Perturbative RGE systematics in precision observables

giuseppe bozzi
University and INFN, Cagliari

with Valerio Bertone and Francesco Hautmann

Phys. Rev. D 105, 096003 (2022)
Phys. Rev. D 111, 074005 (2025)

INFN
Istituto Nazionale di Fisica Nucleare
Sezione di Cagliari

RGEs in QCD

- Generic RGE in QCD:

$$\frac{d\ln R}{d\ln\mu} = \Gamma(\alpha_s(\mu))$$

$$\Gamma(\alpha_s(\mu)) = \sum_{n=0}^{k} \Gamma_n \alpha_s^{n+1}(\mu)$$

$R = (\alpha_s, PDF, TMD)$

Γ = appropriate anomalous dimension

k = highest-order known in Γ-expansion

- Can be solved either analytically or numerically
- Both equally good from the point of view of perturbative accuracy
- **Goal**: devise a strategy to estimate truncation uncertainty in both cases

Analytic solutions

- Well known methodology borrowed from threshold and q_T resummation

Catani, Mangano, Nason, Trentadue NPB 478 (1996)
Catani de Florian Grazzini Nason JHEP 07 (2003)
GB Catani de Florian Grazzini NPB 737 (2006)

- Express $R(\mu)$ in closed analytic form in terms of $R(\mu_0)$

➡ Perturbative series containing terms proportional to $L = \alpha_s(\mu_0)\ln(\mu/u_0)$

- Decompose $L = L_\kappa - \alpha_s(\mu_0)\ln\kappa$ with $L_\kappa = \alpha_s(\mu_0)\ln(\kappa\mu/\mu_0)$

$\kappa\mu$ resummation scale

- Variations of κ in the vicinity of 1 generate sub-leading terms

➡ Estimate of truncation uncertainty

Numerical solutions

- A "shifted kernel" approach:
  - introduce arbitrary sub-leading terms in Γ and solve RGE numerically
  - differences among numerical solutions obtained with different sub-leading terms —> truncation uncertainty
- More specifically:
  - consider a NLO kernel $\Gamma = \alpha_s(\mu)\Gamma_0 + \alpha_s^2(\mu)\Gamma_1$
  - make a RG transformation $\mu \to \xi\mu$ $\longrightarrow$

$\xi\mu$ resummation scale
(one-to-one correspondence between κ and ξ)

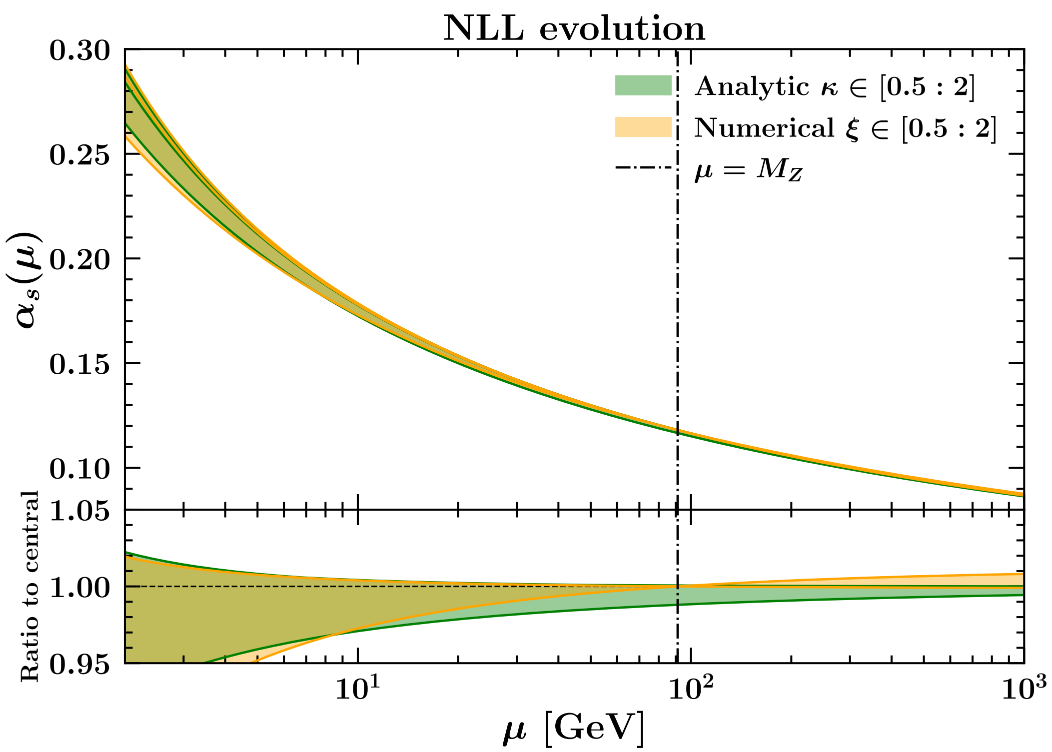
  - express $\alpha_s(\mu) = \alpha_s(\xi\mu) - \alpha_s^2(\xi\mu)\beta_0 \ln\xi + \mathcal{O}(\alpha_s^3)$

➡ $\overline{\Gamma} = \alpha_s(\xi\mu)\Gamma_0 + \alpha_s^2(\xi\mu)\left[\Gamma_1 - \Gamma_0\beta_0 \ln\xi\right] + \mathcal{O}(\alpha_s^3)$

  - solving RGE using Γ or $\overline{\Gamma}$ is equivalent from a perturbative perspective

➡Estimate of truncation uncertainty

Strong coupling



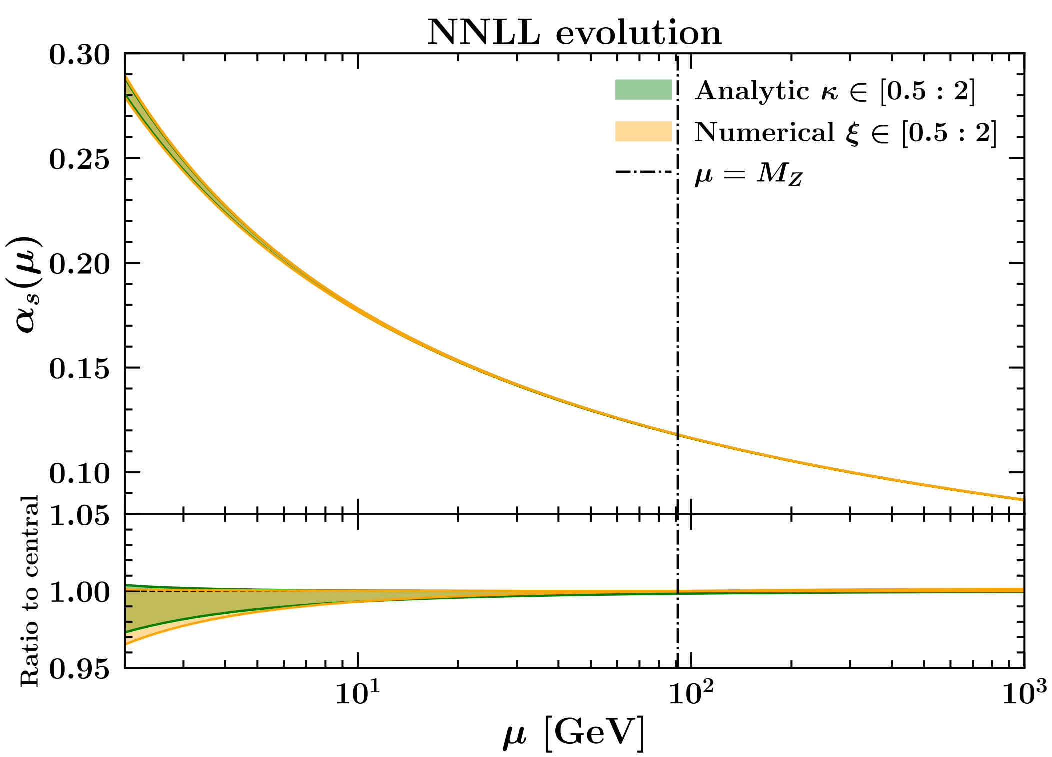
RGE $$\frac{d\ln\alpha_s}{d\ln\mu} = \beta(\alpha_s(\mu)) = \sum_{n=0}^{k}\beta_n\alpha_s^{n+1}(\mu)$$

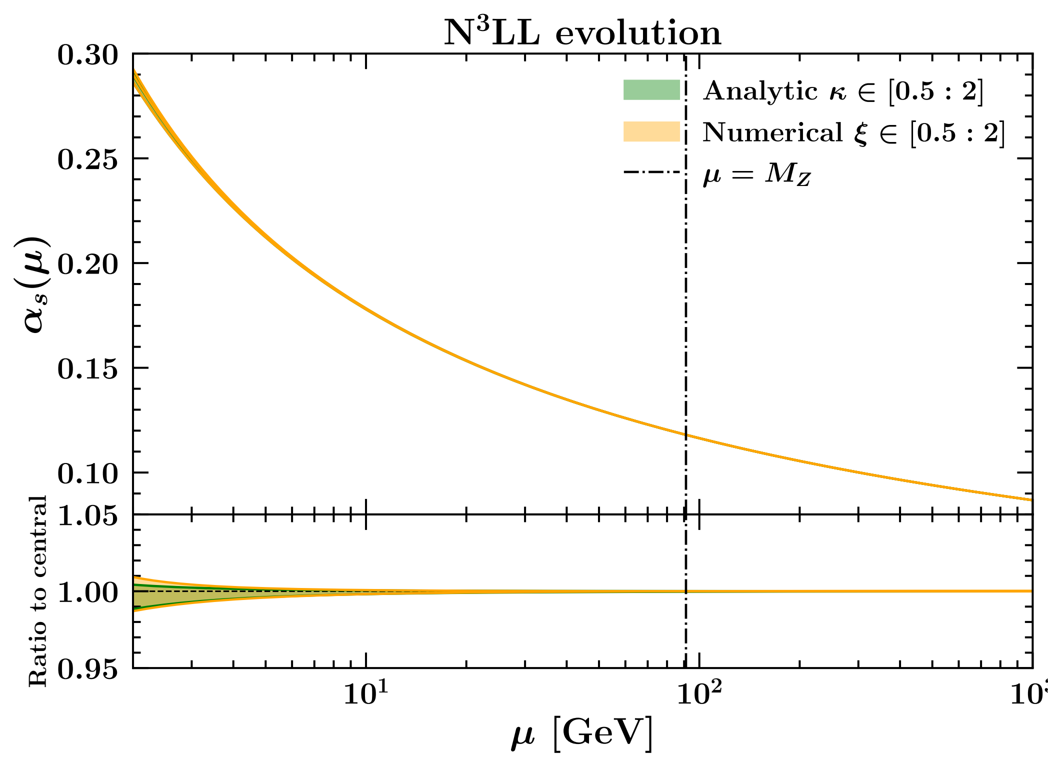
Analytic

$$\alpha_s^{\mathrm{N}^k\mathrm{LL}}(\mu) = a_s(\mu_0)\sum_{l=0}^{k} a_s^l(\mu_0)\, g_{l+1}^{(\beta)}(\lambda,\kappa)$$

