Energy conditions, singularities and exotic spacetimes

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SIGRAV International School
20 February 2025

Outline

Energy conditions

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- Strong energy condition: SEC
- Dominant energy condition: DEC
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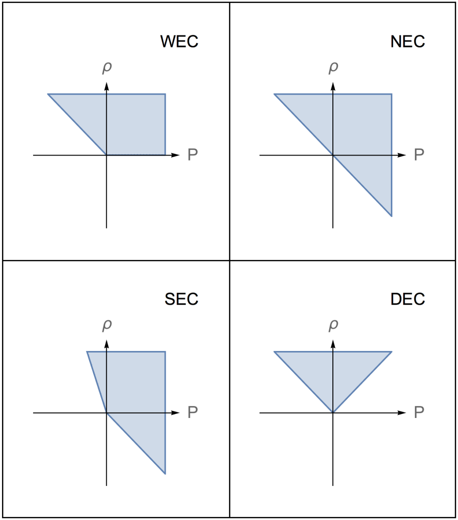
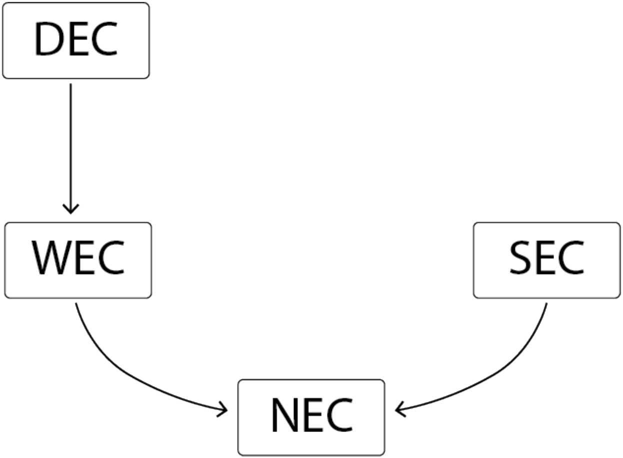
Name	Physical	Geometric	Perfect fluid
WEC	$T_{\mu\nu}U^\mu U^\nu \geq 0$	$G_{\mu\nu}U^\mu U^\nu \geq 0$	$\rho \geq 0$ and $\rho + P \geq 0$
SEC	$\left(T_{\mu\nu} - \frac{Tg_{\mu\nu}}{n-2}\right)U^\mu U^\nu \geq 0$	$R_{\mu\nu}U^\mu U^\nu \geq 0$	$\rho + P \geq 0$ and $(n-3)\rho + (n-1)P \geq 0$
DEC	$T_{\mu\nu}U^\mu \xi^\nu \geq 0$	$G_{\mu\nu}U^\mu \xi^\nu \geq 0$	$\rho \geq \lvert P \rvert$
NEC	$T_{\mu\nu}\ell^\mu \ell^\nu \geq 0$	$R_{\mu\nu}\ell^\mu \ell^\nu \geq 0$	$\rho + P \geq 0$

U^μ and ξ^μ: co-oriented timelike vectors, ℓ^μ: null vector

Visualize perfect fluid energy conditions



Connections



Why do we need energy conditions?

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

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Energy-momentum tensor: matter

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Einstein tensor: spacetime geometry

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Solving Einstein's equations in reverse: you can have any kind of spacetime you want with the right kind of matter.

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- Does our spacetime have singularities? Can they be naked?
- Are wormholes, superluminal communication and closed timelike curves allowed?

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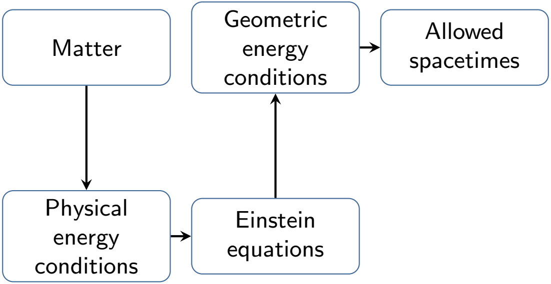

The minimally coupled scalar field

Classical gravitational action integral

$$S = \int d^n x \sqrt{-g} \left[\frac{R}{16\pi G_N} + \frac{1}{2}(\nabla\phi)^2 - V(\phi) \right]$$

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For $V(\phi) = m^2\phi^2/2$ the stress-energy tensor is

$$T_{\mu\nu} = (\nabla_\mu \phi)(\nabla_\nu \phi) - \frac{1}{2} g_{\mu\nu} (m^2\phi^2 - (\nabla\phi)^2)$$

Energy conditions for ϕ

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- **DEC**: For all co-oriented timelike vectors U^μ, ξ^ν the tensor $U^\mu \xi^\nu + \xi^\mu U^\nu - (U^\alpha \xi_\alpha) g^{\mu\nu}$ is positive definite as $U^\alpha \xi_\alpha < 0$.
- **WEC and NEC**: They hold as DEC holds.
- **SEC**:
$$U^\mu U^\nu \left(T_{\mu\nu} - \frac{T}{n-2} g_{\mu\nu} \right) = (U^\mu \nabla_\mu \phi)^2 - \frac{1}{n-2} m^2 \phi^2$$
only obeyed it $m = 0$, but it can be violated when $m > 0$.

