad \bar{p}_\perp) \\ &= (E + p_z, \quad E - p_z, \quad p_x + ip_y, \quad p_x - ip_y) \end{aligned}$$

- We can picture the MRK limit as follows



$$p_i^j = \begin{pmatrix} 0 & \text{mc} & 0 & 0 \\ \text{mc} & 0 & 0 & 0 \\ p_4^+ X_{34} & \text{mc} & \text{mc} & \text{mc} \\ p_4^+ & \text{mc} & \frac{-q_1}{z-1} & \frac{-\bar{q}_1}{\bar{z}-1} \\ p_4^+ / X_{45} & \text{mc} & \frac{q_1 z}{z-1} & \frac{\bar{q}_1 \bar{z}}{\bar{z}-1} \end{pmatrix}$$

mc = fixed by momentum conservation

- The MRK limit is a two-variable problem $z, \bar{z}$;

$q_1, \bar{q}_1, p_4^+$ drop out by normalizing by the tree and $X_{34} \sim X_{45} \sim 1/x \gg 1$

- The NMRK limit is a five-variable problem $z, \bar{z}, w, \bar{w}, X = X_{(45)}$, other variables drop out



CHALLENGE FROM SPURIOUS CANCELLATIONS

- Amplitudes take the form:

$$\mathcal{A}_n^{(\ell)} = \sum_i c_i I_i$$

with c_i rational functions, I_i transcendental master integrals

- For $\mathcal{A}_5^{(2)}$ in the MRK limit we have:

$$c_i \approx \frac{c_{i,-1}}{x} + c_{i,0} + \mathcal{O}(x) , \quad I_i \approx I_{i,0} + x I_{i,1} + \mathcal{O}(x)$$

one spurious order in $x \rightarrow 0$ cancels between rational and transcendental.

- For $\mathcal{A}_6^{(1)}$ in the NMRK limit we have (for the NMHV amplitude):

$$c_i \approx \frac{\cancel{c_{i,-8}}^0}{x^{-8}} + \dots + \frac{\cancel{c_{i,-1}}^0}{x} + c_{i,0} + \mathcal{O}(x) , \quad I_i \approx I_{i,0} + \mathcal{O}(x)$$

Problem: **8 orders of spurious cancellations** in the (N)MRK parameter as $x \rightarrow 0$

CHALLENGE FROM SPURIOUS CANCELLATIONS (2)

- The $\mathcal{A}_5^{(2)}$ coefficients are simple

```
> from antares_results.jjjj.gggggg.mhv import lTerms; lTerms
< [Terms(''+(1<4|5)^2)/((1|2)<1|3)<2|3)'''), Terms(''+(1<4|5)^3)/((1|2)^2<3|4)<3|5)'''), ...]

> len(str(lTerms[0])), max(map(len, map(str, lTerms)))
< 28, 630
```

- The $\mathcal{A}_6^{(1)}$ NMHV coefficients are much more complex

```
> from antares_results.jjjj.gggggg.pmpmpm import coeffs; coeffs['box(1)']
< Terms(''+(-1/2j<1|2)^4[1|2][2|3]<3|1+2|5]^4)/((1|3)^4[4|5][5|6]<1|2+3|4)<3|1+2|6]s_123)''')

> len(str(coeffs['box(1)'])), max(map(len, map(str, coeffs.values())))
< 76, 346853
```

Some coefficients (three mass triangles, bubbles, rational part) are very complicated!

✗ Analytic expansion is a no go. Run out of memory and time after 3 or 4 orders!

✗ Numerical expansion with floating-point numbers is also too complicated.

Say we input $x \approx 10^{-10}$ to have 10 digits to work with, we would lose (at least) 80 digits!

p -ADIC NUMBERS

- You may be familiar with finite field (integers modulo a prime)

von Manteuffel, Schabinger '14; Peraro '16

$$a \in \mathbb{F}_p : a \in \{0, \dots, p-1\} \text{ with } \{+, -, \times, \div\}$$

Limits (and calculus) are not well defined in $\mathbb{F}_p$. We can make things zero, but not small:

$$|a|_0 = 0 \text{ if } a = 0 \text{ else } 1 \quad \text{a.k.a. the trivial absolute value.}$$

- There exists just one more absolute value on the rationals, the p -adic absolute value.

