On the interplay between the BFKL resummation and high–energy factorization in Mueller–Navelet dijet production†

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REF'25

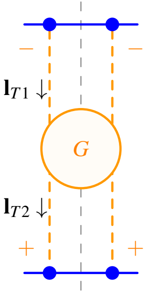
October 13–17, 2025

Supported by BASIS Foundation, grants #24-1-1-16-5 and #25-3-8-28-1

†Based on arXiv:2506.10458

The Regge limit:

$$\Lambda^2 \ll \mu^2 \ll s, \quad Y \sim \ln\left(\frac{s}{\mu^2}\right) \gg 1/\bar{\alpha}_s(\mu_R^2).$$



The Balitsky–Fadin–Kuraev–Lipatov (BFKL) equation:

$$\frac{\partial}{\partial Y} G(\mathbf{l}_{T1}, \mathbf{l}_{T2}, Y) = \int_{\mathbf{q}_T} K(\mathbf{l}_{T1}, \mathbf{q}_T)\, G(\mathbf{q}_T, \mathbf{l}_{T2}, Y),$$

where G is *Green's function*. State–of–the–art (in the QCD):

$$K = \underset{\text{[BFKL '76-78]}}{\overset{\text{LL}\sim\bar{\alpha}_s^n Y^n}{\bar{\alpha}_s(\mu_R^2)\, K^{(0)}}} + \underset{\text{[FL '98]}}{\overset{\text{NLL}\sim\bar{\alpha}_s^{n+1} Y^n}{\bar{\alpha}_s^2(\mu_R^2)\, K^{(1)}}} + \overset{\text{N}^2\text{LL}\sim\bar{\alpha}_s^{n+2} Y^n}{\mathscr{O}(\bar{\alpha}_s^3)}.$$

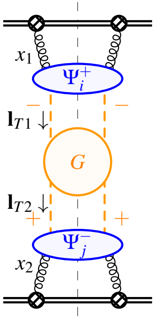
[FL '18; V. Del Duca *et.al.* '21; E. Byrne '24; S. Abreu *et.al.* '25; ...]

Topics beyond the scope of this study:

- Non–forward case [L. Lipatov '85; V. Fadin, R. Fiore '05; ...];
- DGLAP anomalous dimension at small–x [T. Jaroszewicz '82; G. Altarelli *et.al.* '00; R. Ball *et.al.* '18; ...];
- Unitarization: BK equation ['96, 99], JIMWLK hamiltonian ['97-01], ...;
- ...

Motivation

White paper on the BFKL physics [M. Hentschinski *et.al.* '23]: *Mueller–Navelet (MN) dijets* [MN '87; A. Sabio-Vera, F. Schwennsen '07; D. Colferai *et.al.* '10; B. Duclou ´ *et.al.* '14; ...], quarkonia [Z.-G. He, B. Kniehl, M. Nefedov, V. Saleev '19; M. Nefedov '24; ...], $\gamma^\star\gamma^\star$ [...; D. Colferai *et.al.* '23; A. Polizzi, M. Fucilla, A. Papa '25], and many more...



Status of the MN dijets phenomenology:

- Hybrid approach: collinear factorization

$$\sigma = \sum_{i,j} \int \frac{dx_1}{x_1}\, \tilde{f}_i(x_1,\mu_F^2) \int \frac{dx_2}{x_2}\, \tilde{f}_j(x_2,\mu_F^2)\; \hat{\sigma}_{ij}\left(x_{1,2}, \bar{\alpha}_s(\mu_R^2)\right) + \mathscr{O}\left(\frac{\Lambda^\#}{\mu_F^\#}\right),$$

  and BFKL–resummed coefficient function

$$\frac{d}{dY}\,\hat{\sigma}_{ij} = \int_{\mathbf{l}_{T1,2}} \Psi_i^+(\mathbf{p}_{T1},\mathbf{l}_{T1},y_1)\; G(\mathbf{l}_{T1},\mathbf{l}_{T2},Y)\; \Psi_j^-(\mathbf{p}_{T2},\mathbf{l}_{T2},y_2);$$

- Large corrections to the IFs $\sim \ln\left(\mathbf{p}_T^2/\mu_R^2\right)$;
- No applications of the exact solution [G. Chirilli, Y. Kovchegov '13, 14];

$\Big\}$ strong scale–sensitivity

- No phenomenology with Sudakov resummation;
- No uniform description of the data across all rapidities.

The hybrid approach meets TMD [A. Mueller, L. Szymanowski, S. Wallon, B. Xiao, F. Yuan '15; B. Xiao, F. Yuan '18; T. Altinoluk, N. Armesto, A. Kovner, M. Lublinsky '23; ...].

This talk: high–energy factorization + RG–invariant solution.

The high–energy factorization

HEF

The process

$$p\left(P^{+}\right)+p\left(P^{-}\right)\rightarrow \text{dijet}\left(p_1,p_2\right)+X,$$

where $P^{\pm}=\sqrt{S}$ and $p_i=\left(p_i^{+}n_{-}+p_i^{-}n_{+}\right)/2+p_{Ti}$ with $p_i^{\pm}=\left(p_i,n_{\pm}\right)=p_i^0\pm p_i^3$.