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Sketch of the theorem

Let A any local operator that has zero expectation value in the vacuum state $\langle\Omega|A|\Omega\rangle = 0$. Then if A is positive we can write $\|A^{1/2}\Omega\|^2 = \langle\Omega|A|\Omega\rangle = 0$. So $A\Omega = 0$ and then A is identically zero. So A cannot be positive. [Epstein, Glaser, Jaffe, 1965]

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$\Rightarrow$There is a nonzero probability for both positive and negative measurement values, so the spectrum of A extends into the negative half-line.

Quantization and renormalization

We follow algebraic quantization and our main object of interest is the two point function,

$$W_\psi(x,x') \equiv \langle \phi(x)\phi(x') \rangle_\psi \, ,$$

where ψ is a quantum state of interest. The class of states we consider in this paper are the *Hadamard states* whose two point-functions have well-known singularity structures.

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We renormalize the stress-energy tensor following the prescription of Hollands and Wald [Hollands, Wald, 2001].

Quantization and renormalization

First let's define the point-split stress-energy operator

$$T^{\text{split}}_{\mu\nu'}(x,x') = \nabla^{(x)}_{\mu} \otimes \nabla^{(x')}_{\nu'} - \frac{1}{2} g_{\mu\nu'}(x,x') g^{\lambda\rho'}(x,x') \nabla^{(x)}_{\lambda} \otimes \nabla^{(x')}_{\rho'} + \frac{1}{2} m^2 g_{\mu\nu'}(x,x') \mathbb{1} \otimes \mathbb{1} \,,$$

where $g_{\mu\nu'}(x,x')$ is the parallel propagator. Then we can define

$$\langle T^{\text{fin}}_{\mu\nu} \rangle_{\psi}(x) = \lim_{x' \to x} g_{\nu}{}^{\nu'}(x,x') T^{\text{split}}_{\mu\nu'} \circ \overbrace{(W_{\psi} - H_{(k)})(x,x')}^{\text{smooth}} \,,$$

Hadamard parametrix: a bi-distribution that encodes the singularity structure of the two-point function of Hadamard states

Quantization and renormalization

Coincident limit

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$$\langle T^{\text{ren}}_{\mu\nu} \rangle_\psi - \langle T^{\text{ren}}_{\mu\nu} \rangle_{\psi_0} = \left[\!\left[g_\nu{}^{\nu'} \, T^{\text{split}}_{\mu\nu'} \circ (W_\psi - W_{\psi_0}) \right]\!\right] ,$$

where

$$\langle T^{\text{ren}}_{\mu\nu} \rangle_\psi(x) = \langle T^{\text{fin}}_{\mu\nu} \rangle_\psi - \underbrace{Q(x) g_{\mu\nu}(x) + C_{\mu\nu}(x)}_{\text{Renormalization freedom}} .$$

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Massless Minkowski vacuum

$$\langle : T_{\mu\nu} : \rangle_\psi := \left[\!\left[g_\nu{}^{\nu'} \, T^{\text{split}}_{\mu\nu'} \circ (W_\psi - W_\Omega) \right]\!\right] .$$

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If ρ is some contraction of the stress-energy tensor

$$\langle \rho(f)\rangle_\Psi \geq -\langle \mathcal{Q}(f)\rangle_\Psi \,,$$

$$\langle \rho(f)\rangle_\Psi - \langle \rho(f)\rangle_{\Psi_0} \geq -\langle \mathcal{Q}_{\Psi_0}(f)\rangle_\Psi \,.$$

f: non-negative smearing function on spacetime

Derivation of a quantum energy inequality

Average over a timelike geodesic:

$$\langle \rho \circ \gamma \rangle_\psi (f^2) = \int_{-\infty}^{\infty} \mathrm{d}t f^2(t) \langle \rho \rangle_\psi (\gamma(t)) \,.$$

Derivation of a quantum energy inequality

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Then

$$\langle :\!\rho\!:^{\mathrm{split}}_{\psi_0} \rangle_\psi \ (t,t') = \langle \rho^{\mathrm{split}}(t,t') \rangle_\psi - \langle \rho^{\mathrm{split}}(t,t') \rangle_{\psi_0} \, ,$$

Derivation of a quantum energy inequality

$$\langle :\rho:_{\psi_0} \circ \gamma \rangle_\psi (f^2) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{d}t'\, f(t) f(t') \delta(t - t') \langle :\rho:^{\mathrm{split}}_{\psi_0} \rangle_\psi \, (t, t')$$

$$= \int_0^\infty \frac{\mathrm{d}\alpha}{\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{d}t'\, f(t) f(t') e^{i\alpha(t-t')} \langle :\rho:^{\mathrm{split}}_{\psi_0} \rangle_\psi \, (t, t')$$

$$= \int_0^\infty \frac{\mathrm{d}\alpha}{\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{d}t'\, \langle \rho^{\mathrm{split}}(t, t') \rangle_\psi \overline{f_\alpha(t)} f_\alpha(t')$$

$$- \int_0^\infty \frac{\mathrm{d}\alpha}{\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{d}t'\, \langle \rho^{\mathrm{split}}(t, t') \rangle_{\psi_0} \overline{f_\alpha(t)} f_\alpha(t') \,,$$

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$$- \int_0^{\infty} \frac{\mathrm{d}\alpha}{\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{d}t'\, \langle \rho^{\mathrm{split}}(t, t') \rangle_{\psi_0} \overline{f_\alpha(t)} f_\alpha(t')\,,$$

where $f_\alpha(t) = f(t) e^{i\alpha t}$.