Numerical (NLO)

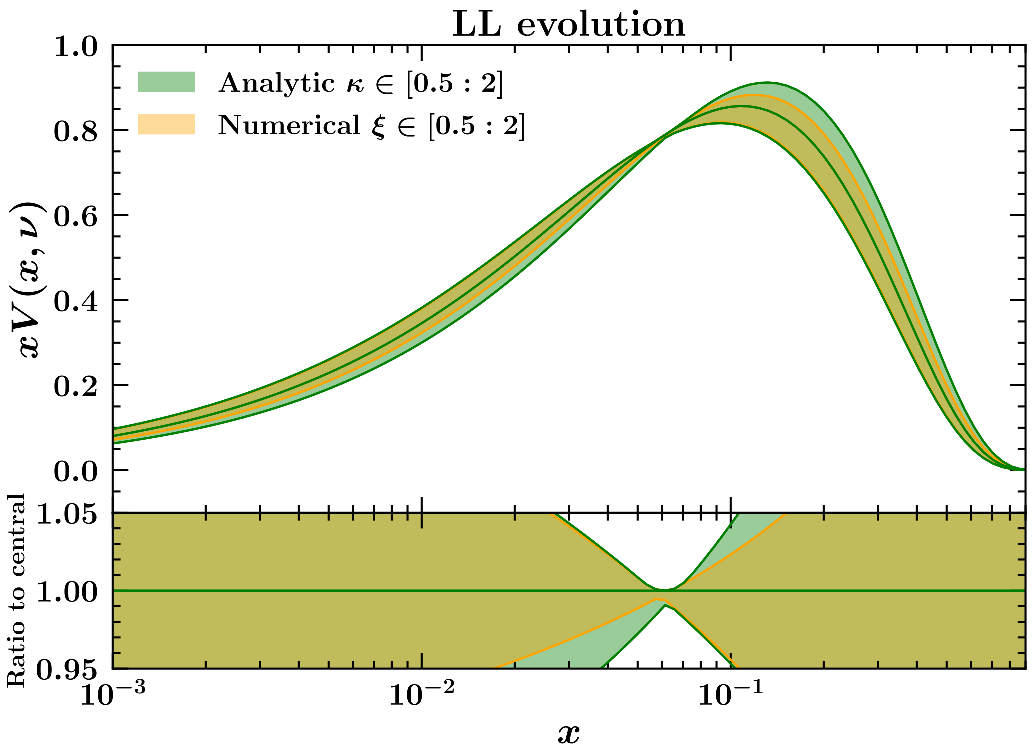
$$\beta(\alpha_s(\mu)) = \alpha_s(\xi\mu)\beta_0[1 + a_s(\xi\mu)(\frac{\beta_1}{\beta_0} - 2\beta_0\ln\xi)] + \mathcal{O}(\alpha_s^3)$$





- Uncertainties of analytic/numerical solutions of the same order
- Good perturbative convergence

PDFs



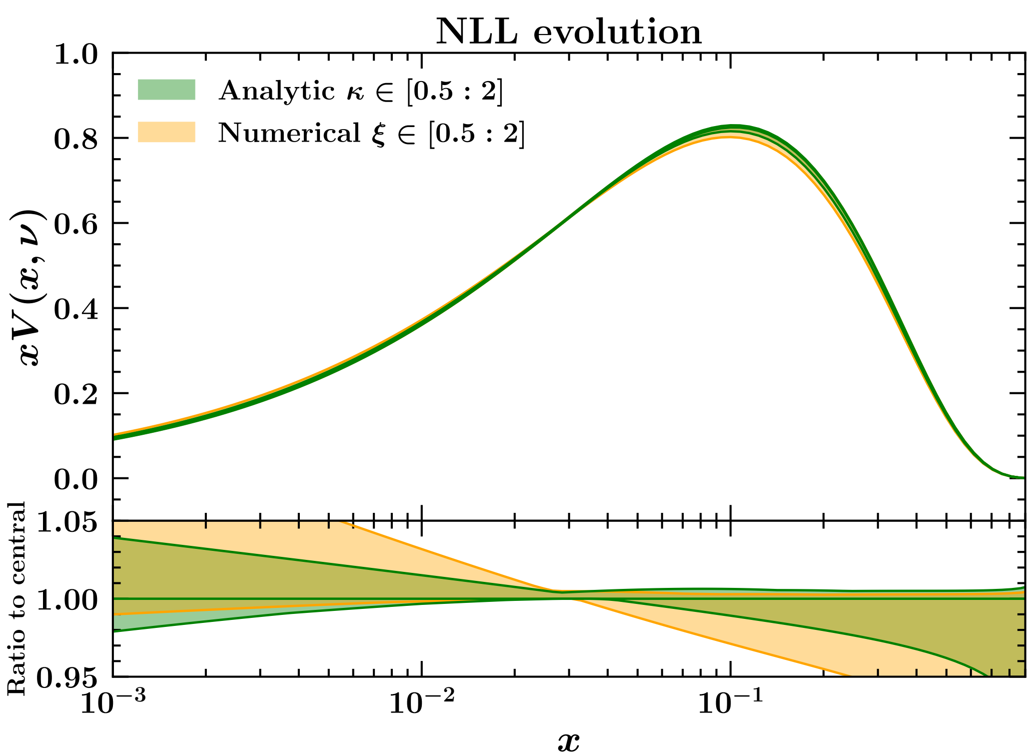
RGE $\dfrac{d\ln f}{d\ln\mu} = \gamma(\alpha_s(\mu)) = \displaystyle\sum_{n=0}^{k} \gamma_n \alpha_s^{n+1}(\mu)$

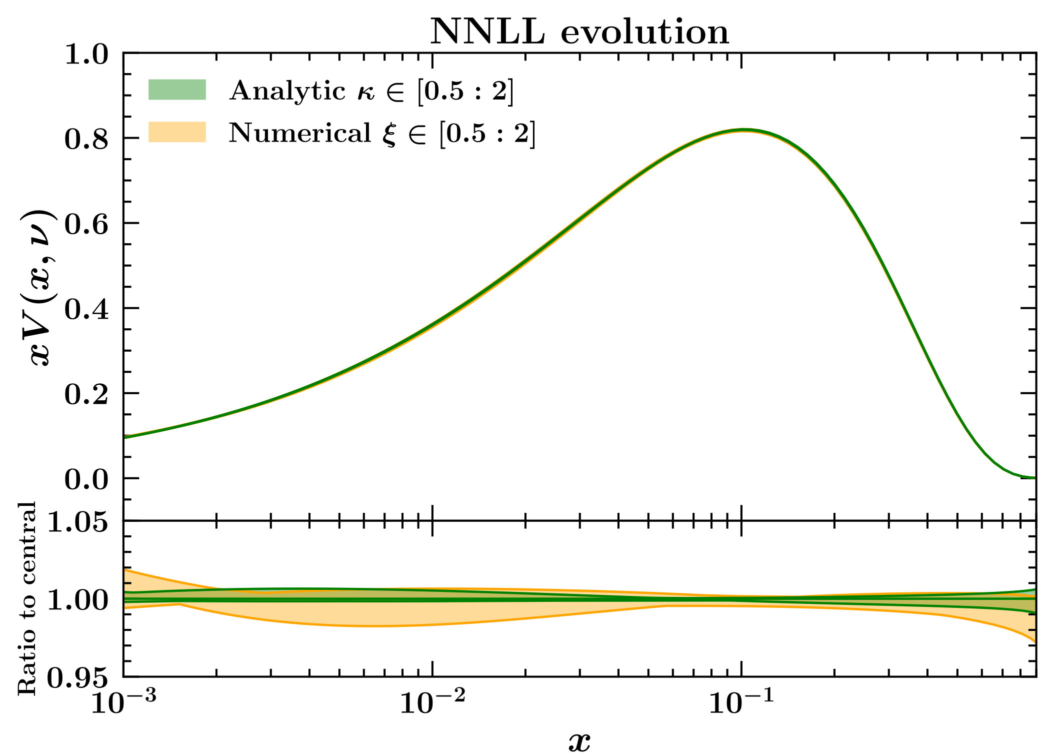
Analytic

$$f^{\mathrm{N}^k\mathrm{LL}}(\mu) = g_0^{(\gamma),\mathrm{N}^k\mathrm{LL}}(\lambda,\kappa)\ \exp\left[\sum_{l=0}^{k} \alpha_s^l(\mu_0) g_{l+1}^{(\gamma)}(\lambda,\kappa)\right] f(\mu_0)$$

Numerical (NLO)

$$\gamma(\alpha_s(\mu)) = \alpha_s(\xi\mu)\gamma_0 + \alpha_s^2(\xi\mu)\left[\gamma_1 - \beta_0\gamma_0 \ln\xi\right] + \mathcal{O}(\alpha_s^3)$$





- Total valence distribution evolved from $\mu = 2$ GeV to $\mu = 100$ GeV
- Similar behavior for other PDFs and same considerations as before