The Regge limit: $p_i^{\pm}\ll P^{\pm}$ and $\Lambda^2\ll \mathbf{p}_{Ti}^2\sim\mu^2\ll S$ or $\boxed{z\sim\mu^2/S\ll 1}$;

Motivation: resum $Y\sim\ln\left(1/z\times\mathbf{q}_T^2/\mu^2\right)$.

HEF

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$$p\,(P^+) + p\,(P^-) \to \text{dijet}\,(p_1, p_2) + X,$$

where $P^\pm = \sqrt{S}$ and $p_i = \left(p_i^+ n_- + p_i^- n_+\right)/2 + p_{Ti}$ with $p_i^\pm = (p_i, n_\pm) = p_i^0 \pm p_i^3$.

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The high–energy factorization (HEF)[S. Catani, M. Ciafaloni, F. Hautmann '90-94; J. Collins, C. Ellis '91]:

$$\sigma = \sum_{i,j} \int \frac{dx_1}{\pi x_1} \int_{\mathbf{q}_{T1}} \Phi_i(x_1, \mathbf{q}_{T1}^2, \mu^2) \int \frac{dx_2}{\pi x_2} \int_{\mathbf{q}_{T2}} \Phi_j(x_2, \mathbf{q}_{T2}^2, \mu^2)\, H_{ij}\left(x_{1,2}, \mathbf{q}_{T1,2}, \bar{\alpha}_s(\mu_R^2)\right)$$

where $x_{1,2} = (p_1^\pm + p_2^\pm)/P^\pm$.

- **Not** limited to the TMD case $\mathbf{p}_{T1,2}^2 \ll \mu^2$;
- **GI** off–shell $q_{1,2}^2 = -\mathbf{q}_{T1,2}^2$ coefficient function H_{ij};
- Unintegrated PDF (UPDF):

$$\Phi_i(x, \mathbf{q}_T^2, \mu^2) = \int_x^1 \frac{dz}{z}\, \tilde{f}_{i'}\left(\frac{x}{z}, \mu^2\right) C_{ii'}(z, \mathbf{q}_T, \mu^2),$$

where $C_{ii'}$ is calculated perturbatively: LP/NLP z, LL/NLL $\ln(1/z)$, $\ln(\mathbf{q}_T^2/\mu^2)$.

The parton Reggeization approach [M. Nefedov, V. Saleev, A. Shipilova '13]

The HEF coefficient function

$$H_{ij}\left(x_{1,2},\mathbf{q}_{T1,2},\bar{\alpha}_s(\mu_R^2)\right)=\int\limits_{p_1,p_2}\overline{\mid A_{ij}^{(\mathrm{EFT})}\mid^2}$$

is calculated with the Reggeized gluons and quarks $i,j=R,Q$:

$$q_1^+\sim|\mathbf{q}_{T1}|\sim\mathcal{O}(z)\gg q_1^-\sim\mathcal{O}(z^2),\quad q_2^-\sim|\mathbf{q}_{T2}|\sim\mathcal{O}(z)\gg q_2^+\sim\mathcal{O}(z^2),\quad q_{1,2}^2=-\mathbf{q}_{T1,2}^2;$$

- Prescriptions:

$$\overline{\mid A_{Rj}^{(\mathrm{EFT})}\mid^2}=\frac{1}{(N_c^2-1)}\left(\frac{q_1^+}{2|\mathbf{q}_{T1}|}\right)^2\mathrm{tr}\left[A_{Rj}^\star A_{Rj}\right],\quad\overline{\mid A_{Qj}^{(\mathrm{EFT})}\mid^2}=\frac{1}{2N_c}\frac{q_1^+}{2}\,\mathrm{tr}\left[\not{n}_-A_{Qj}^\star A_{Qj}\right];$$

- The *induced* term of the effective action [L. Lipatov '95]:

$$S_{\mathrm{ind}}=\frac{i}{g}\int d^4x\,\mathrm{tr}\left[\partial^2R_+(x)\,\partial_-W[v_-(x)]+\partial^2R_-(x)\,\partial_+W[v_+(x)]\right],$$

where $\partial_\mp R_\pm(x)=0$ and

$$W[v_\pm(x)]=P\exp\left[-\frac{ig}{2}\int\limits_{-\infty}^{x_\mp}dx'_\mp\,v_\pm(x_\pm,x'_\mp,\mathbf{x}_T)\right]=\left(1+ig\,\partial_\pm^{-1}v_\pm(x)\right)^{-1}$$

$\Rightarrow$ induced vertices $\supset\partial_\pm^{-1}\to -i/l^\pm\Rightarrow$ *"Green's function"* A_{ij}.

- Effective action for the Reggeized quarks [L. Lipatov, M. Vyazovsky '02].

The parton Reggeization approach [M. Nefedov, V. Saleev, A. Shipilova '13]

Improved KMRW formula [M. Nefedov, V. Saleev '20]:

$$C_{ii'}(x,z,\mathbf{q}_T,\mu^2) = \frac{\alpha_s(\mathbf{q}_T^2)}{2\pi}\,\frac{T_i(x,\mathbf{q}_T^2,\mu^2)}{\mathbf{q}_T^2}\; zP_{ii'}(z)\;\theta\left(\Delta(\mathbf{q}_T^2,\mu^2)-z\right),$$

where $\Delta(\mathbf{q}_T^2,\mu^2) = \mu/(\mu+|\mathbf{q}_T|)$.