Assumptions for ρ^{split}

- Symmetric
- Of positive type

Derivation of a quantum energy inequality

$$\langle :\rho:_{\psi_0} \circ \gamma \rangle_\psi (f^2) \geq -\int_0^\infty \frac{\mathrm{d}\alpha}{\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{d}t' \, \langle \rho^{\mathrm{split}}(t,t') \rangle_{\psi_0} \overline{f_\alpha(t)} f_\alpha(t') \,,$$

[Fewster, 2000]

$$\langle :\rho: \circ \gamma \rangle_\psi (f^2) \geq -\int_0^\infty \frac{\mathrm{d}\alpha}{\pi} \left[(f \otimes f)(\gamma \otimes \gamma)^* (Q \otimes Q\, \tilde{H}_k) \right]^\wedge (-\alpha, \alpha) \,,$$

[Fewster, Smith, 2007]

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But what are these lower bounds?

Examples of QEIs

For minimally coupled scalars on Minkowski spacetime

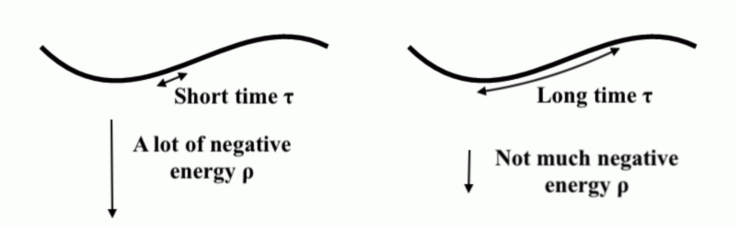
$$\int dt\, f^2 \langle :T_{\mu\nu} U^\mu U^\nu: \rangle_\psi \geq -\frac{1}{16\pi^2} \int f''(t)^2 dt$$

Examples of QEIs

For minimally coupled scalars on Minkowski spacetime

$$\frac{1}{\tau}\int dt\, f^2\langle :T_{\mu\nu}U^\mu U^\nu:\rangle_\psi \geq -\frac{C}{\tau^4}$$

[Ford, Roman, 1995], [Fewster, Eveson, 1998]



Examples of QEIs

For minimally coupled scalars on spacetimes with small curvature

$$\begin{aligned}\int_\gamma dt\, f(t)^2 \langle T^{\text{ren}}_{\mu\nu} U^\mu U^\nu \rangle_\psi (t,0) \quad &\geq \quad -\frac{1}{16\pi^2}\left\{ I_1 + \frac{5}{6} R_{\text{max}} J_2 \right. \\ &+ \quad R''_{\text{max}} \left[\frac{23}{30} J_3 + \left(\frac{43}{40} + 16\pi^2 (24|a| + 11|b|)\right) J_4\right] \\ &+ \quad \left. R'''_{\text{max}} \left[\frac{163\pi + 14}{96\pi} J_5 + \frac{7(2\pi+1)}{192\pi}(4J_6 + J_7)\right] \right\} .\end{aligned}$$

[E-AK, Olum, 2015] The curvature is considered bounded in the following sense

$$|R_{ab}| \leq R_{\text{max}} \qquad |R_{ab,cd}| \leq R''_{\text{max}} \qquad |R_{ab,cde}| \leq R'''_{\text{max}} \,.$$

Examples of QEIs

Other derivations:

- State-dependent bounds for the non-minimally coupled field [Fewster, Osterbrink, 2000], [Fewster, E-AK, 2018]
- Results on Maxwell, Proca and Dirac fields [Fewster, Pfenning, 2003], [Fewster, Mistry, 2003]
- Results for two-dimensional CTFs [Fewster, Hollands, 2005]
- Two-dimensional integrable interacting QFTs [Bostelmann, Cadamuro, Mandrysch, 2023]

Null QEIs

Can we have similar bounds for null geodesics?

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The counterexample

Considered a sequence of vacuum-plus-two-particle states in which the three-momenta of excited modes are unbounded and become more and more parallel to the spatial part of the null vector ℓ^μ. [Fewster, Roman, 2002]

SNEC

Idea

In quantum field theory there is often an ultraviolet cutoff $\ell_{\rm UV}$ which restricts the three-momenta. We can write $G_N \lessapprox \ell_{\rm UV}^2$.