Structure functions

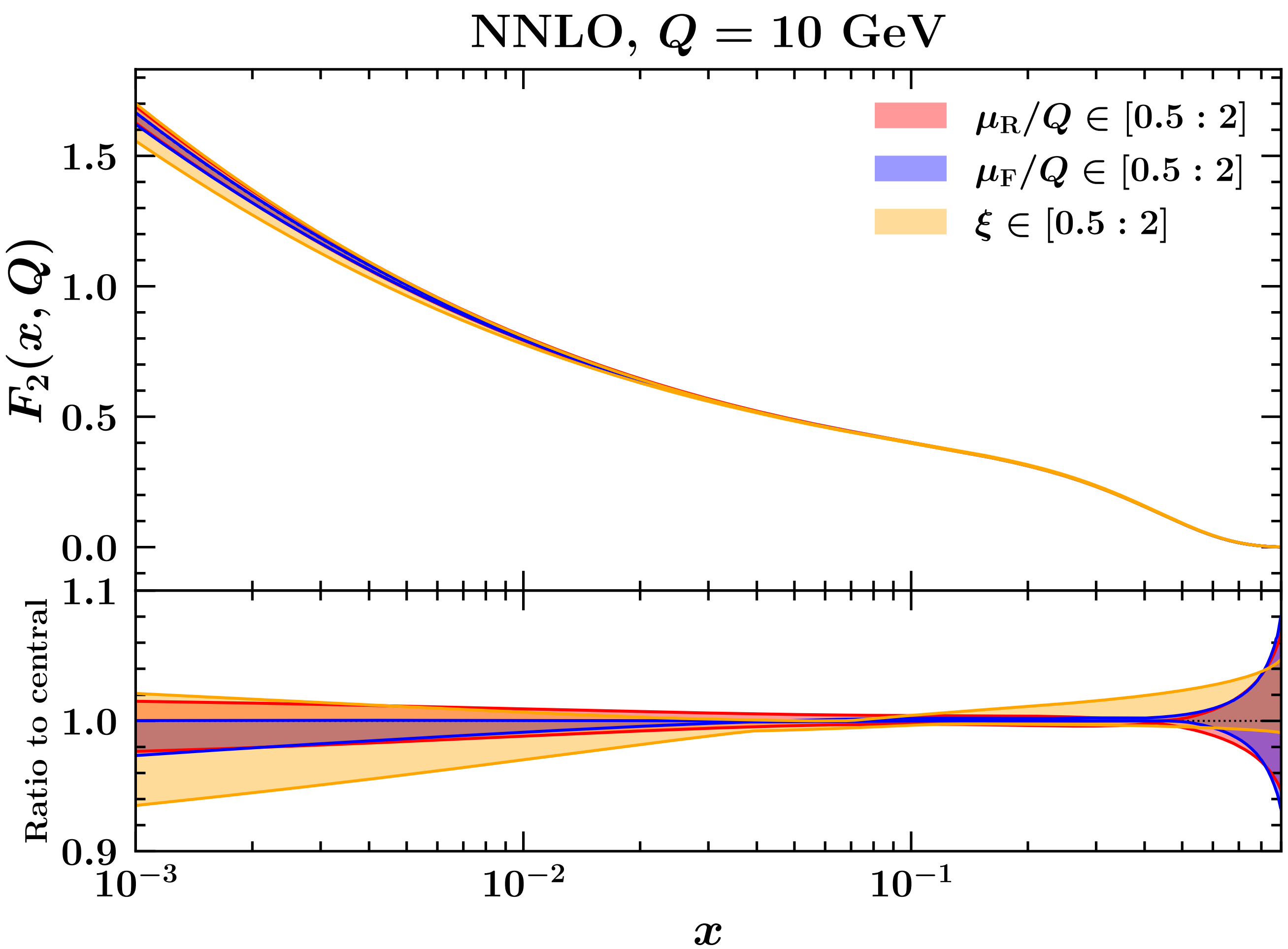
- Assess the impact of $\kappa(\xi)$ variation on physical observables relevant to PDF fits
- Structure function $\mathcal{F} = F_2, F_L, xF_3$ in unpolarized inclusive DIS

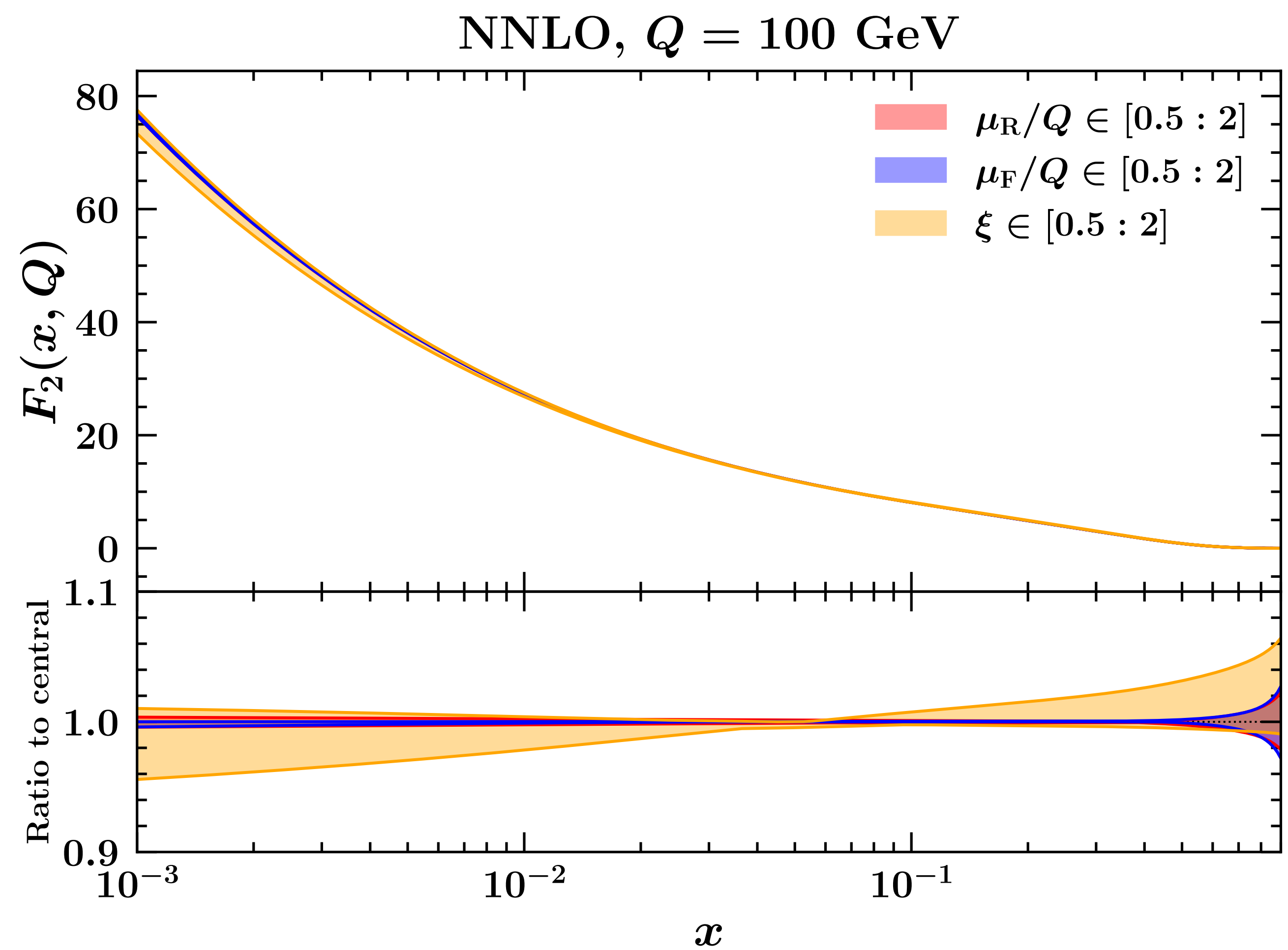
➡ $\mathcal{F}(Q) = C_{\mathcal{F}}\left(\alpha_s(\mu_R;\xi), \mu_R/Q, \mu_F/Q\right) \otimes f(\mu_F;\xi)$

➡(Q virtuality of vector boson, all non-scale dependencies omitted)

- N^kLO partonic $C_{\mathcal{F}}$ consistently combined with N^kLL evolution of α_s and PDFs
- Variations of (μ_R, μ_F, ξ) around $(Q, Q, 1)$ by factor of 2
    - μ_R and μ_F variations give an estimate of corrections to $C_{\mathcal{F}}$
    - ξ variations give an estimate of the uncertainty due to α_s and PDF evolution

F_2

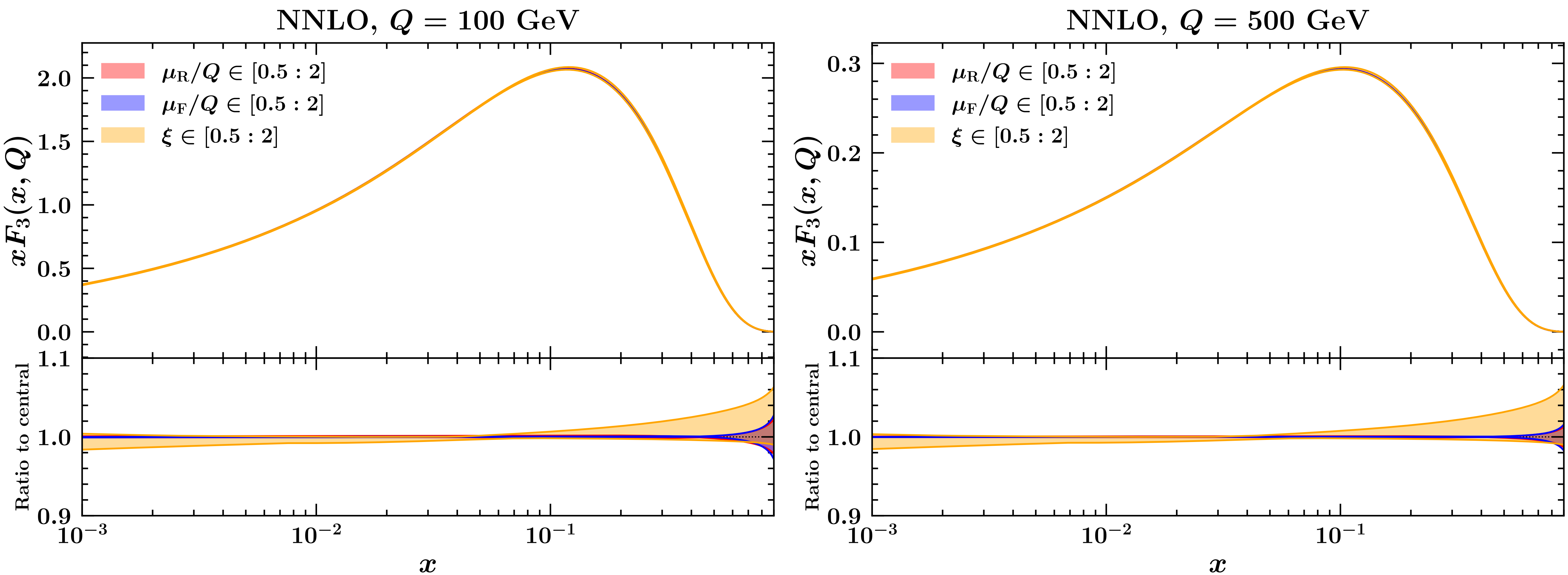




- Input for evolution: $\alpha_s(M_Z) = 0.118$ and MSHT20 at $Q_0 = 2$ GeV
- VFNS with $(m_c, m_b) = (1.4, 4.75)$ GeV
- $Q = 10$ GeV (γ-dominated) and $Q = 100$ GeV (Z-dominated)

- Sizable ξ bands, generally larger than μ_R, μ_F bands
- μ_R, μ_F bands shrink with increasing Q, ξ bands approximately same size