- *à la* Sudakov form factor:

$$-\ln T_i(x,\mathbf{q}_T^2,\mu^2) = \int\limits_{\mathbf{q}_T^2}^{\mu^2} \frac{d\mathbf{q}_T'^2}{\mathbf{q}_T'^2}\,\frac{\bar{\alpha}_s(\mathbf{q}_T'^2)}{2\pi}\left(\tau_i(\mathbf{q}_T'^2,\mu^2)+\Delta\tau_i(x,\mathbf{q}_T'^2,\mu^2)\right);$$

- Expansion w.r.t. $1-\Delta(\mathbf{q}_T^2,\mu^2) = |\mathbf{q}_T|/\mu + \mathcal{O}\left(\mathbf{q}_T^2/\mu^2\right)$:

$$-\ln T_i(\mathbf{q}_T^2,\mu^2) \simeq c_1(\bar{\alpha}_s(\mu_R^2))\times\ln\left(\frac{\mathbf{q}_T^2}{\mu^2}\right) + c_2(\bar{\alpha}_s(\mu_R^2))\times\ln^2\left(\frac{\mathbf{q}_T^2}{\mu^2}\right);$$

- Normalization condition (UPDF $\to$ PDF):

$$\sum_{i'}\int\limits_0^{\mu^2} d\mathbf{q}_T^2\; C_{ii'}(x,z,\mathbf{q}_T,\mu^2) = \delta(1-z).$$

The BFKL equation

The BFKL kernel

Strategy for solving the BFKL equation:

$$\text{the kernel } K \quad \rightarrow \quad \text{eigenfunctions } H_\gamma \quad \rightarrow \quad \text{the Green's function } G$$

The BFKL kernel

Strategy for solving the BFKL equation:

the kernel K $\rightarrow$ eigenfunctions H_γ $\rightarrow$ the Green's function G

At the LO, the kernel is conformal invariant $(\Rightarrow H_\gamma^{(0)}(\mathbf{q}_T) \sim \mathbf{q}_T^{2(\gamma-1)})$:

$$\int_{\mathbf{q}_T} K(\mathbf{l}_T, \mathbf{q}_T)\, H_\gamma^{(0)}(\mathbf{q}_T) = \bar{\alpha}_s(\mathbf{l}_T^2)\, \chi^{(0)}(\gamma)\, H_\gamma^{(0)}(\mathbf{l}_T)$$

- No coupling running: $\bar{\alpha}_s(\mathbf{l}_T^2) = \bar{\alpha}_s(\mu_R^2) + \mathscr{O}(\bar{\alpha}_s^2)$;
- LO *characteristic function:* $\chi^{(0)}(\gamma) = \chi^{(0)}(1-\gamma)$.

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At the NLO, the diagonalization of the kernel breaks:

$$\int_{\mathbf{q}_T} K(\mathbf{l}_T, \mathbf{q}_T)\, H_\gamma^{(0)}(\mathbf{q}_T) = \bar{\alpha}_s(\mathbf{l}_T^2) \left[\chi^{(0)}(\gamma) + \bar{\alpha}_s(\mathbf{l}_T^2)\, \frac{\delta(\gamma)}{4}\right] H_\gamma^{(0)}(\mathbf{l}_T)$$

- Coupling running: $\bar{\alpha}_s(\mathbf{l}_T^2) = \bar{\alpha}_s(\mu_R^2) \left(1 - \bar{\alpha}_s(\mu_R^2)\, \frac{\beta_0}{4\pi} \ln\left(\frac{\mathbf{l}_T^2}{\mu_R^2}\right)\right) + \mathscr{O}(\bar{\alpha}_s^3)$;
- Structure of the $\delta(\gamma)$: $\frac{\delta(\gamma)}{4} = \underbrace{\chi^{(1)}(\gamma)}_{=\chi^{(1)}(1-\gamma)} - \underbrace{\frac{\beta_0}{8\pi}\frac{\partial \chi^{(0)}(\gamma)}{\partial \gamma}}_{\text{antisym. } \gamma \leftrightarrow 1-\gamma}$.

The BFKL equation

The exact solution of the N$^{(l)}$LO BFKL equation constructed in[G. Chirilli, Y. Kovchegov '13, 14]:

- The equation:

$$\frac{\partial}{\partial Y} G(\mathbf{l}_{T1}, \mathbf{l}_{T2}, Y) = \int_{\mathbf{q}_T} K\left(\mathbf{l}_{T1}, \mathbf{q}_T \mid \bar{\alpha}_s(\mu_R^2)\right) G(\mathbf{q}_T, \mathbf{l}_{T2}, Y),$$

initial condition: $G(\mathbf{l}_{T1}, \mathbf{l}_{T2}, Y=0) = \delta^{(2)}(\mathbf{l}_{T1} - \mathbf{l}_{T2})$, the kernel:

$$K\left(\mathbf{l}_T, \mathbf{q}_T \mid \bar{\alpha}_s(\mu_R^2)\right) = \sum_l \bar{\alpha}_s^{l+1}(\mu_R^2) K^{(l)}(\mathbf{l}_T, \mathbf{q}_T), \quad \frac{\partial}{\partial \mu_R^2} K^{(l)}(\mathbf{l}_T, \mathbf{q}_T) = 0;$$