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$$\int d\lambda \langle :T_{\mu\nu}:\ell^\mu \ell^\nu \rangle_\psi f^2 \geq -\frac{4B}{G_N} \|f'\|^2$$

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Numerical constant that expresses the scale of the UV cutoff

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- $\ell_{\mathrm{UV}} \approx$ Planck length $\to$ B order 1 $\to$ A lot of negative energy allowed
- $\ell_{\mathrm{UV}} \gg$ Planck length $\to$ B small $\to$ A little negative energy allowed

DSNEC

SNEC bound dependent on the cutoff. Can we do better? Integrate over two null directions!

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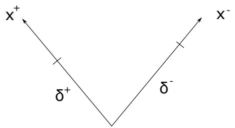

$$\int d^2x^{\pm} f^2(x^{\pm})\langle T_{--}\rangle_\psi \geq -\frac{\mathcal{N}}{(\delta^+)^{n/2-1}(\delta^-)^{n/2+1}}$$

DSNEC

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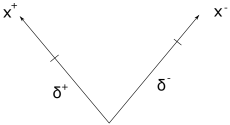

$$\int d^2x^{\pm} f^2(x^{\pm})\langle T_{--}\rangle_\psi \geq -\frac{\mathcal{N}}{(\delta^+)^{n/2-1}(\delta^-)^{n/2+1}}$$

- $\mathcal{N}$: in general depends on number of dimensions, mass and curvature(*)
- The bound diverges for $\delta^{\pm} \to 0$
- There are no theory-dependent parameters as the ℓ_{UV}
- Derivations for non-minimally coupled fields and large N CFTs [Fliss, Freivogel, E-AK, Pardo-Santos, 2023, 2024]

The average null energy condition

If we take the limit $\delta^+ \to 0$ and $\delta^- \to \infty$ while holding $\delta^+\delta^- \equiv \alpha^2$ fixed we get the averaged null energy condition (ANEC):

$$\int_{-\infty}^{\infty} dx^- \langle T_{--} \rangle_\psi \geq 0 \, .$$

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Semiclassical proofs of ANEC:

- [Flanagan, Wald, 1996] studied spacetimes perturbatively close to Minkowski and provided the first proof of ANEC incorporating backreaction.
- [Faulkner et al.,2016] have argued that the ANEC can be derived directly using modular Hamiltonians for a range of interacting QFTs on Minkowski spacetime.
- Using a variation of and argument developed in [Fewster, Olum, Pfenning, 2006], [E-AK, Olum, 2015] proved the achronal ANEC in spacetimes with small curvature

Learn more

- Curiel, Erik. “A primer on energy conditions.” Towards a theory of spacetime theories (2014) ArXiv: 1405.0403
- Kontou, Eleni-Alexandra, and Ko Sanders. “Energy conditions in general relativity and quantum field theory.”(2020) ArXiv: 2003.01815
- Fewster, Christopher J. “Lectures on quantum energy inequalities.” (2012) ArXiv:1208.5399

Singularities

What is a singularity?

- Intuitive definition: A “place” where the curvature diverges.

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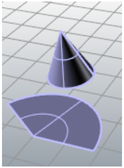
- Problems: Except in highly symmetrical cases (e.g Schwarzschild) we cannot represent the singularity as "place" since the metric is not defined there. The divergence of curvature scalars doesn't cover all singularity cases.



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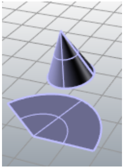

Definition

A spacetime is singular if it possesses at least one incomplete geodesic.

Singularity theorems structure

1. **The initial or boundary condition**
   There exists a trapped surface (null geodesics) or a spatial slice with negative expansion (timelike goedesics)

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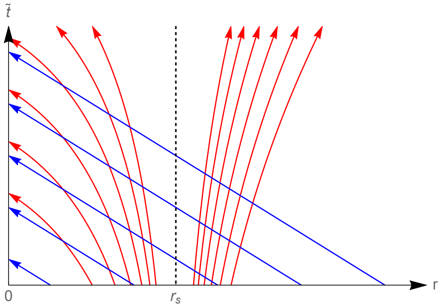

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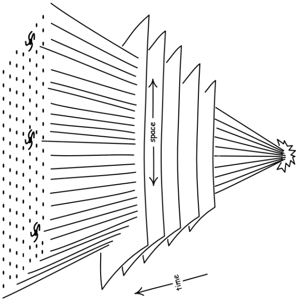

Singularity theorems structure

2. **The energy condition**
   Restriction on the stress-energy tensor expressing "physical" properties of matter.
   Null geodesics: Geometric form of the NEC → Null Convergence Condition $R_{\mu\nu}\ell^{\mu}\ell^{\nu} \geq 0$
   Timelike geodesics: Geometric form of the (SEC) → Timelike Convergence Condition $R_{\mu\nu}U^{\mu}U^{\nu} \geq 0$

Singularity theorems structure

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   Null geodesics: Geometric form of the NEC → Null Convergence Condition $R_{\mu\nu}\ell^{\mu}\ell^{\nu} \geq 0$
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3. **Causality condition**
   There is a Cauchy surface: spacelike hypersurface which intersects causal geodesics once and only once