Ostrowski's theorem 1916

- Let's start from p -adic integers, instead of working modulo p , expand in powers of p

$$a \in \mathbb{Z}_p : a_0 p^0 + a_1 p^1 + a_2 p^2 + \dots + \mathcal{O}(p^n)$$

In some sense we are correcting the finite field result with more (subleading) information.

- p -adic numbers $\mathbb{Q}_p$ allow for negative powers of p , (would be division by zero in $\mathbb{F}_p$!)

$$a \in \mathbb{Q}_p : a_{-\nu} p^{-\nu} + \dots + a_0 + a_1 p^1 + \dots + \mathcal{O}(p^n)$$

GDL, Page '22

- The p -adic absolute value is defined as $|a|_p = p^\nu$.

Think of p as a small quantity, ϵ , even if it is a large prime (by the real absolute value, $|\cdot|_\infty$).



THE p -ADIC (N)MRK LIMIT

- The space of p -adic numbers is an **ultrametric** space, the triangle inequality is strengthened to:

$$d(x, z) \leq \max \{d(x, y), d(y, z)\}$$

This leads to better stability properties: adding two numbers can never result in a larger number!

- A general kinematic evaluation at a $(2^{31} - 1)$ -adic phase space point

```
> from lips import Particles; from singular import Field
> oPs = Particles(6, field=Field("padic", 2 ** 31 - 1, 9), seed=0) # create psp
> (1j * coeffs['bubble(1)'])(oPs) # evaluate the coefficient(s)

< 490010355 + 1085079429*2147483647 + 1676653899*2147483647^2 + 726358851*2147483647^3 + 1074867770*2147483
```

- Manipulate phase space: set the (N)MRK parameter controlling the rapidity gap to be $x \approx p$

$$0 \leq \text{leading NMRK behavior} \leq p - 1 + \mathcal{O}(2147483647^1)$$

- ✓ We still lose 1 digit per spurious pole (8 in total), but the result is now exact.



ANALYTIC RECONSTRUCTION

FUNDAMENTALS OF ANALYTIC RECONSTRUCTION

- Analytic reconstruction is a powerful alternative to symbolic manipulations:
 - ★ cancellations happen numerically, avoiding intermediate bottlenecks
 - ★ the cost is largely driven by the complexity of the final results
- We have a ring in 5 independent variables over a field $\mathbb{F}(= \mathbb{Q}_p)$

$$R_{NRMK} = \mathbb{F}[z, \bar{z}, w, \bar{w}, X(= X_{45})]$$

we need to recover rational functions from numerical samples:

$$\{z, \bar{z}, w, \bar{w}, X\} \in \mathbb{F}^5 \rightarrow \text{BlackBox} \rightarrow c_i \in \mathbb{F} \rightarrow c_i = \frac{\mathcal{N}(z, \bar{z}, w, \bar{w}, X)}{\mathcal{D}(z, \bar{z}, w, \bar{w}, X)}$$

The real power of the approach is with polynomial quotient rings.

- The complexity is not driven just by the number of variables, but also by the singularities

$$\begin{aligned} \mathcal{D}_{\Delta_{3m}} = & -4(-1 + w)w(-1 + \bar{w})\bar{w}X^2(-1 + z)z(-1 + \bar{z})\bar{z} + \\ & (Xz(\bar{w} + \bar{z} - \bar{w}\bar{z} + X\bar{z}) + w(\bar{w} - X(-1 + z)\bar{z} + \bar{w}X(1 - \bar{z} + z(-1 + 2\bar{z}))))^2 \end{aligned}$$

alone has degree 10. It appears up to cubic pole, making denominators exceed degree 30.