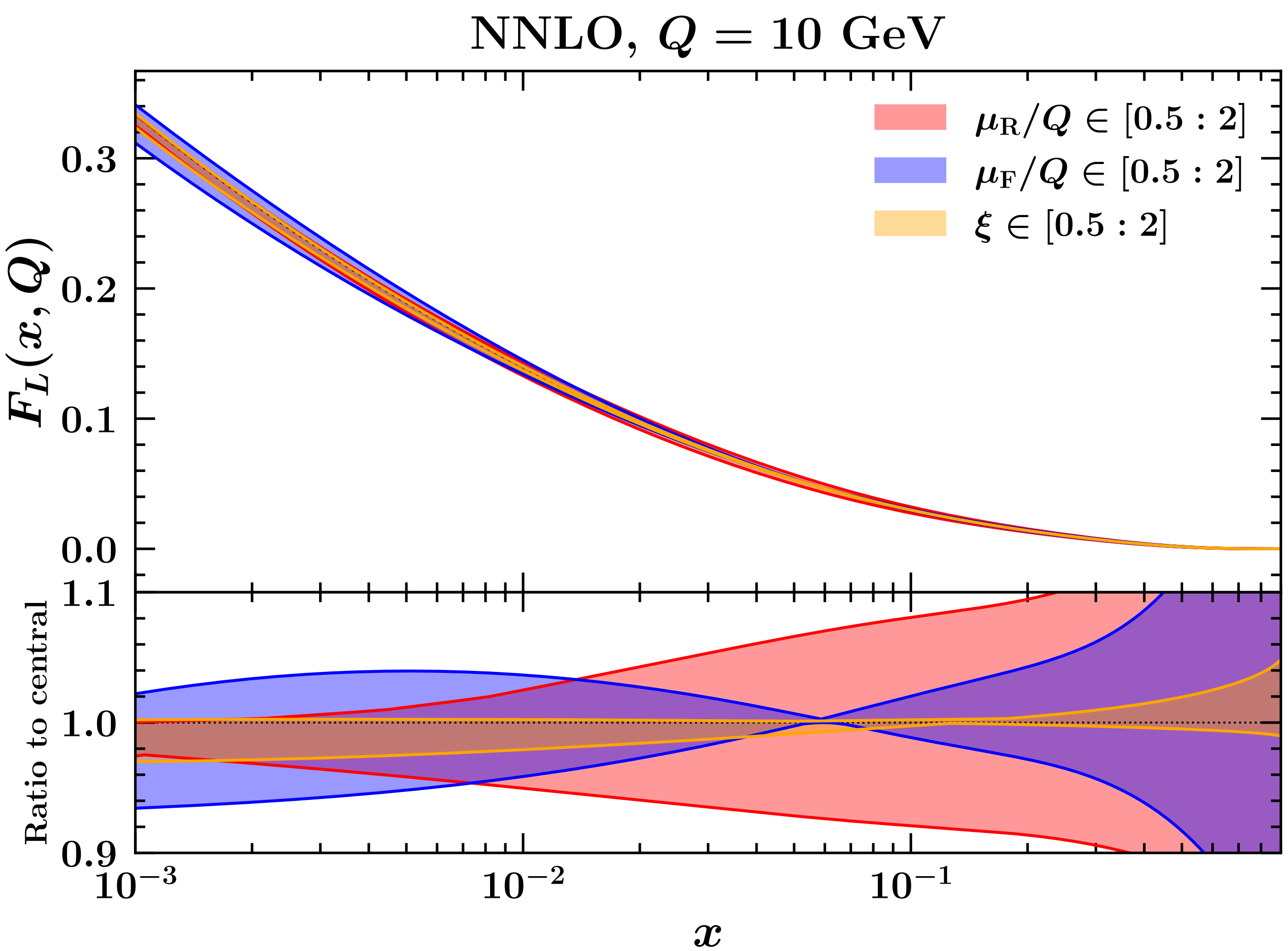
xF_3

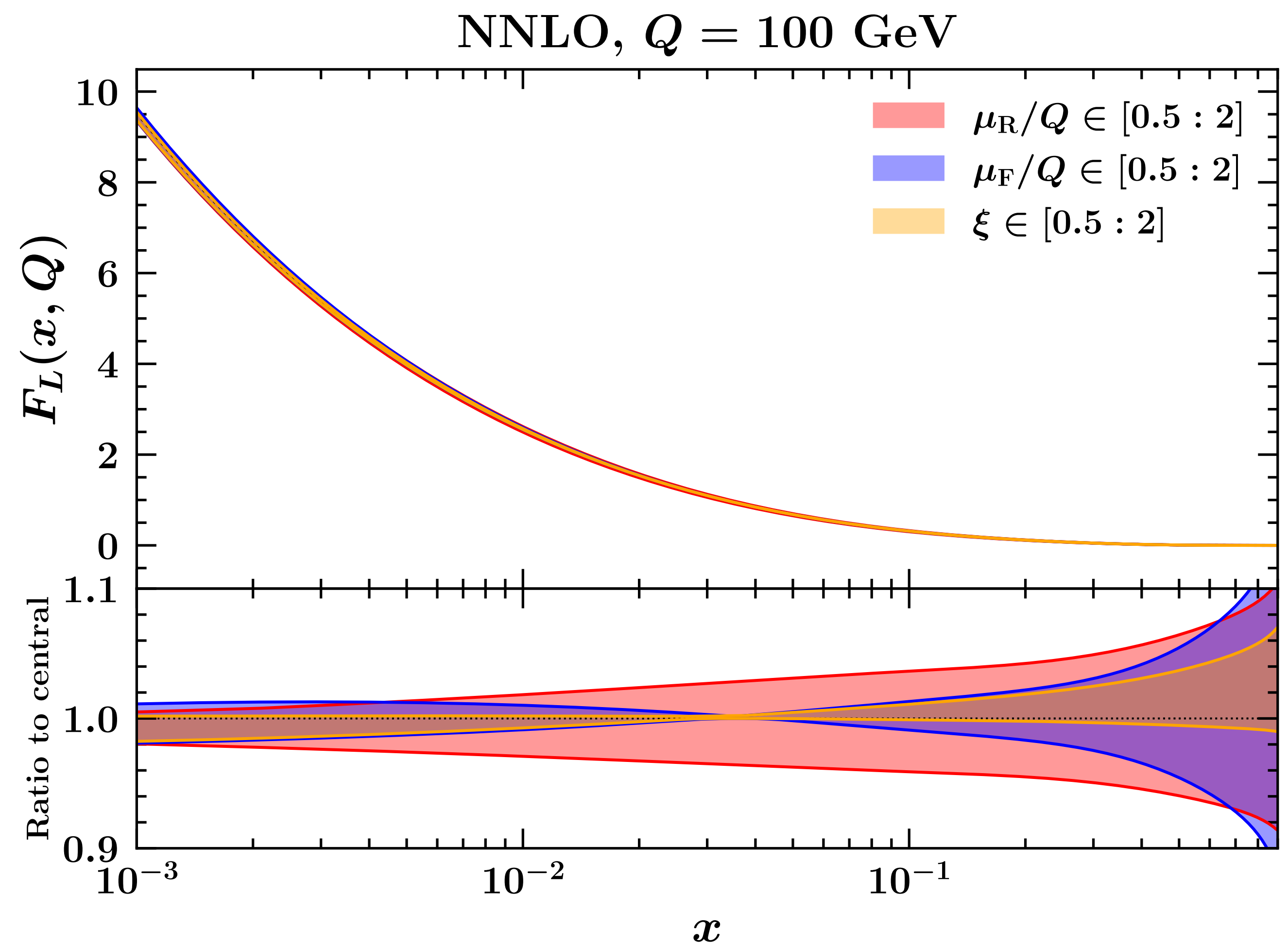


- Input for evolution: $\alpha_s(M_Z) = 0.118$ and MSHT20 at $Q_0 = 2$ GeV
- VFNS with $(m_c, m_b) = (1.4, 4.75)$ GeV
- $Q = 100, 500$ GeV (xF_3 suppressed by Z propagator below M_Z)

- Sizable ξ bands, generally larger than μ_R, μ_F bands
- μ_R, μ_F bands shrink with increasing Q, ξ bands approximately same size

F_L





- Input for evolution: $\alpha_s(M_Z) = 0.118$ and MSHT20 at $Q_0 = 2$ GeV
- VFNS with $(m_c, m_b) = (1.4, 4.75)$ GeV
- $Q = 10$ GeV (γ-dominated) and $Q = 100$ GeV (Z-dominated)

- Starts at $\mathcal{O}(\alpha_s)$: μ_R, μ_F bands generally larger than ξ bands
- μ_R, μ_F bands shrink with increasing Q, ξ bands approximately same size