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- Diagonalization of the kernel (order-by-order in $\bar{\alpha}_s$):

$$\int_{\mathbf{q}_T} K\left(\mathbf{l}_T, \mathbf{q}_T \mid \bar{\alpha}_s(\mu_R^2)\right) H_\gamma(\mathbf{q}_T, \mu_R^2) \;=\; \bar{\alpha}_s(\mu_R^2)\, \chi(\gamma)\, H_\gamma(\mathbf{l}_T, \mu_R^2),$$

  where

$$\chi(\gamma) = \sum_l \bar{\alpha}_s^l(\mu_R^2)\, \chi^{(l)}(\gamma), \qquad \boxed{H_\gamma(\mathbf{l}_T, \mu_R^2) = \sum_l \bar{\alpha}_s^l(\mu_R^2)\, H_\gamma^{(l)}(\mathbf{l}_T, \mu_R^2)};$$

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- Diagonalization of the kernel (order-by-order in $\bar{\alpha}_s$):

$$\int_{\mathbf{q}_T} K\left(\mathbf{l}_T, \mathbf{q}_T \mid \bar{\alpha}_s(\mu_R^2)\right) H_\gamma(\mathbf{q}_T, \mu_R^2) = \bar{\alpha}_s(\mu_R^2) \chi(\gamma) H_\gamma(\mathbf{l}_T, \mu_R^2),$$

  where

$$\chi(\gamma) = \sum_l \bar{\alpha}_s^l(\mu_R^2) \chi^{(l)}(\gamma), \qquad \boxed{H_\gamma(\mathbf{l}_T, \mu_R^2) = \sum_l \bar{\alpha}_s^l(\mu_R^2) H_\gamma^{(l)}(\mathbf{l}_T, \mu_R^2)};$$

- The RG–invariant expansion of the kernel:

$$K\left(\mathbf{l}_T, \mathbf{q}_T \mid \bar{\alpha}_s(\mu_R^2)\right) = \bar{\alpha}_s(\mu_R^2) \int \frac{d\gamma}{2\pi i} \chi(\gamma) H_\gamma(\mathbf{l}_T, \mu_R^2) H_\gamma^\star(\mathbf{q}_T, \mu_R^2),$$

  where $\chi(\gamma) = \chi(1-\gamma)$, so $K(\mathbf{l}_T, \mathbf{q}_T) \equiv K(\mathbf{q}_T, \mathbf{l}_T)$.

The BFKL equation

The RG–invariant solution (for $n \neq 0$):

$$G(\mathbf{l}_{T1}, \mathbf{l}_{T2}, Y) = \sum_n \int \frac{d\gamma}{2\pi i} \exp\left[\bar{\alpha}_s(\mu_R^2)\,\chi(n,\gamma)\,Y\right] H_{n,\gamma}(\mathbf{l}_{T1}, \mu_R^2)\, H^\star_{n,\gamma}(\mathbf{l}_{T2}, \mu_R^2).$$

- The characteristic function (symm. and real), see $\delta(n,\gamma)$ in[A. Kotikov, L. Lipatov '00]:

$$\chi^{(0)}(n,\gamma) = 2\psi(1) - 2\,\mathrm{Re}\,\psi\left(\frac{n}{2}+\gamma\right), \quad \chi^{(1)}(n,\gamma) = \frac{\beta_0}{8\pi}\frac{\partial \chi^{(0)}(n,\gamma)}{\partial\gamma} + \frac{\delta(n,\gamma)}{4},$$

- The eigenfunctions:

$$H^{(0)}_{n,\gamma}(\mathbf{l}_T) = \frac{1}{\sqrt{\pi}}\,\mathbf{l}_T^{2(\gamma-1)}\,e^{in\phi_l}, \quad H^{(1)}_{n,\gamma}(\mathbf{l}_T, \mu_R^2) = \sum_m c_m(n,\gamma)\ln^m\left(\frac{\mathbf{l}_T^2}{\mu_R^2}\right) H^{(0)}_{n,\gamma}(\mathbf{l}_T),$$

the coefficients:

$$c_0 = 0, \quad c_1(n,\gamma) = \partial_\gamma c_2(n,\gamma), \quad c_2(n,\gamma) = \frac{\beta_0}{8\pi}\frac{1}{\partial_\gamma \ln\chi^{(0)}(n,\gamma)}, \quad c_{m>2} = 0.$$

Completeness and orthonormality:

$$\sum_n \int \frac{d\gamma}{2\pi i} H_{n,\gamma}(\mathbf{l}_{T1}, \mu_R^2) H^\star_{n,\gamma}(\mathbf{l}_{T2}, \mu_R^2) = \delta^{(2)}(\mathbf{l}_{T1} - \mathbf{l}_{T2}), \quad \int d^2\mathbf{l}_T\, H_{n,\gamma}(\mathbf{l}_T, \mu_R^2) H^\star_{n',\gamma'}(\mathbf{l}_T, \mu_R^2) = \delta_{nn'}\,\delta(\gamma-\gamma').$$

Agrees with the solution[A. Grabovsky '13] up to NLL, details coming soon. . .

