Renormalized stress-energy tensor

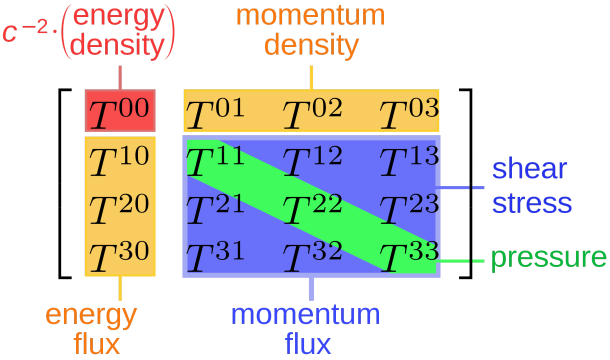
Elizabeth Winstanley

School of Mathematical and Physical Sciences
The University of Sheffield

Stress-energy tensor (SET)

Classical Einstein equations

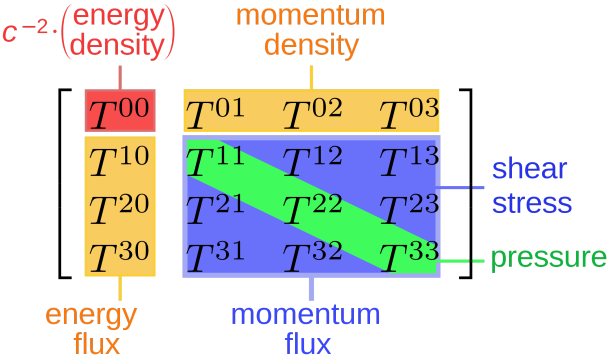
$$G_{\lambda\rho} + \Lambda g_{\lambda\rho} = 8\pi T_{\lambda\rho}$$



Stress-energy tensor (SET) expectation value

Semi-classical Einstein equations

$$G_{\lambda\rho} + \Lambda g_{\lambda\rho} = 8\pi \langle \hat{T}_{\lambda\rho} \rangle$$



Quantum scalar field $\hat{\Phi}$

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Field equation

$$\left[\nabla_\lambda \nabla^\lambda - \mu^2 - \xi R \right] \Phi = 0$$

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- μ – scalar field mass

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- ξ – coupling to Ricci scalar curvature R

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- ξ – coupling to Ricci scalar curvature R
  - $\xi = 0$ – minimal coupling

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- μ – scalar field mass
- ξ – coupling to Ricci scalar curvature R
  - $\xi = 0$ – minimal coupling
  - $\xi = \frac{1}{6}$ – conformal coupling

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Field operators

Quantum scalar field $\hat{\Phi}$

Field equation

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Field operators

- Stress-energy tensor $\hat{T}_{\lambda\rho}$

Quantum scalar field $\hat{\Phi}$

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$$\left[\nabla_\lambda \nabla^\lambda - \mu^2 - \xi R \right] \Phi = 0$$

Field operators

- Stress-energy tensor $\hat{T}_{\lambda\rho}$
- Square of the field $\hat{\Phi}^2$

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Vacuum polarization $\langle \hat{\Phi}^2 \rangle$

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- Simplest nontrivial expectation value

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Vacuum polarization $\langle \hat{\Phi}^2 \rangle$

- Simplest nontrivial expectation value
- Simpler to compute than SET

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Field operators

- Stress-energy tensor $\hat{T}_{\lambda\rho}$
- Square of the field $\hat{\Phi}^2$

Vacuum polarization $\langle \hat{\Phi}^2 \rangle$

- Simplest nontrivial expectation value
- Simpler to compute than SET
- Has some physical features in common with SET

SET renormalization on Minkowski space-time

QFT on Minkowski space-time: Mode approach

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Minkowski space-time

$$ds^2 = \eta_{\lambda\rho}\, dx^\lambda\, dx^\rho = -dt^2 + dx^2 + dy^2 + dz^2$$

QFT on Minkowski space-time: Mode approach

Minkowski space-time

$$ds^2 = \eta_{\lambda\rho}\, dx^\lambda\, dx^\rho = -dt^2 + dx^2 + dy^2 + dz^2$$

Klein-Gordon equation

$$\left[\partial_\lambda \partial^\lambda - \mu^2\right] \Phi = 0$$

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Plane wave solutions

$$\phi_{\mathbf{p}} = \frac{1}{\sqrt{16\pi^3 \left|\omega\right|}} \exp\left(-i\omega t\right) \exp\left(i\mathbf{p}.\mathbf{x}\right)$$

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Plane wave solutions

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- ω – frequency
- $\mathbf{p}$ – momentum

$$\omega^2 - |\mathbf{p}|^2 = \mu^2$$

Canonical quantization

Canonical quantization

Classical scalar field

$$\Phi = \int_{\omega > 0} \mathrm{d}^3\mathbf{p} \, \left[a_{\mathbf{p}} \phi_{\mathbf{p}} + a_{\mathbf{p}}^{\dagger} \phi_{\mathbf{p}}^{*} \right]$$