Singularity theorems structure

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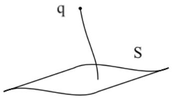
Proof structure

1. Initial condition: Geodesics start focusing
2. Energy condition: Focusing continues
3. Causality condition: No focal points

⇒ Geodesic incompleteness

Focal points



Timelike geodesic normal to spacelike hypersurface S

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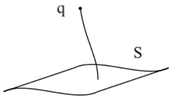

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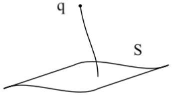

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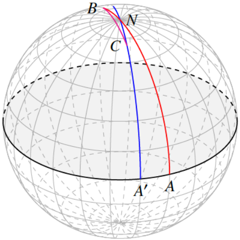

- N: focal point
- AN equal to $A'N$
- ANB longer than $A'CB$

The index form

Problem

Maximise proper time among timelike curves joining surface S to point q.

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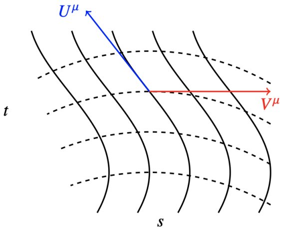
$$U^\mu = \frac{\partial \gamma_s^\mu}{\partial t}, \qquad V^\mu = \frac{\partial \gamma_s^\mu}{\partial s}$$



$$L[\gamma] = \int_0^t |\dot\gamma(t)|\, dt, \qquad |\dot\gamma(t)| = \sqrt{g_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu}$$

The index form

The first variation of length

$$\left.\frac{dL[\gamma_s]}{ds}\right|_{s=0} = 0$$

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Focal point test

$I[V] \geq 0$ for some $V^\mu \implies \exists$ focal point in $(0, t]$

Hawking's singularity theorem

Pick a V: Let e_i with $i = 1, 2, \ldots n$ be an orthonormal basis of $T_{\gamma(0)}S$, with $e_0^\mu = U^\mu$. Then, take f a smooth function with $f(0) = 1$ and $f(t) = 0$, so that $V_i = fe_i$ and sum over i

$$\sum_{i=1}^{n} I[fe_i] = -\int_0^t \left((n-1)\dot{f}^2 - f^2 R_{\mu\nu} U^\mu U^\nu\right) dt - \underbrace{K}_{Expansion} |_{\gamma(0)} \geq 0$$

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Condition for the formation of a focal point

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- Choose a function $f(t) = 1 - t/\tau$

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$$\overbrace{\int_0^\tau}^{(n-1)/\tau} \left((n-1)\dot{f}^2 - \overbrace{f^2 R_{\mu\nu} U^\mu U^\nu}^{+}\right) dt \leq \overbrace{-K}^{|K|}$$

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The causality condition implies the timelike geodesics have no focal points and thus they have length at most τ.

Penrose's singularity theorem

Differences between timelike and null case

- Action integral instead of length
- Co-dimension 2 hypersurface instead of Cauchy surface
- Mean normal curvature instead of expansion
- The affine parameter is fixed by requiring $K^\mu d\gamma_\mu / d\lambda = 1$ on S

Penrose's singularity theorem

$$\overbrace{\int_0^\ell}^{(n-2)/\ell}\left((n-2)f'^2 - \overbrace{f^2 R_{\mu\nu}\ell^\mu\ell^\nu}^{+}\right)d\lambda \leq \overbrace{-(n-2)K}^{(n-2)|K|}$$

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- Step 3: Use the semiclassical Einstein equation

$$8\pi G_N \langle {:}T_{\mu\nu}{:} U^\mu U^\nu \rangle_\psi = R_{\mu\nu} U^\mu U^\nu$$

Singularity theorems with weaker conditions

Theorem [Fewster, E-AK, 2019]

1. Energy condition

$$\int_0^\tau f^2 R_{\mu\nu} U^\mu U^\nu \, dt \geq -Q_m \|f^{(m)}\|^2 - Q_0 \|f\|^2$$

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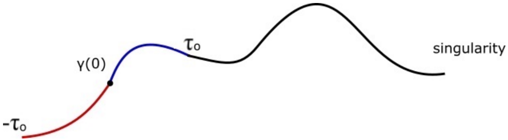

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Comments

- Same for Penrose's theorem
- Still classical! Condition only *inspired* by QEIs

Idea of the proof

Condition for the formation of a focal point

$$\int_0^\tau \left((n-1)\dot{f}^2 - f^2 R_{\mu\nu} U^\mu U^\nu\right) dt \leq -K \,, f(0) = 1 \,, f(\tau) = 0$$

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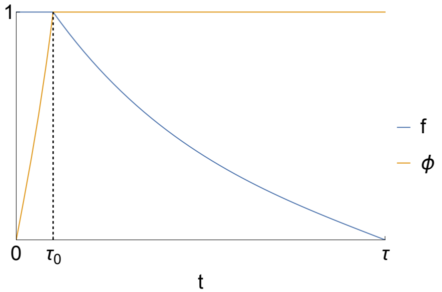