Resummation of the collinear poles

Collinear–improvement: $\chi(n,\gamma)+\Delta\chi(n,\gamma)$.

- *A (anti-) collinear poles:*

$$\chi^{(l)}(n,\gamma)=\sum_{k=1}^{2l+1}\kappa_{l,k}\,d_0^{-k}\,\bar{\alpha}_s^l+\mathcal{O}(\gamma)+\{\gamma\to 1-\gamma\},\quad d_m(n,\gamma)=m+n/2+\gamma;$$

- The ω–shift scheme[G. Salam '98]:

$$d_0^{-k}\left(n,\gamma\pm\frac{1}{2}\bar{\alpha}_s\,\chi(n,\gamma)\right)\bar{\alpha}_s^l\supset\sum_{l'=l}^{\infty}d_0^{-k-2(l-l')}\,\bar{\alpha}_s^{l'}$$

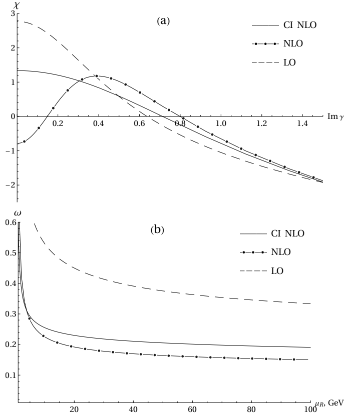
due to $s_0=|\mathbf{l}_{T1}||\mathbf{l}_{T2}|$ (BFKL: $|\mathbf{l}_{T1}|\sim|\mathbf{l}_{T2}|$) $\longrightarrow s_0=\mathbf{l}_{T1,2}^2$ (DGLAP: $\mathbf{l}_{T1,2}^2\gg\mathbf{l}_{T2,1}^2$) in $Y=\ln\left(\frac{s}{s_0}\right)$;

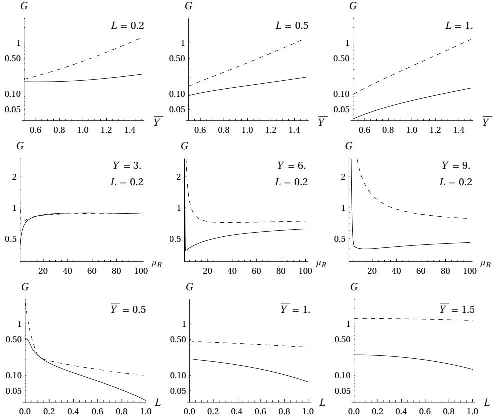
- The "all–poles" approximation[A. Sabio-Vera '05; A. Sabio-Vera, F. Schwennsen '07]:

$$\begin{aligned}\bar{\alpha}_s\,\Delta\chi(n,\gamma)\quad=\quad&\sum_{m=0}^{\infty}\Bigg[\kappa_{1,2}\,\bar{\alpha}_s-d_m+\sqrt{2\bar{\alpha}_s\left(\kappa_{0,1}+\kappa_{1,1}\,\bar{\alpha}_s\right)+\left(\kappa_{1,2}\,\bar{\alpha}_s-d_m\right)^2}\\&-\sum_{l=0}^{1}\sum_{k=1}^{2l+1}\kappa_{l,k}\,d_m^{-k}\,\bar{\alpha}_s^{l+1}\Bigg]+\{\gamma\to 1-\gamma\};\end{aligned}$$

- Doesn't break the diagonalization of the kernel: $\Delta\chi\sim\mathcal{O}(\bar{\alpha}_s^3)$.

The Green's function



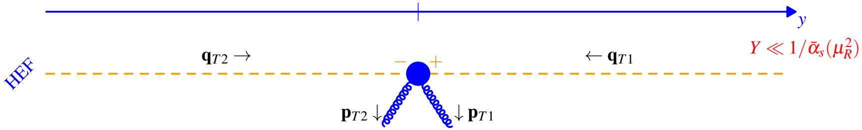


The collinearly–improved NLO:

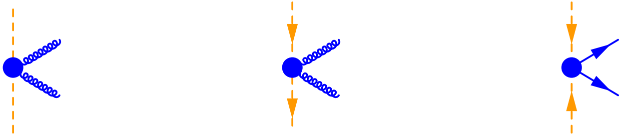
- The intercept: $\omega = \bar{\alpha}_s(\mu_R^2)\,\chi(0, 1/2)$;
- The ω is scale–independent at $\mu_R \gtrsim 20$ GeV;
- grows slower w.r.t. $\bar{Y} = \bar{\alpha}_s(\mu_R^2)\,Y$ than the LO one;
- is approximately scale-independent at $\bar{Y} \sim 1$;
- doesn't oscillate w.r.t. $L = \ln(|\mathbf{l}_{T1}|/|\mathbf{l}_{T2}|)$.