Canonical quantization

Quantum scalar field

$$\hat{\Phi} = \int_{\omega>0} \mathrm{d}^3\mathbf{p} \, \left[\hat{a}_{\mathbf{p}} \phi_{\mathbf{p}} + \hat{a}^{\dagger}_{\mathbf{p}} \phi^{*}_{\mathbf{p}} \right]$$

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$$\left[\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{p}'} \right] = \delta(\mathbf{p} - \mathbf{p}') \qquad \left[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'} \right] = 0 \qquad \left[\hat{a}^{\dagger}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{p}'} \right] = 0$$

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Vacuum state

$$\hat{a}_{\mathbf{p}} |0\rangle = 0$$

Minkowski space-time renormalization: Part 1

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Vacuum polarization

$$\langle 0|\hat{\Phi}^2|0\rangle$$

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Vacuum polarization

$$\langle 0|\hat{\Phi}^2|0\rangle = \int d^3\mathbf{p}\, d^3\mathbf{p}' \left\langle 0 \left| \left[\hat{a}_{\mathbf{p}'}\phi_{\mathbf{p}'} + \hat{a}^\dagger_{\mathbf{p}'}\phi^*_{\mathbf{p}'}\right] \left[\hat{a}_{\mathbf{p}}\phi_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\phi^*_{\mathbf{p}}\right] \right| 0 \right\rangle$$

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Vacuum polarization

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$$= \int \mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}'\,\phi_{\mathbf{p}'}\phi^*_{\mathbf{p}}\langle 0|\hat{a}_{\mathbf{p}'}\hat{a}^\dagger_{\mathbf{p}}|0\rangle = \int \mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}'\,\phi_{\mathbf{p}'}\phi^*_{\mathbf{p}}\langle 0|\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}'} + \delta(\mathbf{p}' - \mathbf{p})|0\rangle$$

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$$\begin{aligned}\langle 0|\hat{\Phi}^2|0\rangle &= \int d^3\mathbf{p}\, d^3\mathbf{p}' \left\langle 0 \left| \left[\hat{a}_{\mathbf{p}'}\phi_{\mathbf{p}'} + \hat{a}^\dagger_{\mathbf{p}'}\phi^*_{\mathbf{p}'}\right] \left[\hat{a}_{\mathbf{p}}\phi_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\phi^*_{\mathbf{p}}\right] \right| 0 \right\rangle \\ &= \int d^3\mathbf{p}\, d^3\mathbf{p}'\, \phi_{\mathbf{p}'}\phi^*_{\mathbf{p}} \langle 0|\hat{a}_{\mathbf{p}'}\hat{a}^\dagger_{\mathbf{p}}|0\rangle = \int d^3\mathbf{p}\, d^3\mathbf{p}'\, \phi_{\mathbf{p}'}\phi^*_{\mathbf{p}} \langle 0|\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}'} + \delta(\mathbf{p}'-\mathbf{p})|0\rangle \\ &= \int d^3\mathbf{p}\, \left|\phi_{\mathbf{p}}\right|^2\end{aligned}$$

Minkowski space-time renormalization: Part 1

Vacuum polarization

$$\begin{aligned}\langle 0|\hat{\Phi}^2|0\rangle &= \int d^3\mathbf{p}\, d^3\mathbf{p}' \left\langle 0 \left| \left[\hat{a}_{\mathbf{p}'}\phi_{\mathbf{p}'} + \hat{a}^\dagger_{\mathbf{p}'}\phi^*_{\mathbf{p}'}\right] \left[\hat{a}_{\mathbf{p}}\phi_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\phi^*_{\mathbf{p}}\right] \right| 0 \right\rangle \\ &= \int d^3\mathbf{p}\, d^3\mathbf{p}'\, \phi_{\mathbf{p}'}\phi^*_{\mathbf{p}} \langle 0|\hat{a}_{\mathbf{p}'}\hat{a}^\dagger_{\mathbf{p}}|0\rangle = \int d^3\mathbf{p}\, d^3\mathbf{p}'\, \phi_{\mathbf{p}'}\phi^*_{\mathbf{p}} \langle 0|\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}'} + \delta(\mathbf{p}'-\mathbf{p})|0\rangle \\ &= \int d^3\mathbf{p}\, \left|\phi_{\mathbf{p}}\right|^2 = \frac{1}{16\pi^3}\int d^3\mathbf{p}\, \frac{1}{|\omega|}\end{aligned}$$