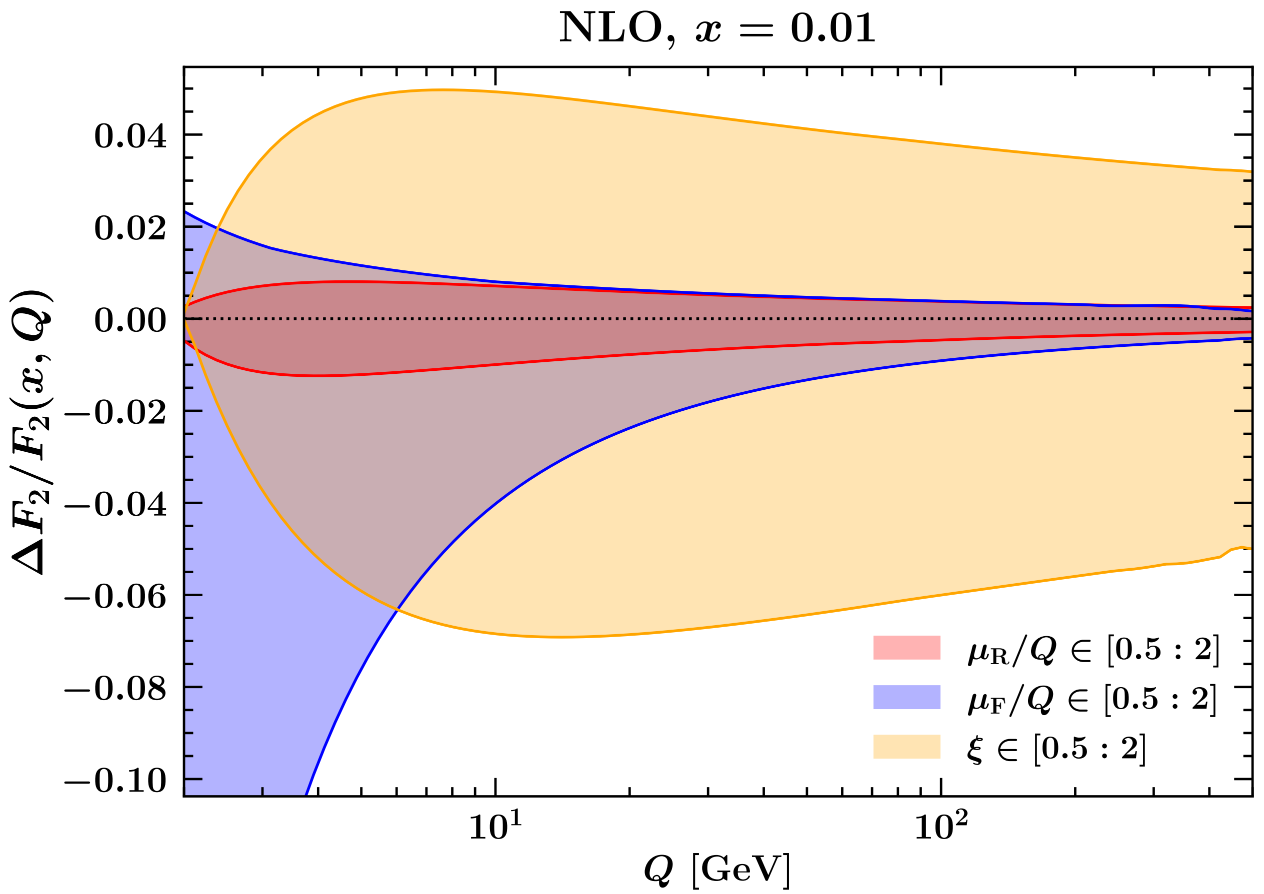
ξ vs. μ_R, μ_F

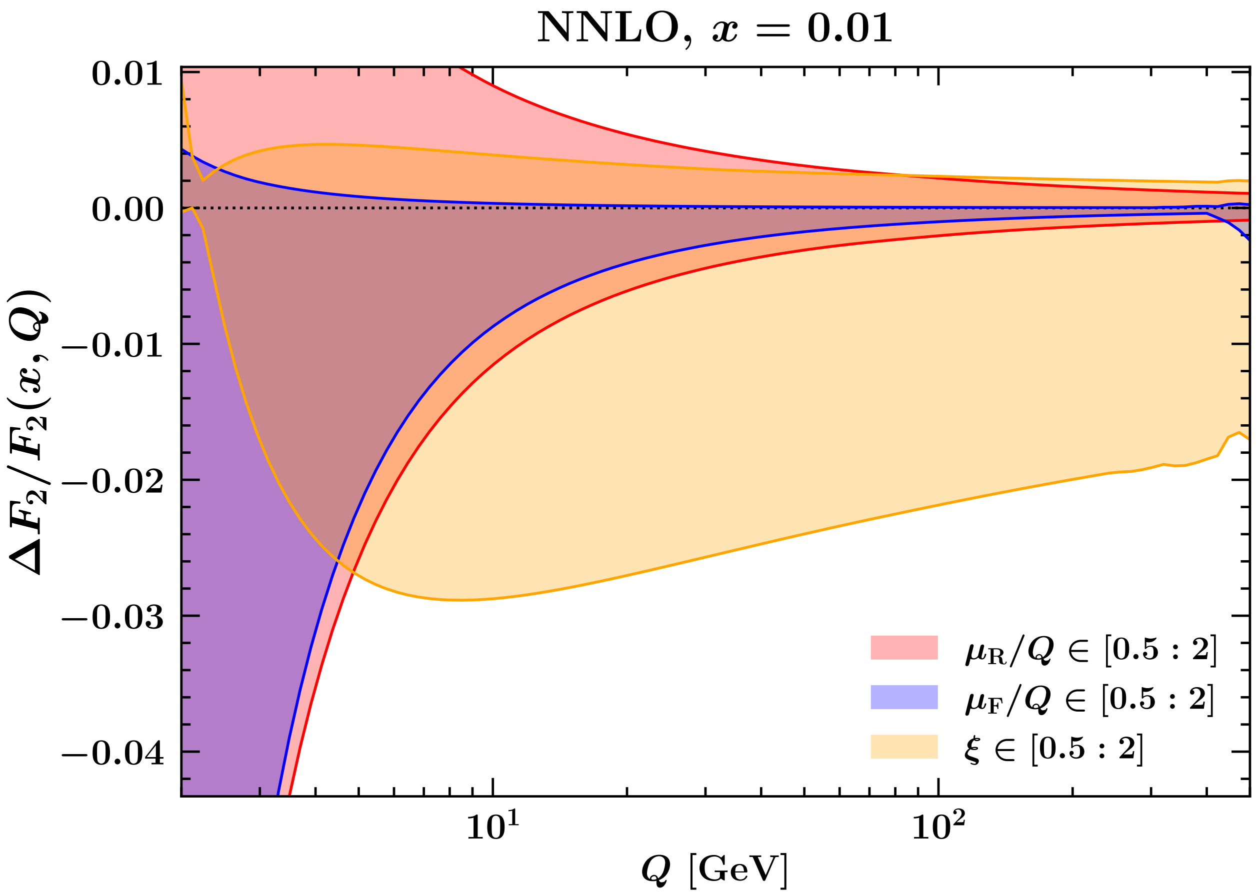
- Consider $f(\mu,\xi)$, a PDF evolved from μ_0 to μ at NLL with a choice of ξ

$$\Delta f(\mu,\xi) = f(\mu,\xi) - f(\mu,1) = \left(\alpha_s^2(\mu) - \alpha_s^2(\mu_0)\right) \ln\xi \left(\gamma_1 + \frac{\beta_1}{2\beta_0}\gamma_0 - \frac{1}{2}\beta_0\gamma_0 \ln\xi\right) f(\mu) + \mathcal{O}(\alpha_s^3)$$

- Very distinctive behaviour:
    - Δf is $\mathcal{O}(\alpha_s^2)$, *i.e.* sub-leading at NLL accuracy.
    - Proportional to $\ln\xi$, thus approaches zero as $\xi \to 1$ as expected
    - Proportional to $\alpha_s^2(\mu) - \alpha_s^2(\mu_0)$
        - For $\mu \sim \mu_0$, Δf small <u>no matter how small is μ</u> (how large is $\alpha_s(\mu)$)
        - For $\mu \gg \mu_0$, Δf dominated by $a_s(\mu_0)$, <u>no matter how large is μ</u>
- General scaling at N^{k}LL: $\alpha_s^{k+1}(\mu) - \alpha_s^{k+1}(\mu_0)$
- ξ uncertainties scale **differently** from $\mu_{R,F}$ uncertainties (that scale as $\alpha_s^{k+1}(\mu_{R,F})$)
"cumulative" uncertainty due to $\mu_0 \to \mu$ evolution

Uncertainty scaling in F_2





- μ_R and μ_F uncertainties large at small Q, decrease with growing Q
- ξ uncertainty grows and remain sizable at large Q
- Similar effect for other structure functions
- Impact expected in **PDF fits —>** **Francesco's talk**

TMDs

- TMDs (in b_T space) obey ***two*** evolution equations:

$$\frac{\partial \ln F}{\partial \ln \sqrt{\zeta}} = K(\mu_0) - \int_{\mu_i}^{\mu_f} \frac{d\mu'}{\mu'} \gamma_K(\alpha_s(\mu')) \qquad \frac{\partial \ln F}{\partial \ln \mu} = \gamma_F(\alpha_s(\mu)) - \gamma_K(\alpha_s(\mu)) \ln \frac{\sqrt{\zeta}}{\mu}$$

- The solution is:

$$F(\mu_f, \zeta_f) = e^{S(\mu_f, \zeta_f; \mu_i, \zeta_i)} F(\mu_i, \zeta_i)$$

$$S(\mu_f, \zeta_f; \mu_i, \zeta_i) = K(\mu_i) \ln \frac{\sqrt{\zeta_f}}{\sqrt{\zeta_i}} + \int_{\mu_i}^{\mu_f} \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu')) \ln \frac{\sqrt{\zeta_f}}{\mu'} \right]$$

- Boundary conditions for evolution:

$$\mu_i = \sqrt{\zeta_i} = \mu_b = \frac{2e^{-\gamma_E}}{b_T}$$

$$\mu_f = \sqrt{\zeta_f} = M$$

- Nullifies scale logs
- M invariant mass of the partonic system
- Exact choice of ζ is immaterial ($\zeta_1 \zeta_2 = M^4$)
- μ_f variations accounted for by κ variations

TMDs

$$F(M, M^2) = e^{S(M,\mu_b)} F(\mu_b, \mu_b^2)$$

$$S(M, \mu_b) = K(\mu_b) \ln \frac{M}{\mu_b} + \int_{\mu_b}^{M} \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu')) \ln \frac{M}{\mu'} \right]$$

- A perturbative expansion of the kernels:

$$K(a_s(\mu)) = \sum_{n=0}^{\infty} a_s^{n+1}(\mu) K^{(n)} \quad \gamma_F(a_s(\mu)) = \sum_{n=0}^{\infty} a_s^{n+1}(\mu) \gamma_F^{(n)} \quad \gamma_K(a_s(\mu)) = \sum_{n=0}^{\infty} a_s^{n+1}(\mu) \gamma_K^{(n)}$$

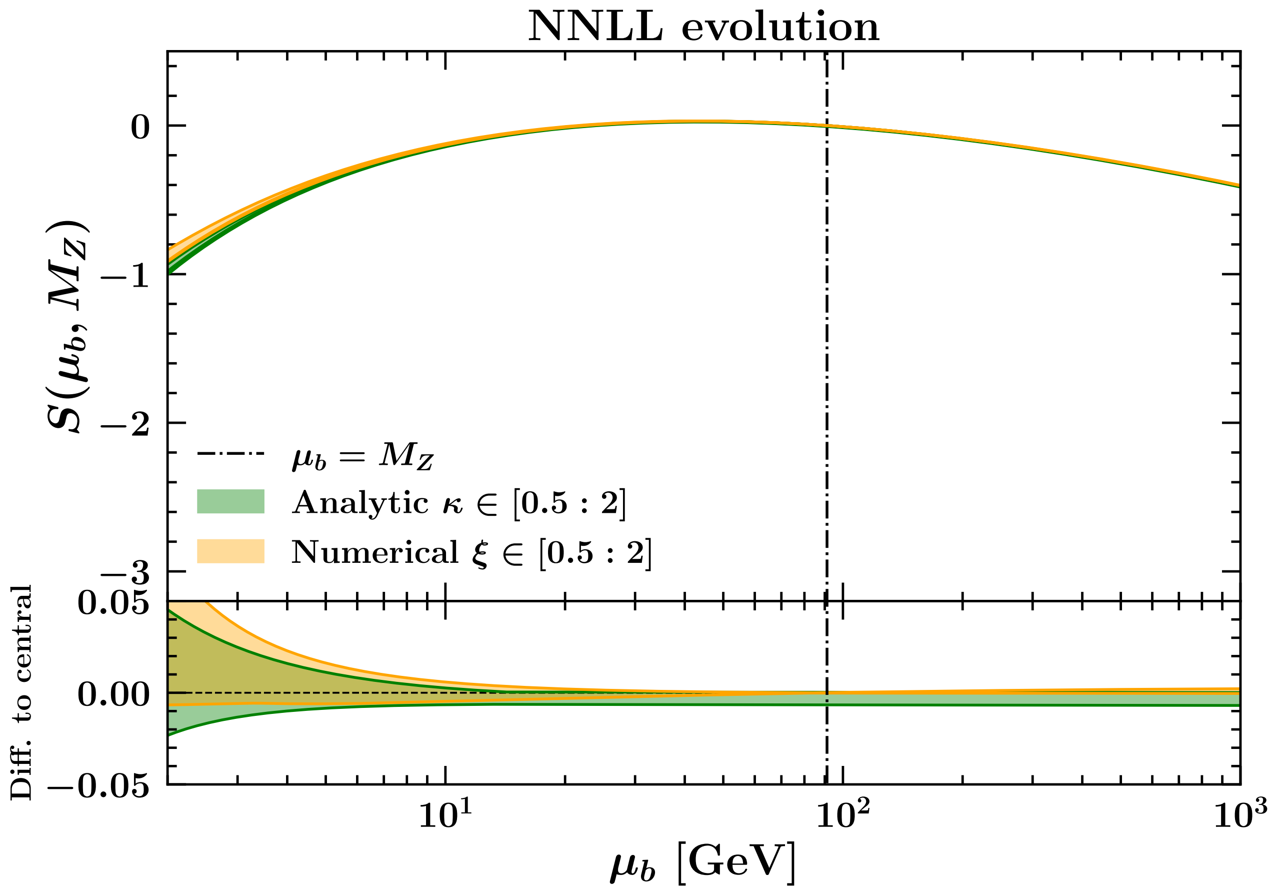
allows to compute S **numerically** or **analytically** (using analytic solutions of α_s):