Mueller–Navelet dijets production

Matching scheme



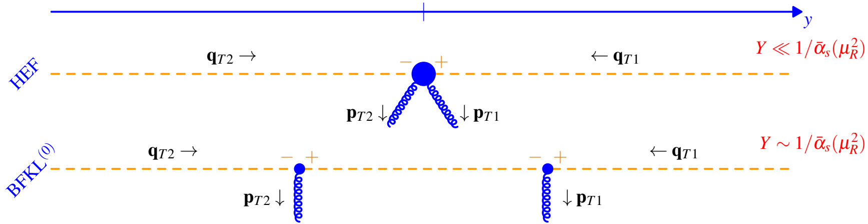
Examples:



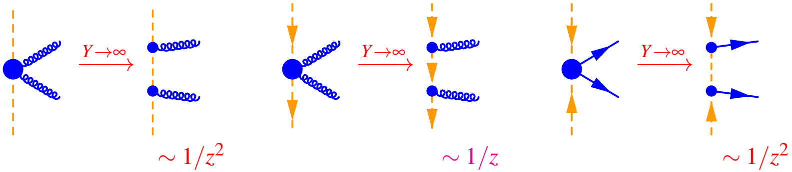
The coefficient function for $Y \ll 1/\bar{\alpha}_s(\mu_R^2)$:

$$H_{ij}^{(\mathrm{HEF})}$$

Matching scheme



Examples $(Y \sim \ln(1/z))$:

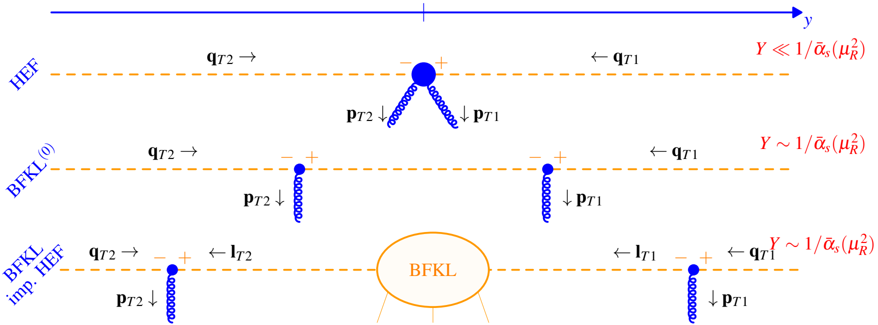


The coefficient function for $Y \sim 1/\bar{\alpha}_s(\mu_R^2)$:

$$H_{ij}^{(\mathrm{BFKL},0)} = \lim_{Y\to\infty} H_{ij}^{(\mathrm{HEF})} = \int\limits_{\mathbf{l}_{T1,2}} \Psi_i^+(\mathbf{p}_{T1}, \mathbf{l}_{T1}, y_1)\, \delta^{(2)}(\mathbf{l}_{T1} - \mathbf{l}_{T2})\, \Psi_j^-(\mathbf{p}_{T2}, \mathbf{l}_{T2}, y_2) + \mathscr{O}(1/z)$$

Reminder: $G(\mathbf{l}_{T1}, \mathbf{l}_{T2}, Y = 0) = \delta^{(2)}(\mathbf{l}_{T1} - \mathbf{l}_{T2})$.

Matching scheme

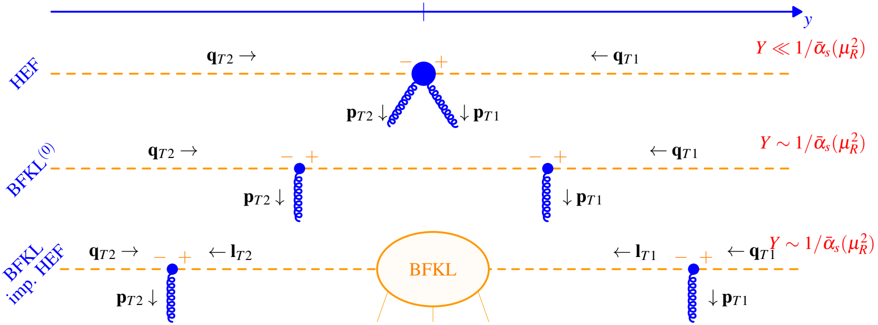


The resummed coefficient function for $Y \sim 1/\bar{\alpha}_s(\mu_R^2)$:

$$H_{ij}^{(\mathrm{BFKL})} = \int_{\mathbf{l}_{T1,2}} \Psi_i^{+}(\mathbf{p}_{T1},\mathbf{l}_{T1},y_1)\, G(\mathbf{l}_{T1},\mathbf{l}_{T2},Y)\, \Psi_j^{-}(\mathbf{p}_{T2},\mathbf{l}_{T2},y_2)$$

Note: the UPDFs resum $\ln^{1,2}\left[\left(\mathbf{p}_{T1,2} \pm \mathbf{l}_{T1,2}\right)^2/\mathbf{p}_{T1,2}^2\right]$ to the IFs $\Psi_{i,j}^{\pm}$.