By comparison the most complicated singularity for $\mathcal{A}_5^{(2)}$ was $(z - \bar{z})$

LEAST COMMON DENOMINATOR

(i.e. geometry at codimension one)

- We can determine the least common denominators (LCDs),

$$\mathcal{D} = \prod_j \mathcal{D}_j^{q_{ij}}(z, \bar{z}, w, \bar{w}, X).$$

from a univariate slice $\vec{\lambda}(t)$ and guesses for the possible $\mathcal{D}_j$.

- The curve $\vec{\lambda}(t)$ must intersect all varieties $V(\langle \mathcal{D}_j \rangle)$, e.g.

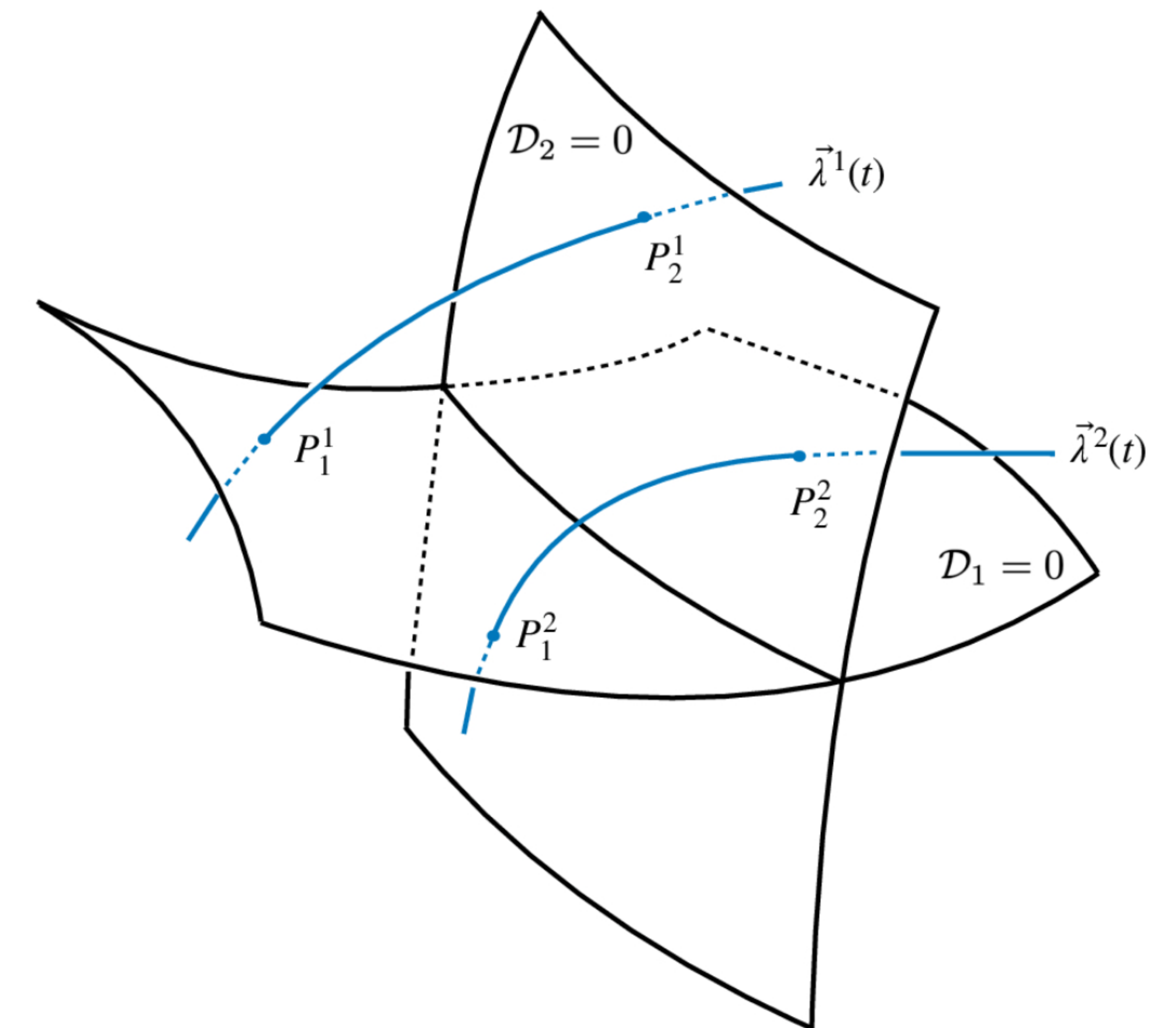
$$z \rightarrow z + c_z t, \quad \bar{z} \rightarrow \bar{z} + c_{\bar{z}} t,$$