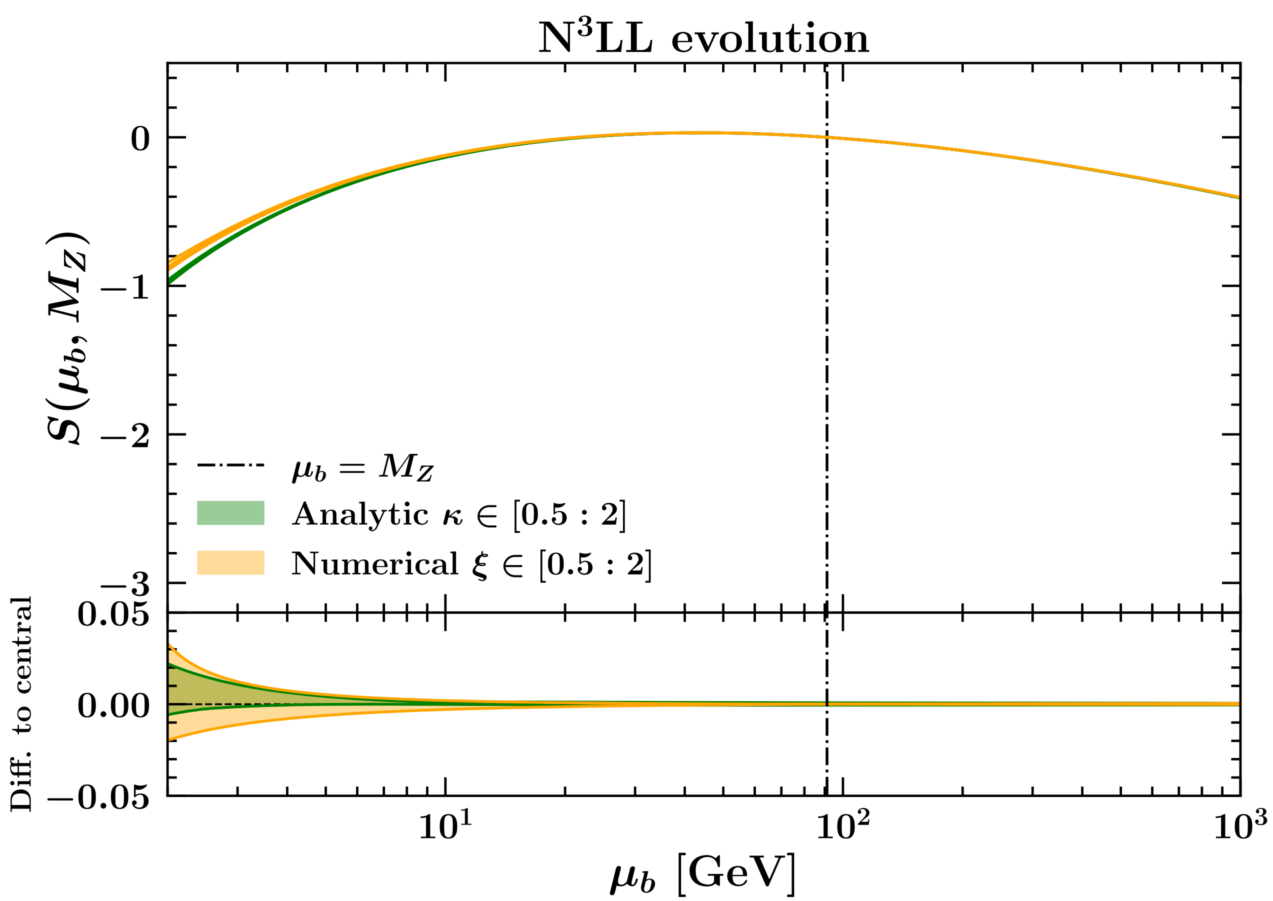
$$\exp(S) = g_0(\alpha_s(\mu_0)) \exp \left[L g_1(\lambda) + g_2(\lambda, \kappa) + \alpha_s(\mu_0) g_3(\lambda, \kappa) + \alpha_s^2(\mu_0) g_4(\lambda, \kappa) + \ldots \right]$$

- Same λ as in α_s : agreement with GB Catani de Florian Grazzini NPB 737 (2006) by identifying $\mu_{\mathrm{res}} = \mu_0/\kappa$ and $\mu_R = \mu_0$ (μ_0 = boundary condition for α_s)

- Variation of the **resummation scale in $\boldsymbol{\alpha_s}$** propagate into the Sudakov and produce the usual **$\boldsymbol{q_T}$-resummation** μ_{res} **scale**.

Sudakov





- Uncertainties of analytic/numerical solutions of the same order
- Good perturbative convergence
- From 3-4% in the small-μ_b region (where α_s is large) to sub-percent level at high μ_b values

Matching to PDFs

- At small b_T, matching of TMDs onto collinear PDFs: $F(\mu_b, \mu_b^2) = C(\alpha_s(\mu_b)) \otimes f(\mu_b)$

➡ "standard" implementation in TMD factorisation (S computed numerically)

$$F(M, M^2) = e^{S(M,\mu_b)} C(\alpha_s(\mu_b)) \otimes f(\mu_b)$$

- Estimate of theoretical uncertainties through variations of ξ in α_s

- <u>Connection with q_T resummation</u>: absorb DGLAP and C coefficient evolution into a modified Sudakov $\tilde{S}$

$$F(M, M^2) = e^{\tilde{S}(M,\mu_b)} \otimes \tilde{C}(\alpha_s(\mu_0)) \otimes f(\nu_0)$$

- Agreement with customary resummation formalism if ν_0 identified with the scale at which PDFs are measured (fitted) (just as μ_0 for α_s): $\nu_0 \sim \mathcal{O}$ (GeV)

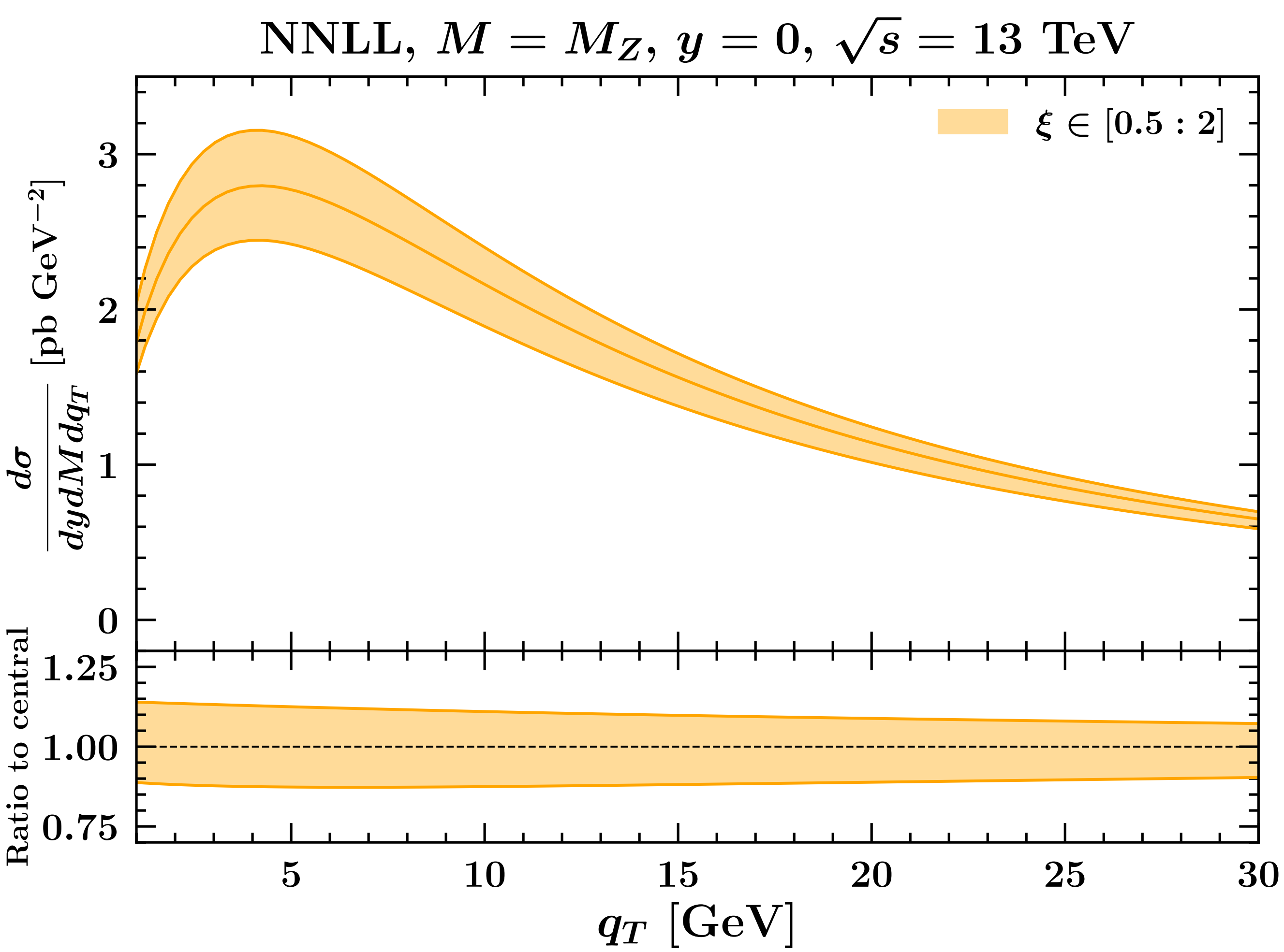
- correspondence of (μ_R, μ_F) in q_T resummation with (μ_0, ν_0) in this formalism