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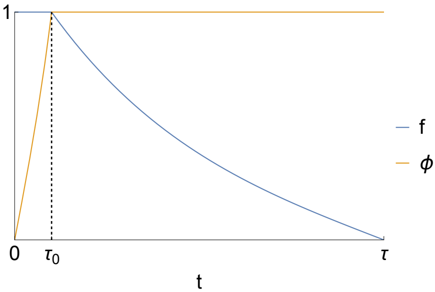

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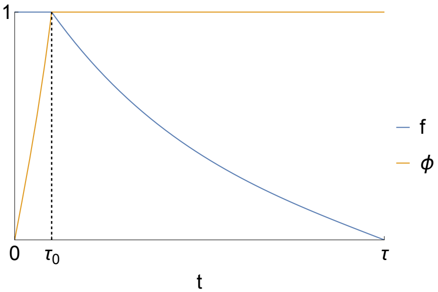

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The problems (timelike)

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Problems

- State dependence
- Only valid for Minkowski

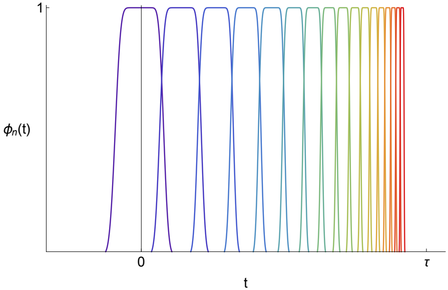
Solutions (timelike)

- State dependence: pick a class of states where

$$\left|\langle : \Phi^2 : \circ \gamma \rangle_\psi\right| \leq \phi_{\max}^2 \,,$$

- Only valid for Minkowski: Series of functions with compact support

$$f^2 = \sum_{i=1}^{n} (f\phi_n)^2$$



Solutions (timelike)

$$\int_\gamma \langle :\rho_U: \rangle_\psi f^2 dt \geq -Q_m |||f|||^2 - Q_0 ||f||^2$$

$$|||f|||^2 \equiv \sum_j^m c_j(T_0, \tau) ||f^{(j)}||^2 , \quad Q_m = \frac{\hbar S_{2m-2}}{(2\pi)^{2m-2}} , \quad \text{and} \quad Q_0 = \frac{4\pi M^2 \phi_{\max}^2}{m-1}$$

- T_0: Curvature scale
- τ: Maximum time for singularity
- M: Mass of the field
- m: $n/2$
- $\phi^2_{\max}$: Maximum field value

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Solutions?

Could we have a singularity theorem with DSNEC?

Timelike semiclassical singularity theorem

Assumptions:

- Minimally coupled scalar quantum field
- Bounded field value ϕ_{max}
- Geodesic with *controlled* curvature
- Semiclassical Einstein equation

$$8\pi G_N \langle :\rho_U: \rangle = R_{\mu\nu} U^\mu U^\nu$$

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2. The initial extrinsic curvature of S satisfies

$$K \leq -\nu(M, \phi_{\max}, T_0, \tau_0, \tau)$$

3. There exists a Cauchy surface.

$\Rightarrow$ The spacetime is timelike geodesically incomplete.

Cosmological application

- We use the ΛCDM model and data from [PLANCK, 2018]: $\Omega_{m0} = 0.31$ and $\Omega_{\Lambda 0} = 0.69$

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Parameter $\phi_{\max}$ is determined using the square root of the Wick square in a Minkowski spacetime KMS state of a temperature that is 1% of the reduced Compton temperature T_{Compton} which defines a scale beyond which the model cannot be trusted.

$$\phi_{\max}^2 \sim 10^{-2} \frac{c^4}{G_N} (M\ell_{\text{Pl}})^2 K_1(100)$$

Cosmological application

Particle	optimal τ in s	ν_* in s^{-1}	$\min T_0$ in s
Pion	2.02×10^{20}	3.57×10^{-20}	1.05×10^{-10}
Proton	4.16×10^{18}	1.73×10^{-18}	1.51×10^{11}
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- The SEC was last satisfied when $t_* = 2.41 \times 10^{17}$s,
  $H_* = 3.14 \times 10^{-18}\ \mathrm{s}^{-1}$

Null semiclassical singularity theorem

Assumptions:

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2. The mean normal curvature of P satisfies

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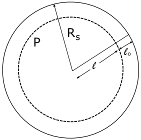
3. There exists a non-compact Cauchy surface.

$\Rightarrow$ The spacetime is null geodesically incomplete.