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$$\begin{aligned}
\langle 0|\hat{\Phi}^2|0\rangle &= \int d^3\mathbf{p}\, d^3\mathbf{p}' \left\langle 0 \left| \left[\hat{a}_{\mathbf{p}'}\phi_{\mathbf{p}'} + \hat{a}^\dagger_{\mathbf{p}'}\phi^*_{\mathbf{p}'} \right] \left[\hat{a}_{\mathbf{p}}\phi_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\phi^*_{\mathbf{p}} \right] \right| 0 \right\rangle \\
&= \int d^3\mathbf{p}\, d^3\mathbf{p}'\, \phi_{\mathbf{p}'}\phi^*_{\mathbf{p}} \langle 0|\hat{a}_{\mathbf{p}'}\hat{a}^\dagger_{\mathbf{p}}|0\rangle = \int d^3\mathbf{p}\, d^3\mathbf{p}'\, \phi_{\mathbf{p}'}\phi^*_{\mathbf{p}} \langle 0|\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}'} + \delta(\mathbf{p}' - \mathbf{p})|0\rangle \\
&= \int d^3\mathbf{p}\, \left|\phi_{\mathbf{p}}\right|^2 = \frac{1}{16\pi^3} \int d^3\mathbf{p}\, \frac{1}{|\omega|} \to \infty
\end{aligned}$$

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Minkowski space-time renormalization: Part 1

Vacuum polarization

$$\begin{aligned}
\langle 0|\hat{\Phi}^2|0\rangle &= \int \mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}' \left\langle 0 \middle| \left[\hat{a}_{\mathbf{p}'}\phi_{\mathbf{p}'} + \hat{a}^\dagger_{\mathbf{p}'}\phi^*_{\mathbf{p}'}\right]\left[\hat{a}_{\mathbf{p}}\phi_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\phi^*_{\mathbf{p}}\right] \middle| 0 \right\rangle \\
&= \int \mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}'\,\phi_{\mathbf{p}'}\phi^*_{\mathbf{p}}\langle 0|\hat{a}_{\mathbf{p}'}\hat{a}^\dagger_{\mathbf{p}}|0\rangle = \int \mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}'\,\phi_{\mathbf{p}'}\phi^*_{\mathbf{p}}\langle 0|\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}'} + \delta(\mathbf{p}'-\mathbf{p})|0\rangle \\
&= \int \mathrm{d}^3\mathbf{p}\,\left|\phi_{\mathbf{p}}\right|^2 = \frac{1}{16\pi^3}\int \mathrm{d}^3\mathbf{p}\,\frac{1}{|\omega|} \to \infty
\end{aligned}$$

Normal ordering

Annihilation operators are *always* to the right of creation operators

Minkowski space-time renormalization: Part 1

Vacuum polarization

$$\langle 0|\hat{\Phi}^2|0\rangle = \int \mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}' \left\langle 0\left|\left[\hat{a}_{\mathbf{p}'}\phi_{\mathbf{p}'} + \hat{a}^\dagger_{\mathbf{p}'}\phi^*_{\mathbf{p}'}\right]\left[\hat{a}_{\mathbf{p}}\phi_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\phi^*_{\mathbf{p}}\right]\right|0\right\rangle$$
$$= \int \mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}'\,\phi_{\mathbf{p}'}\phi^*_{\mathbf{p}}\langle 0|\hat{a}_{\mathbf{p}'}\hat{a}^\dagger_{\mathbf{p}}|0\rangle = \int \mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}'\,\phi_{\mathbf{p}'}\phi^*_{\mathbf{p}}\langle 0|\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}'} + \delta(\mathbf{p}'-\mathbf{p})|0\rangle$$
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Normal ordering

Annihilation operators are *always* to the right of creation operators

Renormalized vacuum SET

$$\langle 0|\hat{T}_{\lambda\rho}|0\rangle := 0$$

QFT on Minkowski space-time: Green function approach

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Vacuum Feynman Green function

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Vacuum Feynman Green function

$$-\mathrm{i}G_{\mathrm{F}}(x,x')$$

QFT on Minkowski space-time: Green function approach

Vacuum Feynman Green function

$\mathsf{T}\left[\hat{\Phi}(x),\hat{\Phi}(x')\right]$ – time-ordered product

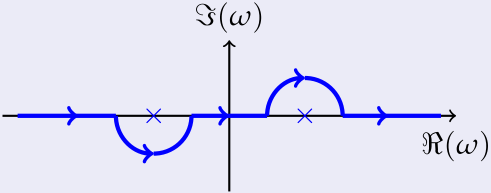
$$-\mathrm{i}G_{\mathrm{F}}(x,x') = \langle 0|\mathsf{T}\left[\hat{\Phi}(x),\hat{\Phi}(x')\right]|0\rangle$$

QFT on Minkowski space-time: Green function approach

Vacuum Feynman Green function

$\mathsf{T}\left[\hat{\Phi}(x), \hat{\Phi}(x')\right]$ – time-ordered product

$$
\begin{aligned}
-\mathrm{i}G_{\mathrm{F}}(x,x') &= \langle 0|\mathsf{T}\left[\hat{\Phi}(x), \hat{\Phi}(x')\right]|0\rangle \\
&= -\frac{\mathrm{i}}{16\pi^4}\int \mathrm{d}\omega\, \mathrm{d}^3\mathbf{p}\, \frac{e^{-\mathrm{i}\omega(t-t')}e^{\mathrm{i}\mathbf{p}.(\mathbf{x}-\mathbf{x}')}}{-\omega^2+|\mathbf{p}.\mathbf{p}|+\mu^2}
\end{aligned}
$$