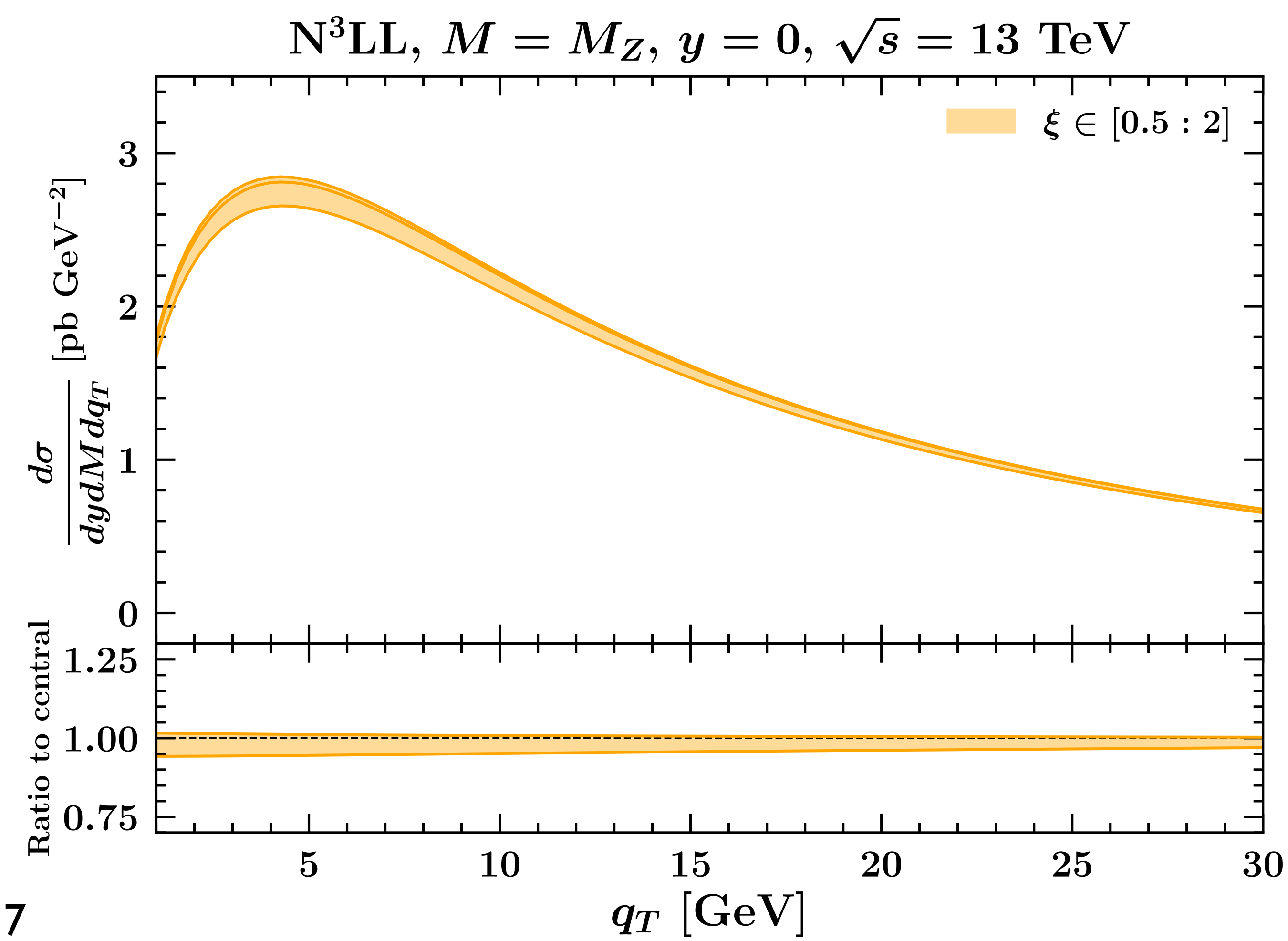
DY spectrum at $q_T \ll M$

$$\frac{d\sigma}{dq_T}(q_T, M, s) = \sigma_0 H(\alpha_s(M)) \int_0^\infty db \frac{b}{2} J_0(b q_T) F_1(b, M, M^2) F_2(b, M, M^2)$$

- Boundary conditions: $\alpha_s(M_Z) = 0.118$, $f(Q_0 = 2\text{ GeV})$ from MSHT
- Accuracy of evolutions consistent with accuracy of cross section
- Role of μ_R and μ_F played by M_Z and Q_0 —> may *not* be changed
- In this formalism, the only parameter that encapsulates theory uncertainties is ξ, accounting for α_s, PDF and TMD evolution at once

➡ mild dependence on q_T, good perturbative convergence



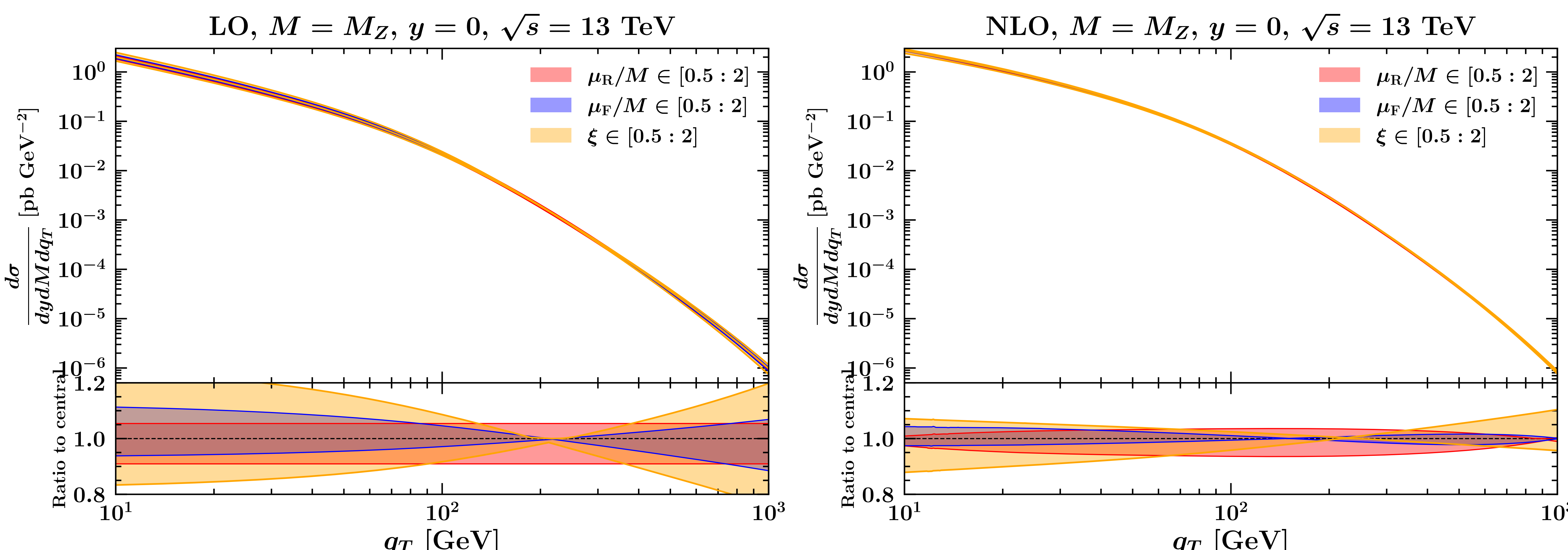


DY spectrum at $q_T \approx M$

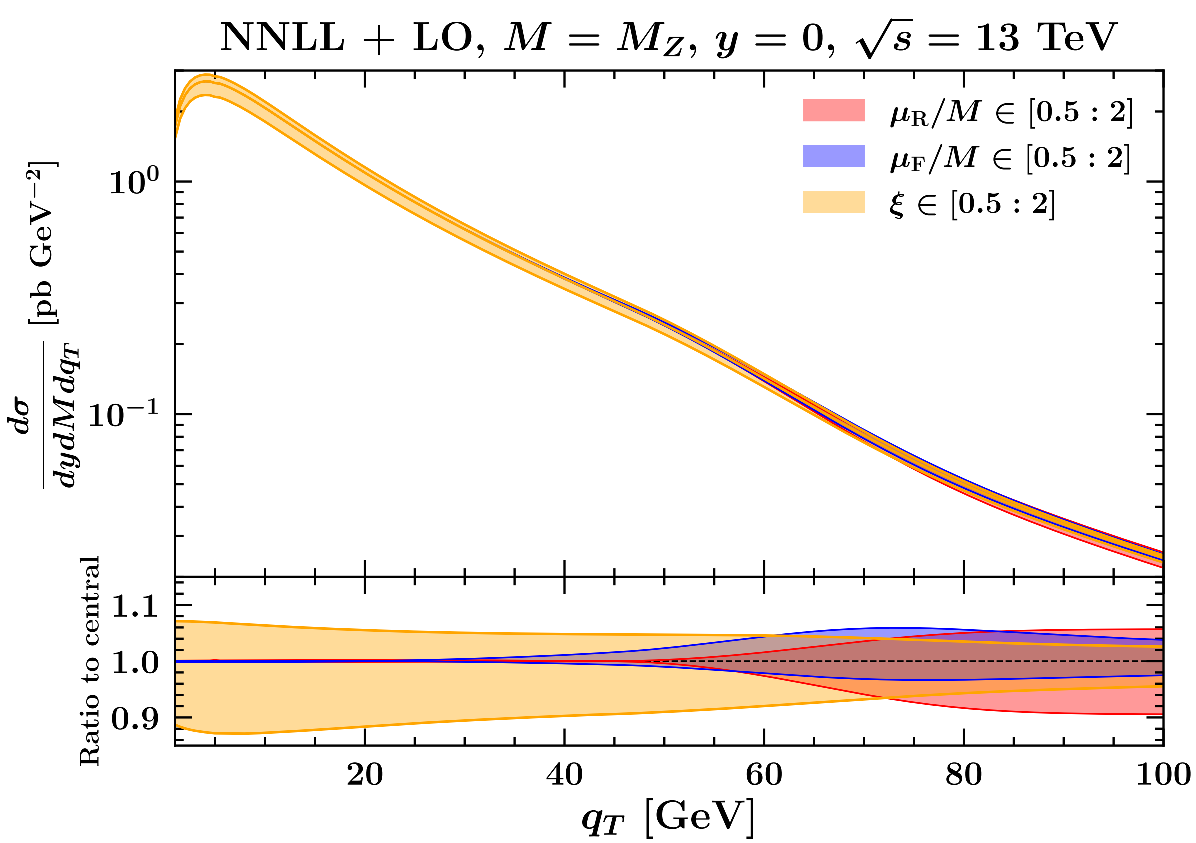
$$\frac{d\sigma}{dq_T}(q_T, M, s) = \frac{d\hat{\sigma}}{dq_T}(q_T, M, \alpha_s(\mu_R), \mu_R, \mu_F) \otimes f_1(\mu_F) f_2(\mu_F)$$

- Collinear factorisation
- Computational details equivalent to DIS structure functions