Black hole application

Toy model of evaporating black holes



Affine distance $\rightarrow$ Coordinate distance

- $\ell \rightarrow yR_s$
- $\ell_0 \rightarrow xR_s$

Strategy: compare K of Schwarzschild geometry to $\nu(B, \ell_0, \ell)$ from theorem

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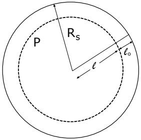

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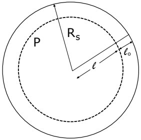

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Strategy: compare K of Schwarzschild geometry to $\nu(B, \ell_0, \ell)$ from theorem
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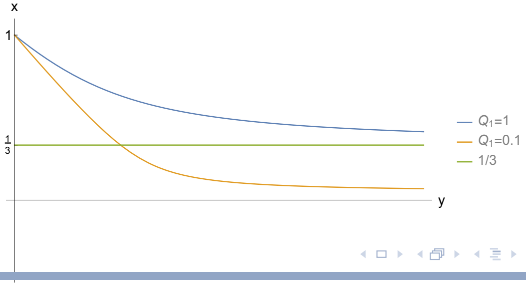
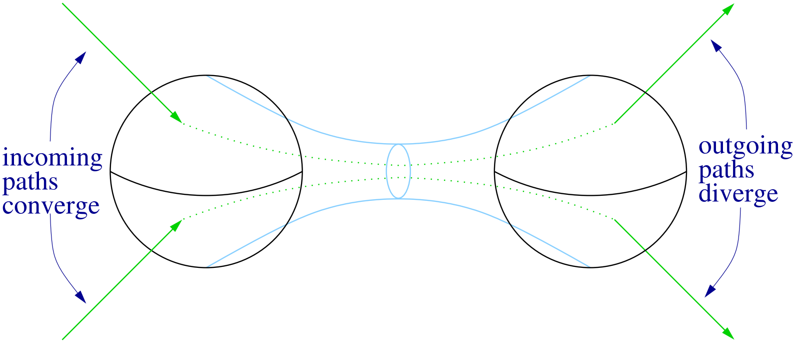


Learn more

- Senovilla, José MM. "Singularity theorems and their consequences." ArXiv: 1801.04912 (1998)
- O'Neill, B. "Semi-Riemannian geometry with applications to relativity." Pure and Applied Mathematics/Academic Press, Inc (1983)
- Fewster, Christopher, and Kontou Eleni-Alexandra, "Singularity theorems with weakened energy conditions" In preparation (2025)

Exotic spacetimes

Wormholes and energy conditions



Wormholes and energy conditions

- Wormholes violate pointwise energy conditions [Tipler, 1977]
- [Morris, Thorne, Yurtsever, 1988] showed that the wormhole's stress-energy tensor must violate the average weak energy condition
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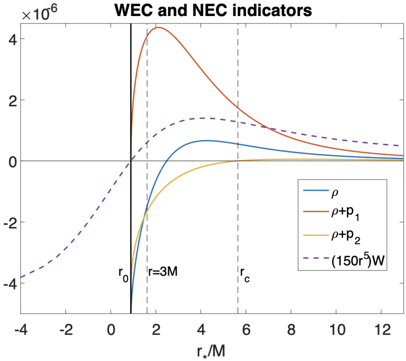
> Constructing and sustaining an (asymptotically flat) wormhole requires ANEC violation

ANEC violations

1. Violations of the ANEC on chronal null geodesics.
2. Violations of the non-self consistent ANEC, meaning that it doesn't satisfy the (classical or semiclassical) Einstein equation.
3. Violations of the order of Planck scale.

ANEC violations

1. A quantum scalar field in a Schwarzschild spacetime around a black hole [Visser, 1996], [Levi, Ori, 2016]



ANEC violations

2. Conformal transformation [Visser, 1994]

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \,,$$

$$\tilde{T}^{\mu}{}_{\nu} = \Omega^{-4}\left(T^{\mu}{}_{\nu} - 8\alpha Z^{\mu}{}_{\nu} \ln \Omega\right),$$

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$$8\alpha \ln \Omega \underbrace{\int_\gamma d\lambda \, Z^{\mu}{}_{\nu} \ell_\mu \ell^\nu}_{J_\gamma} > \int_\gamma d\lambda \, T^{\mu}{}_{\nu} \ell_\mu \ell^\nu \,,$$

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For a scalar field $\alpha = 1/(2880\pi^2)$ so:

- $J_\gamma < 0$: enormous contraction
- $J_\gamma > 0$: enormous dilation

$\Rightarrow$ Consistency with the (semiclassical) Einstein equation cannot be imposed.

ANEC violations

3. Violations that occur of the order of Planck length: outside the scope of semiclassical gravity.
   [Flanagan, Wald, 1996]: Calculated the ANEC integral in a perturbative way. Non-negative if the integral is transversely smeared over a few Planck lengths:
   $$S(\mathbf{x}) \propto \frac{1}{1+\frac{\mathbf{x}^4}{\Lambda_T}}$$
   Λ_T: Length of the order of a few Planck lengths

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Wormholes?

1. Not asymptotic flatness
2. Not in the realm of semiclassical gravity
3. Wormholes that don't lead to causality violations? Long wormholes!