Matching scheme



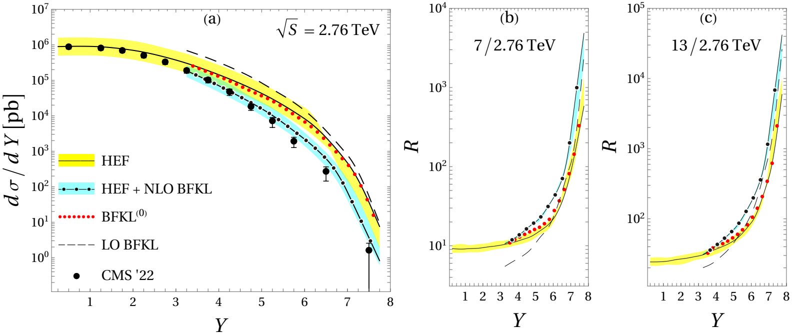
The matching scheme:

$$H_{ij}^{(\mathrm{HEF+BFKL})} = H_{ij}^{(\mathrm{HEF})} + H_{ij}^{(\mathrm{BFKL})} - H_{ij}^{(\mathrm{BFKL},0)},$$

where

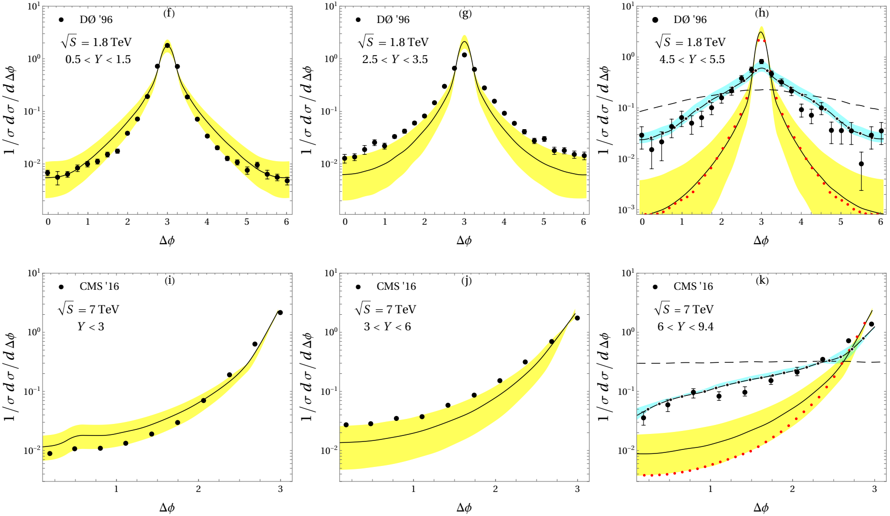
$$\begin{aligned} H_{ij}^{(\mathrm{BFKL})} - H_{ij}^{(\mathrm{BFKL},0)} &= 0 \quad \text{for} \quad Y \ll 1/\bar{\alpha}_s(\mu_R^2), \\ H_{ij}^{(\mathrm{HEF})} - H_{ij}^{(\mathrm{BFKL},0)} &= 0 \quad \text{for} \quad Y \sim 1/\bar{\alpha}_s(\mu_R^2). \end{aligned}$$

Results



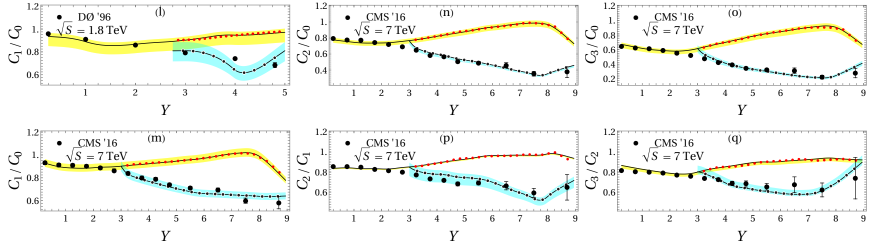
- The HEF describes the data well at low $Y < 2-3$;
- The HEF gets subtracted by the BFKL$^{(0)}$ starting from $Y > 3-3.5$;
- The NLO BFKL–improved HEF is in agreement with the data at large $Y > 3.5$;
- The BFKL–improved HEF ratio $R(Y)$ grows significantly faster than the HEF one.

Results



- The HEF describes the data well at low Y;
- The NLO BFKL–improved HEF is in agreement with the data at large Y;
- Compare with [A. Sabio-Vera, F. Schwennsen '07; B. Ducloué *et.al.* '14].

Results



The angular coefficients:

$$\frac{1}{\sigma}\frac{d\sigma}{dY\, d\Delta\phi} = \frac{1}{2\pi}\left[1 + 2\sum_{n\geq 0} C_n(Y)\cos\left(n(\Delta\phi - \pi)\right)\right]$$

- Low Y: no suppression by $C_{k>n}/C_n$, is covered by the HEF;
- Large Y: in the limit $Y \to \infty$

  $$G \sim e^{\bar{\alpha}_s(\mu_R^2)\,\chi(n=0,1/2)\,Y},$$

  suppression by $C_{k>n}/C_n$, interplay between the Sudakov & BFKL resummations;
- Compare with[A. Sabio-Vera, F. Schwennsen '07; B. Duclou ´ et.al. '14].