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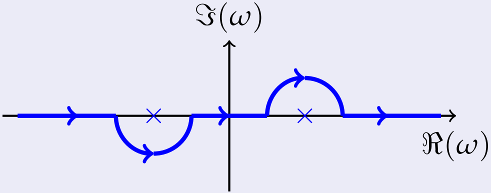

Klein-Gordon equation

$$\left[\partial_\lambda \partial^\lambda - \mu^2\right]\left[-\mathrm{i}G_{\mathrm{F}}(x,x')\right] = -\delta(x-x')$$

QFT on Minkowski space-time: Green function approach

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Feynman Green function

$$-\mathrm{i}G_{\mathrm{F}}(x,x') = -\frac{\mathrm{i}\mu}{8\pi\sqrt{2\sigma}}\mathsf{H}_1^{(2)}\left(\mu\sqrt{2\sigma}\right)$$

QFT on Minkowski space-time: Green function approach

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- Synge world function

$$2\sigma(x,x') = \eta_{\lambda\rho}\left(x^\lambda - x^{\lambda'}\right)\left(x^\rho - x^{\rho'}\right)$$

QFT on Minkowski space-time: Green function approach

Klein-Gordon equation

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Feynman Green function

$$-\mathrm{i}G_{\mathrm{F}}(x,x') = -\frac{\mathrm{i}\mu}{8\pi\sqrt{2\sigma}}\mathsf{H}_1^{(2)}(\mu\sqrt{2\sigma})$$
$$= \frac{1}{8\pi^2\sigma} - \frac{\mu^2}{8\pi^2}\log\left[2\mu^2\sigma\right] + \frac{\mu^2}{16\pi^2}\left[1 - 2\mathsf{C} + \mathrm{i}\pi\right] + \dots$$

- Synge world function

$$2\sigma(x,x') = \eta_{\lambda\rho}\left(x^\lambda - x^{\lambda'}\right)\left(x^\rho - x^{\rho'}\right)$$

- Hankel function $\mathsf{H}_1^{(2)}$, Euler's constant C

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Feynman Green function

$$\begin{aligned}-\mathrm{i}G_{\mathrm{F}}(x,x') &= -\frac{\mathrm{i}\mu}{8\pi\sqrt{2\sigma}}\mathsf{H}_1^{(2)}\left(\mu\sqrt{2\sigma}\right) \\ &= \frac{1}{8\pi^2\sigma} - \frac{\mu^2}{8\pi^2}\log\left[2\mu^2\sigma\right] + \frac{\mu^2}{16\pi^2}\left[1 - 2\mathsf{C} + \mathrm{i}\pi\right] + \ldots\end{aligned}$$

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- Synge world function

$$2\sigma(x,x') = \eta_{\lambda\rho}\left(x^\lambda - x^{\lambda'}\right)\left(x^\rho - x^{\rho'}\right)$$

- Hankel function $\mathsf{H}_1^{(2)}$, Euler's constant C

QFT on Minkowski space-time: Green function approach

Klein-Gordon equation

$$\left[\partial_\lambda \partial^\lambda - \mu^2\right]\left[-\mathrm{i}G_{\mathrm{F}}(x,x')\right] = -\delta(x-x')$$

Feynman Green function

$$\begin{aligned}-\mathrm{i}G_{\mathrm{F}}(x,x') &= -\frac{\mathrm{i}\mu}{8\pi\sqrt{2\sigma}}\mathsf{H}_1^{(2)}(\mu\sqrt{2\sigma}) \\ &= \frac{1}{8\pi^2\sigma} - \frac{\mu^2}{8\pi^2}\log\left[2\mu^2\sigma\right] + \frac{\mu^2}{16\pi^2}\left[1 - 2\mathsf{C} + \mathrm{i}\pi\right] + \ldots\end{aligned}$$

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Minkowski space-time renormalization: Part 2

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Vacuum SET

$$\langle 0|\hat{T}_{\lambda\rho}|0\rangle$$

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second order differential operator $\mathcal{T}_{\lambda\rho}$

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Renormalization in Minkowski space-time

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Approaches to renormalization