The idea of long wormholes

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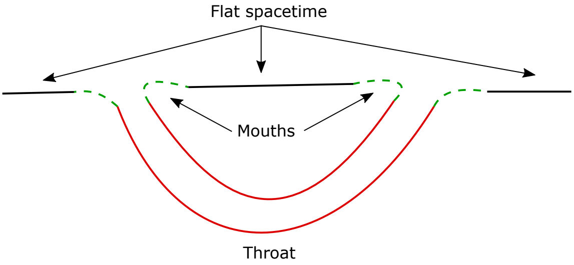
- AANEC obeyed in semiclassical gravity: are there wormholes that don't have complete achronal null geodesics?
- 'Long': it takes longer to go through the wormhole than through the ambient space.
- Those wormholes need a source of negative energy and the ANEC should be violated on the chronal null geodesics.
- [Maldacena, Milekhin, Popov, 2018] Idea: In the presence of the magnetic field, a massless charged fermion gives rise to q two-dimensional fields. The corresponding two dimensional field moves on the spatial circle and gives rise to a negative Casimir-like vacuum energy.

The wormhole of Maldacena, Milekhin and Popov



[Maldacena, Milekhin, Popov, 2018] It only uses matter predicted from the standard model and not any speculative particles: it is a solution of an Einstein–Maxwell theory with charged massless fermions

The wormhole of Maldacena, Milekhin and Popov

1. The mouths are described by the magnetically charged black hole solutions:

$$ds^2 = -\left(1 - \frac{2MG_N}{r} + \frac{r_e^2}{r^2}\right) dt^2 + \left(1 - \frac{2MG_N}{r} + \frac{r_e^2}{r^2}\right)^{-1} dr^2 + d\Omega_2^2\,,$$

where $r_e^2 = \pi q^2 \ell_{\mathrm{pl}}^2 / g^2$ and d is the distance between the two mouths

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$$ds^2 = r_e^2 \left[-(\rho^2+1) d\tau^2 + \frac{d\rho^2}{\rho^2+1} + d\Omega_2^2 \right] \,,$$

   The matching conditions are

$$\tau = \frac{t}{\ell}, \quad \rho = \frac{\ell(r-r_e)}{r_e^2}, \quad \text{with } 1 \ll \rho, \quad \frac{r-r_e}{r_e} \ll 1, \quad 1 \ll \frac{\ell}{r_e} \,,$$

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3. Asymptotically flat region

Length of the wormhole

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The maximum value of d is found by minimizing the energy of the wormhole: $d = \pi\ell/2.35$, so $\rho_0 = 4.13$ The stress-energy tensor inside the wormhole is

$$\langle T_{tt} \rangle = -\frac{q}{12\pi^2 r_e^2 \ell^2} A ,$$

SNEC constraints

The coordinate ρ is an affine parameter so SNEC becomes

$$\int_0^\infty d\rho\, f(\rho)^2 \frac{2\ell^2}{(1+\rho^2)^2} \langle T_{tt} \rangle_\omega \geq -\frac{4B}{G_N} \int_0^\infty d\rho\, f'(\rho)^2 \,.$$

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For $f(\rho)$ a normalized Gaussian function of width σ

$$C \int_0^\infty d\rho\ e^{-\frac{\rho^2}{2\sigma^2}}\ \frac{1}{(1+\rho^2)^2} \leq \frac{1}{\sigma^4} \int_0^\infty d\rho\ \frac{\rho^2}{4}\ e^{-\frac{\rho^2}{2\sigma^2}}\ , \quad C = \frac{g^2}{24B\pi^3 q} A$$

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Setting $\sigma = \rho_0 = 4.13$,

$$Bq \gtrapprox 1.3 \times 10^{-2}\,.$$

If $B = 1/32\pi^2$, we need $q \lessapprox 1$ to saturate SNEC. But $q \gg 1$ so that the wormhole is stabilized [Freivogel, E-AK, Krommydas, 2020]

DSNEC constraints

Flat metric

$$ds^2 = -r_e^2 ds^- ds^+ \, ,$$

where $s^- = \tau - \rho$ and $s^+ = \tau + \rho$ for $\rho + 1 \approx 1$.

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We have $\delta^- = \ell$ and $\delta^+ = r_e$, the AdS_2 radius. To violate the DSNEC we need $q \gtrsim r_e/\ell$. As $r_e \ll \ell$ the DSNEC is easily violated.

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$$\langle T_{--} \rangle_{\text{SNEC}} \geq -\frac{4B}{\boxed{\ell_{\text{pl}}^2 \ell^2}} , \qquad \langle T_{--} \rangle_{\text{DSNEC}} \geq -\frac{N}{\boxed{\ell^3 r_e}} .$$

[E-AK, 2024]

Learn more

- Visser, Matt. "Lorentzian wormholes. from Einstein to Hawking." Woodbury (1995)
- Kontou, Eleni-Alexandra. "Wormhole restrictions from quantum energy inequalities." ArXiv: 2405.05963 (2024)

Conclusions and questions

What to remember

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- Classical energy conditions are violated in QFT but average restrictions still hold.
- Singularities are generally predicted in the context of semiclassical gravity.
- The self-consistent achronal averaged null energy condition seems to be fundamental in semiclassical gravity and it is sufficient to rule out causality violations.

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- Why are null QEIs different than the timelike ones?
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- Are there meaningful constraints to long wormholes?