Conclusions & outlook

HEF + RG–invariant & coll. imp. NLO BFKL

- Low Y: the HEF describes the data, large Y: the BFKL resummation is necessary
  $\Rightarrow$ new evidences for the BFKL dynamics in the MN dijets;
- The matching scheme interpolates predictions between the two descriptions
  $\Rightarrow$ uniform description of the data across all values of the rapidity difference;
- No strong scale–sensitivity due to the NLL Sudakov and NLO eigenfunctions
  $\Rightarrow$ no need for the BLM procedure.

Outlook:

- NLO HEF [A. van Hameren, M. Nefedov '25];
- Resummation of the transverse logarithms [A. Kovner, M. Lublinsky, V. Skokov, Z. Zhao '23; ...];
- Sub–eikonal corrections [T. Altinoluk, D. Dumitru '15; I. Balitsky, A. Tarasov '15; G. Chirilli '21; M. Nefedov '21; ...].

Thank you for your attention!

NLO characteristic function

The NLO characteristic function:

$$\chi^{(1)}(n,\gamma) = \mathscr{S}\chi^{(0)}(n,\gamma) + \frac{3}{2}\zeta(3) - \frac{\beta_0}{8N_c}\left(\chi^{(0)}(n,\gamma)\right)^2$$
$$-\frac{\pi^2\cos(\pi\gamma)}{4(1-2\gamma)\sin^2(\pi\gamma)}\left[\left(3+\left(1+\frac{N_\mathrm{f}}{N_c^3}\right)\frac{2+3\gamma(1-\gamma)}{(1+2\gamma)(3-2\gamma)}\right)\delta_n^0 - \left(1+\frac{N_\mathrm{f}}{N_c^3}\right)\frac{\gamma(1-\gamma)}{2(1+2\gamma)(3-2\gamma)}\delta_n^2\right]$$
$$+\frac{1}{4}\left[\psi''\left(\gamma+\frac{n}{2}\right)+\psi''\left(1-\gamma+\frac{n}{2}\right)-2\left(\phi(n,\gamma)+\phi(n,1-\gamma)\right)\right],$$

where $\mathscr{S} = (4-\pi^2+5\beta_0/N_c)/12$ and $\beta_0 = (11N_c - 2N_\mathrm{f})/3$. The function ϕ is:

$$\begin{aligned}\phi(n,\gamma) &= \sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{d_m}\Big(\psi'(m+n+1)-\psi'(m+1)\\ &+ (-1)^{m+1}\left(\beta'(m+n+1)-\beta'(m+1)\right)+\frac{\psi(m+1)-\psi(m+n+1)}{d_m}\Big),\end{aligned}$$

where

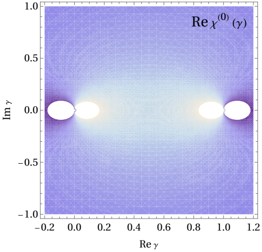
$$\beta(z) = \frac{1}{2}\left(\psi\left(\frac{z+1}{2}\right)-\psi\left(\frac{z}{2}\right)\right).$$

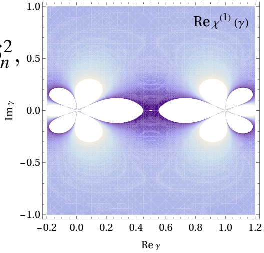
The characteristic function

A (anti-) collinear poles:

$$\chi^{(l)}(n,\gamma) = \sum_{k=1}^{2l+1} \kappa_{l,k}\, d_0^{-k}\, \bar{\alpha}_s^l(\mu_R^2) + \mathscr{O}(\gamma) + \{\gamma \to 1-\gamma\},$$

where $d_m = d_m(n,\gamma) = m + n/2 + \gamma$ and





$$\begin{aligned}
\kappa_{0,1} &= 1,\\
\kappa_{1,1} &= \mathscr{S} - \frac{\pi^2}{24} + \frac{\beta_0}{4N_c}\,\mathrm{H}_n + \frac{1}{8}\left(\psi'\left(\frac{n+1}{2}\right) - \psi'\left(\frac{n+2}{2}\right)\right)\\
&+ \frac{1}{2}\,\psi'(n+1) - \frac{1}{36}\left(67 + 13\frac{N_f}{N_c^3}\right)\delta_n^0 - \frac{47}{1800}\left(1+\frac{N_f}{N_c^3}\right)\delta_n^2,\\
-\kappa_{1,2} &= \frac{\beta_0}{8N_c} + \frac{1}{2}\,\mathrm{H}_n + \frac{1}{12}\left(11 + 2\frac{N_f}{N_c^3}\right)\delta_n^0 + \frac{1}{60}\left(1+\frac{N_f}{N_c^3}\right)\delta_n^2,\\
\kappa_{1,3} &= \frac{1}{24}\left(1+\frac{N_f}{N_c^3}\right),
\end{aligned}$$

where $\mathscr{S} = (4 - \pi^2 + 5\beta_0/N_c)/12$, $\beta_0 = (11N_c - 2N_f)/3$, and $\mathrm{H}_n = \psi(n+1) - \psi(1)$.