➡ ξ bands of similar size of μ_R, μ_F bands, good perturbative convergence



DY spectrum matched



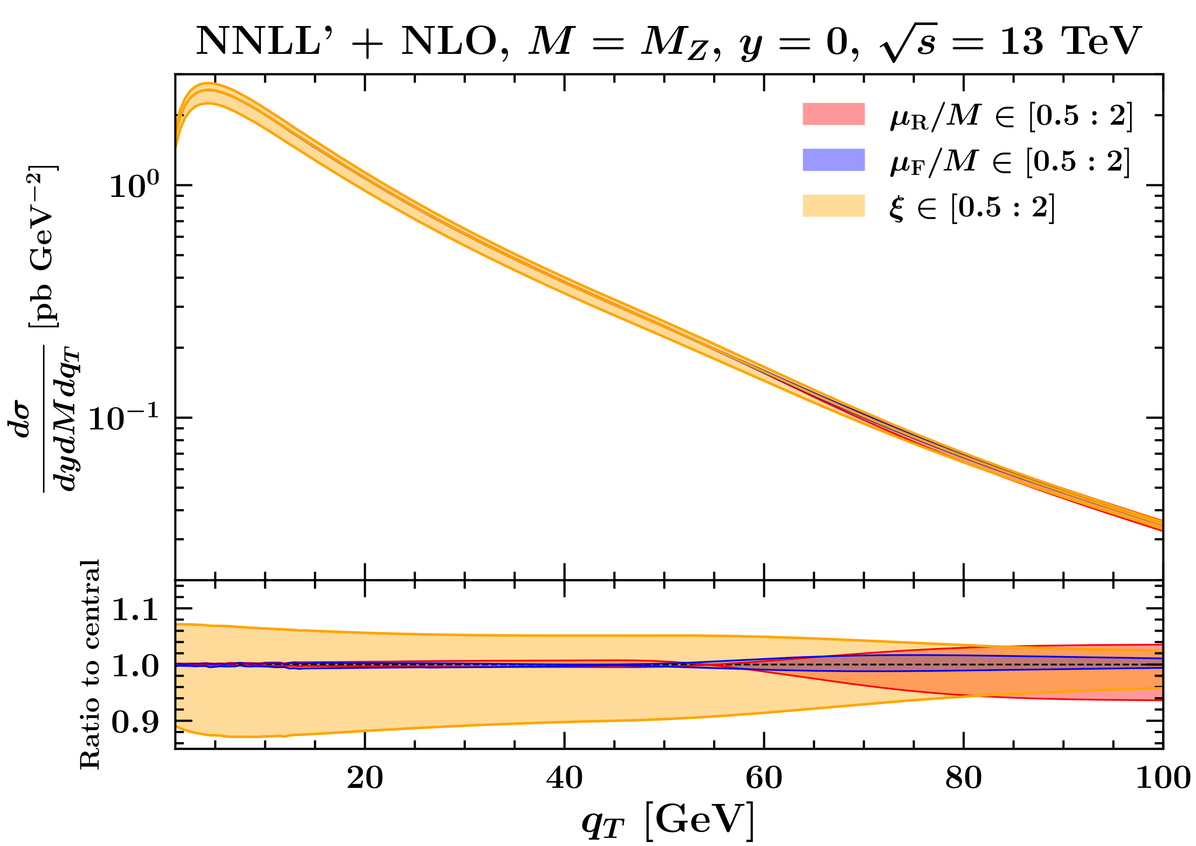
- Additive matching
- Damping function

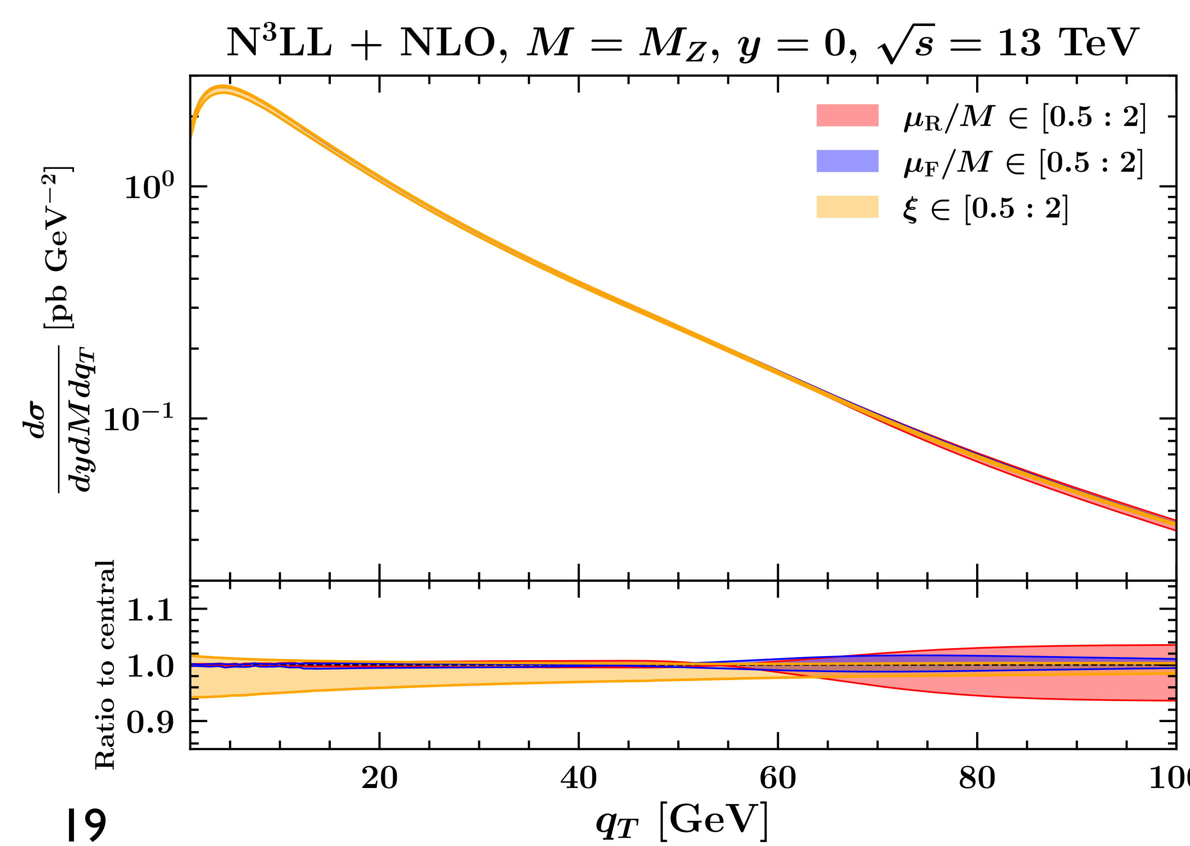
$$f(q_T, M) = \begin{cases} 1 & q_T < kM, \\ \exp\left[-\frac{(kM - q_T)^2}{\delta^2 M^2}\right] & q_T > kM, \end{cases}$$

$(k = 0.5,\ \delta = 0.25)$

➡ predominance of ξ bands at small and intermediate q_T (μ_R, μ_F bands take over at large q_T)

➡ ~5% uncertainty at N3LL

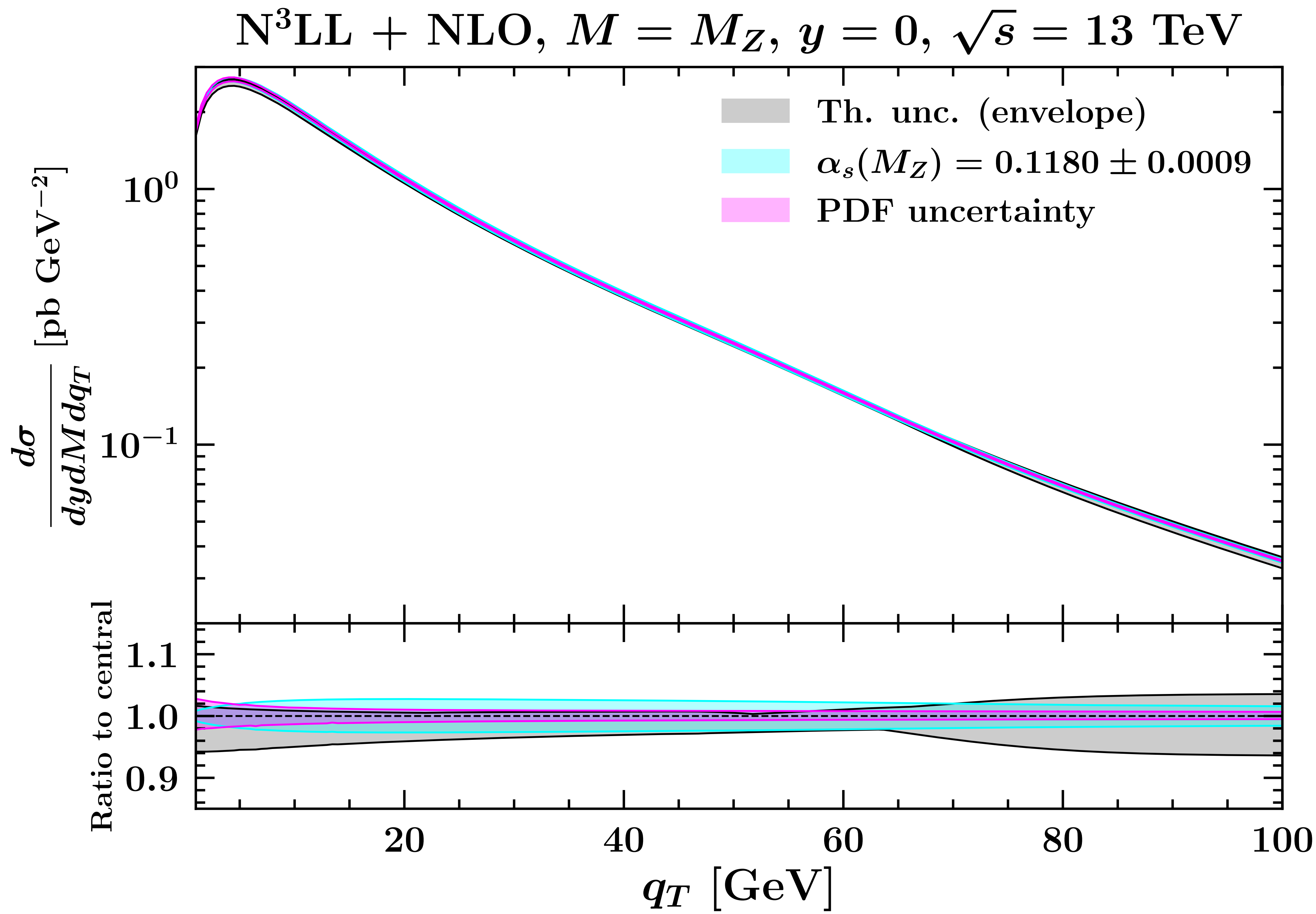




DY spectrum matched

- (Gray) Envelope of theoretical uncertainties
- (Cyan) Band associated Variation of boundary condition for α_s
- (Magenta) Band produced by error members of MSHT2020 at $\nu_0 = 2$ GeV

➡ bands of similar size, theory band generally larger



Conclusions

- **RGE solutions in pQCD carry an uncertainty**:
  - direct consequence of the perturbative **truncation** of the anomalous dimensions
  - it can be estimated by introducing **resummation scales** (in both analytic and numerical solutions)
- We have studied the cases of α_s, **PDFs**, and **TMDs**.
- Study of F_2, F_L, xF_3 as application to physical observables relevant for PDF fits
  - **distinctive behaviour** of resummation scale compared to μ_R and μ_F
  - relevant to **PDF extractions**.
- Formalism extended to **TMD double-logarithmic evolution**:
  - q_T-resummation formalism recovered
  - phenomenological application on Drell-Yan production at small q_T