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Homework

Extend this to curved space-time

Renormalization in curved space-time

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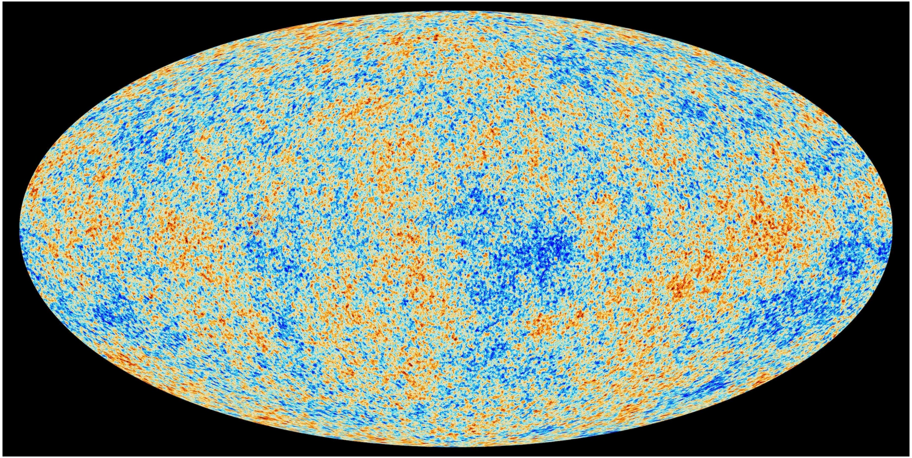
Form of this in curved space-time?

Adiabatic renormalization

Fulling & Parker *Ann. Phys.* **87** 176 (1974)
Parker & Fulling *PRD* **9** 341 (1974)
Fulling, Parker & Hu *PRD* **10** 3905 (1974)
Birrell *Proc. Roy. Soc.* **B361** 513 (1978)
Bunch *JPA* **11** 603 (1978)
Bunch *JPA* **13** 1297 (1980)
Anderson & Parker *PRD* **36** 2963 (1987)
del Rio & Navarro-Salas *PRD* **91** 064031 (2015)

Cosmological space-times



[Image: ESA and the Planck Collaboration]

Scalar field on cosmological space-times

Scalar field on cosmological space-times

Flat FLRW

$$\mathrm{d}s^2 = g_{\lambda\rho}\,\mathrm{d}x^\lambda\,\mathrm{d}x^\rho = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right]$$

Scalar field on cosmological space-times

Flat FLRW

$$ds^2 = g_{\lambda\rho}\,dx^\lambda\,dx^\rho = -dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right]$$

Scalar field modes

$$\phi_{\mathbf{p}}(t, \mathbf{x}) = h_{\mathbf{p}}(t)\exp\left(i\mathbf{p}.\mathbf{x}\right)$$

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Quantum scalar field

$$\hat{\Phi} = \int d^3\mathbf{p}\,\frac{1}{\sqrt{16\pi^3 a^3}}\left[\hat{b}_{\mathbf{p}}\phi_{\mathbf{p}} + \hat{b}^\dagger_{\mathbf{p}}\phi^*_{\mathbf{p}}\right]$$

VP on flat FLRW

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Vacuum polarization

VP on flat FLRW

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$$\langle \hat{\Phi}^2 \rangle$$

VP on flat FLRW

Vacuum polarization

$$\langle \hat{\Phi}^2 \rangle = \frac{1}{16\pi^3 a^3(t)} \int d^3\mathbf{p} \, |\phi_{\mathbf{p}}|^2$$

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Remove high frequency divergences?

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Remove high frequency divergences?

WKB approximation

$$h_{\mathbf{p}}(t) = \frac{1}{\sqrt{\Omega_{\mathbf{p}}(t)}} \exp\left[-i \int \Omega_{\mathbf{p}}(t) \, dt \right]$$

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Adiabatic expansion

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General $a(t)$

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General $a(t)$

Each time derivative adds an adiabatic order

$$\Omega_{\mathbf{p}} = \Omega_0 + \Omega_2 + \Omega_4 + \dots$$

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Adiabatic order zero

$$\Omega_0 = \sqrt{\frac{|\mathbf{p}|^2}{a(t)^2} + \mu^2} = \omega(t)$$

Adiabatic renormalization

Adiabatic renormalization

VP

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VP

$$\langle \hat{\Phi}^2 \rangle = \frac{1}{16\pi^3 a^3(t)} \int \mathrm{d}^3\mathbf{p} \quad \left| h_{\mathbf{p}}(t) \right|^2$$

Adiabatic renormalization

VP

$$\langle \hat{\Phi}^2 \rangle = \frac{1}{16\pi^3 a^3(t)} \int d^3\mathbf{p} \left[\left| h_{\mathbf{p}}(t) \right|^2 - \Omega_{\mathbf{p}}^{-1} \right]$$

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$\Omega_{\mathbf{p}}^{-1}$

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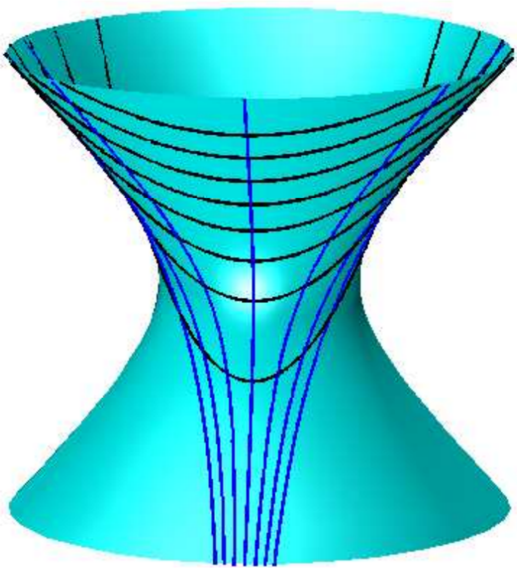
SET