$$w \rightarrow w + c_{\bar{w}} t, \quad \bar{w} \rightarrow \bar{w} + c_{\bar{w}} t, \quad X \rightarrow X + c_X t$$

Thiele interpolation yields $\mathcal{D}(t)$, do univariate factorization and match to factors from multivariate guesses.

- Open-source implementation in [ANTARES](#), [LIPS](#), [SYNGULAR](#)

`Ring.univariate_slice` and `num_func.get_lcd`



Space has dimension 5,
 $\mathcal{D}_j = 0$ have dimension 4,
 $\vec{\lambda}(t)$'s have dimension 1.

Poles & Zeros	$\Leftrightarrow$	Irreducible Varieties	$\Leftrightarrow$	Prime Ideals
<i>Physics</i>		<i>Geometry</i>		<i>Algebra</i>

SUMMARY & OUTLOOK

TOWARDS THE NMHV 2-EMISSION CEV

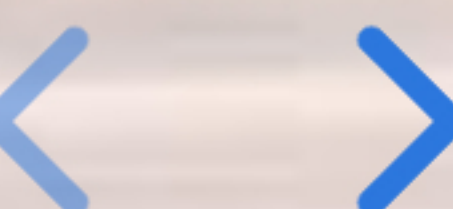
- Much more can be said on reconstruction, in brief:
 - ★ the LCD form of the coefficients is too complex (would require millions of evaluations to fit);
 - ★ simplifications arise from partial fraction decompositions and **computational algebraic geometry**.
- Status:
 - ★ all amplitude coefficients have been reconstructed in the NMRK limit;
 - ★ after reconstruction, no more spurious cancellations in the NMRK parameter.
- Checks:
 - ★ The MRK limit ($X_{45} \rightarrow \text{large}$) reproduces known results;
 - ★ We obtain the same result from $g^+ g^- g^+ g^- g^+ g^-$ and $g^+ g^+ g^- g^+ g^- g^-$ (distinct in general kinematics);
 - ★ Reproduce known $\mathcal{N} = 4$ and $\mathcal{N} = 1$ SUSY results.
- To do:
 - ★ Split result into contributions to trajectory, impact factors (known) and identify the (new) vertex.
- Outlook:
 - ★ The proposed method provides a scalable solution to more complex processes, this calculation was performed entirely on a laptop.



THANK YOU FOR YOUR ATTENTION!

These slides are powered by:

[markdown](#), [html](#), [revealjs](#), [hugo](#), [mathjax](#), [github](#)



BACKUP SLIDES



MULTIVARIATE PARTIAL FRACTIONS

GDL, Maître ('19)
GDL, Page ('22)

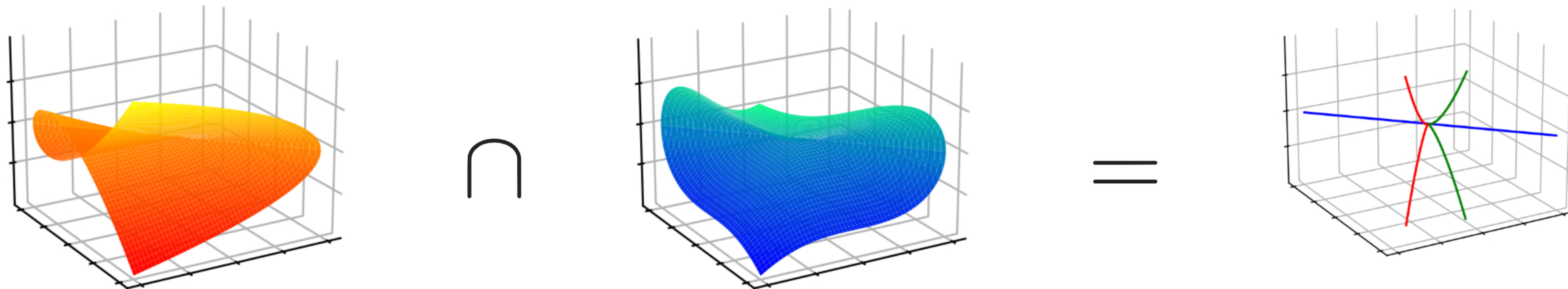
- We want to determine whether a partial fraction decomposition is possible