Need to subtract terms up to and including adiabatic order four

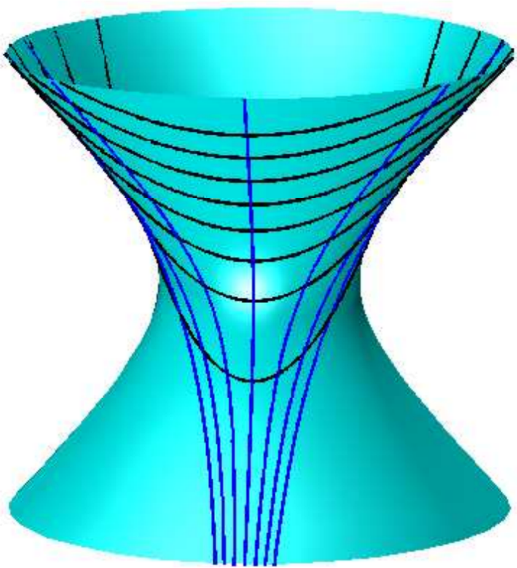
Example: de Sitter space-time



[Figure: Moschella *Sem. Poincaré* **1** 1 (2005)]

Example: de Sitter space-time

- Maximally symmetric space-time



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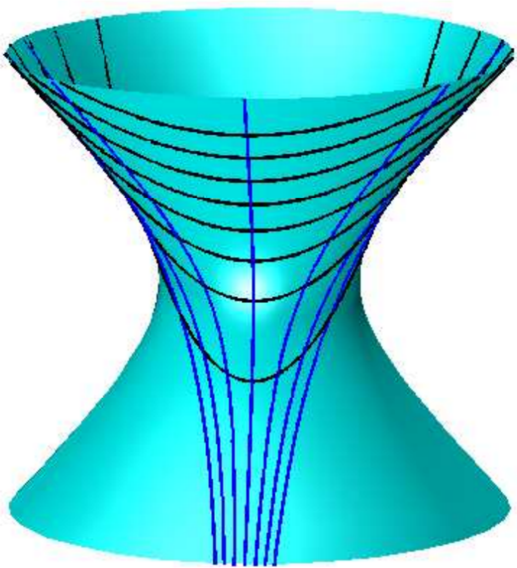

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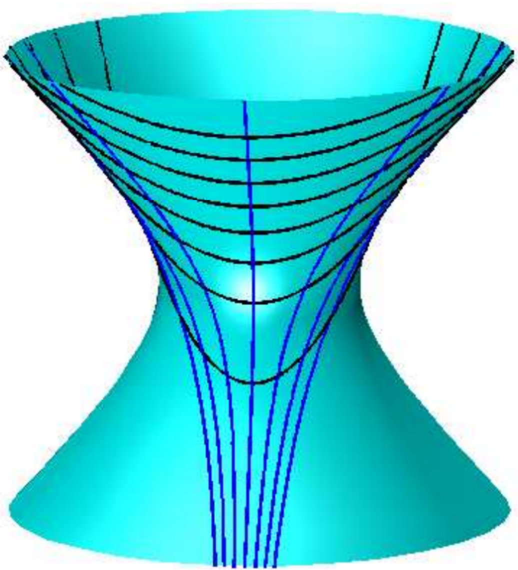

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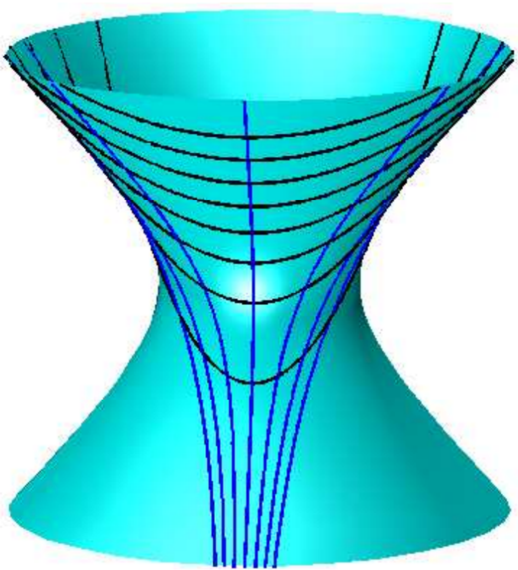
Scale factor

$$a(t) = e^{Ht} \qquad H = \text{constant}$$



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Example: de Sitter space-time

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Massless, conformally coupled scalar field modes

$$\ddot{h}_{\mathbf{p}} + \left[\frac{|\mathbf{p}|^2}{e^{2Ht}} - \frac{H^2}{4} \right] h_{\mathbf{p}} = 0$$