$$\frac{\mathcal{N}}{\mathcal{D}_1 \mathcal{D}_2} \stackrel{?}{=} \frac{\mathcal{N}_2}{\mathcal{D}_1} + \frac{\mathcal{N}_1}{\mathcal{D}_2}$$

without knowing $\mathcal{N}$ analytically. The complexity should not depend on $\mathcal{N}$ (besides numerical evaluations).
The complexity will depend on the irreducible polynomials $\mathcal{D}_1, \mathcal{D}_2$.

- Multivariate partial fraction decompositions follow from varieties where pairs of denominator factors vanish

$$\frac{\mathcal{N}}{\mathcal{D}_1 \mathcal{D}_2} \stackrel{?}{=} \frac{\mathcal{N}_2}{\mathcal{D}_1} + \frac{\mathcal{N}_1}{\mathcal{D}_2} \iff \mathcal{N} \in \langle \mathcal{D}_1, \mathcal{D}_2 \rangle \text{ i.e. } \mathcal{N} \stackrel{?}{=} \mathcal{N}_1 \mathcal{D}_1 + \mathcal{N}_2 \mathcal{D}_2$$



$$\langle xy^2 + y^3 - z^2 \rangle + \langle x^3 + y^3 - z^2 \rangle = \langle xy^2 + y^3 - z^2, x^3 + y^3 - z^2 \rangle = \langle 2y^3 - z^2, x - y \rangle \cap \langle y^3 - z^2, x \rangle \cap \langle z^2, x + y \rangle$$

This is a primary decomposition, it is the equivalent for polynomials of say: $12 = 2^2 \times 3$

If $\mathcal{N}$ vanishes on all branches, then the partial fraction decomposition exists.

ITERATED POLE SUBTRACTION

- After we determine valid partial fraction decompositions, determine a numerator at a time, e.g.

$$c_i = \frac{\mathcal{N}_2}{\mathcal{D}_1} + \frac{\mathcal{N}_1}{\mathcal{D}_2}$$

Isolate $\mathcal{N}_2$ by taking points in the limit $\mathcal{D}_1 \rightarrow 0$.

- To do this, we need to nest p -adic limits:
 - ★ set $x \propto p^5$, get 5 digits for the leading NMRK behaviour
 - ★ set $\mathcal{D}_1 \propto p$, as long as its pole degree is less than 5, get a value for the residue.
- Example of explicit construction with [syngular](#) (on GitHub), a Python extension to [Singular](#)

```
> from syngular import Field, Ring, Ideal, RingPoint
> ring = Ring('0', ('z', 'zb', 'w', 'wb', 'X'), 'dp')
> I = Ideal(ring, [ '(-4*(-1+w)*w*(-1+wb)*wb*X**2*(-1+z)*z*(-1+zb)*zb+(X*z*(wb+zb-wb*zb+X*zb)+w*(w
> I.squash() # just expand the polynomial in this case
> point = RingPoint(ring, field=Field("padic", 2 ** 31 - 1, 9)) # a dictionary {'z': number, ...
> point.singular_variety(I, valuations=(1, ), seed=0) # push the point on the surface
> point(I.generators[0])
< 26429729*2147483647 + ... + 0(2147483647^9)
```