Solution in terms of Hankel functions

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$$\Omega_{\mathbf{p}}^2 = \frac{3\dot{\Omega}_{\mathbf{p}}^2}{4\Omega_{\mathbf{p}}^2} - \frac{\ddot{\Omega}_{\mathbf{p}}}{2\Omega_{\mathbf{p}}} + \frac{|\mathbf{p}|^2}{e^{2Ht}} - \frac{H^2}{4}$$

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$$\langle \hat{T}^{\rho}_{\lambda} \rangle = \mathrm{Diag}\{-E, P, P, P\}$$

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Homework

Extend adiabatic renormalization to

- Black hole space-times

Hadamard renormalization

DeWitt *Phys. Rept.* **19** 295 (1975)
Christensen *PRD* **14** 2490 (1976)
Wald *CMP* **54** 1 (1977)
Christensen *PRD* **17** 946 (1978)
Decanini & Folacci *PRD* **78** 044025 (2008)

Overall strategy

Stress-energy tensor operator $\hat{T}_{\lambda\rho}$

- Involves products of field operators at the same space-time point
- Expectation values are divergent

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- $\mathcal{T}_{\lambda\rho}\left[-\mathrm{i}G_{\mathrm{F}}(x,x')\right]$ finite for $x' \neq x$
- Divergences as $x' \to x$ are purely geometric and independent of the quantum state

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Renormalized expectation value

- Subtract off appropriate divergent terms $G_{\mathrm{S}}(x,x')$

$$\langle \hat{T}_{\lambda\rho}(x)\rangle_{\mathrm{ren}} = \lim_{x'\to x}\left[\mathcal{T}_{\lambda\rho}\left\{-\mathrm{i}\left[G_{\mathrm{F}}(x,x') - G_{\mathrm{S}}(x,x')\right]\right\}\right]$$

Hadamard renormalization

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Minkowski space-time

$$-\mathrm{i}G_{\mathrm{S}}(x,x') = \frac{1}{8\pi^2\sigma(x,x')} - \frac{\mu^2}{8\pi^2}\log\left[2\mu^2\sigma(x,x')\right]$$

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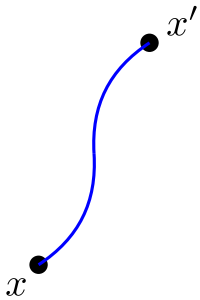

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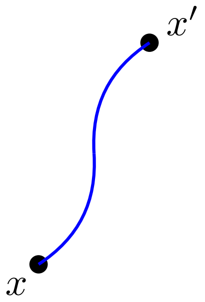

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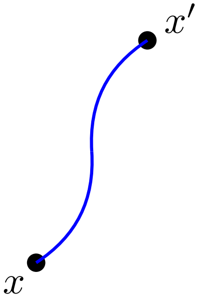

[Decanini & Folacci *PRD* **78** 044025 (2008)]

Hadamard parameters

[Decanini & Folacci *PRD* **73** 044027 (2006); *PRD* **78** 044025 (2008)
Ottewill & Wardell *PRD* **84** 104039 (2011)]

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$U(x,x')$

$$U(x,x') = \frac{\Delta(x,x')}{8\pi^2}$$

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[Visser *PRD* **47** 2395 (1993)]

[Decanini & Folacci *PRD* **73** 044027 (2006); *PRD* **78** 044025 (2008)
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$$\begin{aligned} V_1(x,x') = {} & \frac{1}{8}\mu^4 + \frac{1}{4}\left(\xi - \frac{1}{6}\right)\mu^2 R \\ & - \frac{1}{24}\left(\xi - \frac{1}{5}\right)\nabla_\lambda \nabla^\lambda R \\ & + \frac{1}{8}\left(\xi - \frac{1}{6}\right)^2 R^2 - \frac{1}{720} R_{\lambda\rho} R^{\lambda\rho} \\ & + \frac{1}{720} R_{\lambda\rho\tau\kappa} R^{\lambda\rho\tau\kappa} + \ldots \end{aligned}$$

[Decanini & Folacci *PRD* **73** 044027 (2006); *PRD* **78** 044025 (2008)

Renormalized SET

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Regularized Green function

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Trace

[Decanini & Folacci *PRD* **78** 044025 (2008)]

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$$G_{\lambda\rho} + \Lambda g_{\lambda\rho} = 8\pi \langle \hat{T}_{\lambda\rho} \rangle_{\rm ren} \qquad \Longrightarrow \qquad \nabla^{\lambda} \langle \hat{T}_{\lambda\rho} \rangle_{\rm ren} = 0$$

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Trace

$$\langle \hat{T}_{\lambda}^{\lambda} \rangle_{\rm ren}$$

[Decanini & Folacci *PRD* **78** 044025 (2008)]

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Massless $\mu = 0$, conformally coupled $\xi = \frac{1}{6}$, scalar field

$$\langle \hat{T}^{\lambda}_{\lambda} \rangle_{\text{ren}} = 2v_1$$

[Decanini & Folacci *PRD* **78** 044025 (2008)]

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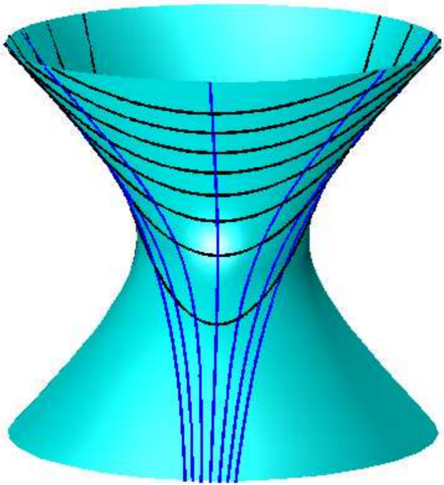
$$\langle \hat{T}_{\lambda}^{\lambda} \rangle_{\text{ren}} = 2v_1 = \frac{1}{2880\pi^2}\left[\nabla_{\lambda}\nabla^{\lambda} R - R_{\lambda\rho} R^{\lambda\rho} + R_{\lambda\rho\tau\kappa} R^{\lambda\rho\tau\kappa}\right]$$

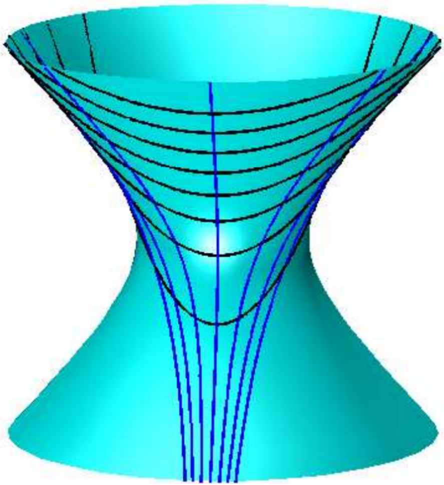
[Decanini & Folacci *PRD* **78** 044025 (2008)]

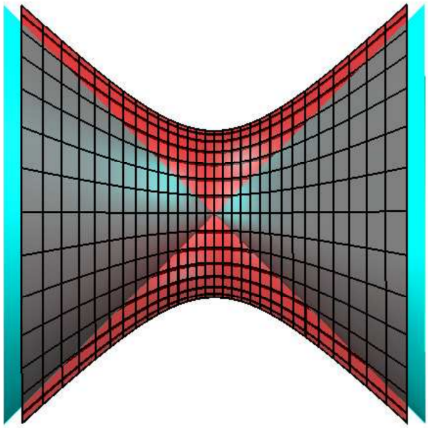
Example: de Sitter space-time



[Figures: Moschella *Sem. Poincaré* **1** 1 (2005)]

Example: (Anti-)de Sitter space-time





[Figures: Moschella *Sem. Poincaré* **1** 1 (2005)]

Example: (Anti-)de Sitter space-time

Massless, conformally coupled, scalar field

[Page *PRD* **25** 1499 (1982); Allen, Folacci & Gibbons *PLB* **189** 304 (1987)]

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Massless, conformally coupled, scalar field

Green function

$$-\mathrm{i}G_{\mathrm{F}}(x,x')$$

[Page *PRD* **25** 1499 (1982); Allen, Folacci & Gibbons *PLB* **189** 304 (1987)]

Example: (Anti-)de Sitter space-time

Massless, conformally coupled, scalar field

Green function

$$-\mathrm{i}G_{\mathrm{F}}(x,x') = \frac{\Lambda}{48\pi^2}\left[\csc\sqrt{\frac{\Lambda\sigma(x,x')}{6}}\right]^2$$

[Page *PRD* **25** 1499 (1982); Allen, Folacci & Gibbons *PLB* **189** 304 (1987)]

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$$-\mathrm{i}G_{\mathrm{S}}(x,x') = \frac{U(x,x')}{\sigma(x,x')} + V(x,x')\log\left[\frac{\sigma(x,x')}{L^2}\right]$$

[Page *PRD* **25** 1499 (1982); Allen, Folacci & Gibbons *PLB* **189** 304 (1987)]

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[Page *PRD* **25** 1499 (1982); Allen, Folacci & Gibbons *PLB* **189** 304 (1987)]

(Anti-)de Sitter space-time

Massless, conformally coupled, scalar field

[Page *PRD* **25** 1499 (1982); Allen, Folacci & Gibbons *PLB* **189** 304 (1987)]

(Anti-)de Sitter space-time

Massless, conformally coupled, scalar field

Regularized Green function

$$W(x, x')$$

[Page *PRD* **25** 1499 (1982); Allen, Folacci & Gibbons *PLB* **189** 304 (1987)]

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$$W(x,x') = \frac{\Lambda}{48\pi^2}\left[\csc\sqrt{\frac{\Lambda\sigma(x,x')}{6}}\right]^2 - \frac{1}{8\pi^2\sigma(x,x')}\left[\frac{2\Lambda\sigma(x,x')}{3}\right]^{\frac{3}{4}}\left[\csc\sqrt{\frac{2\Lambda\sigma(x,x')}{3}}\right]^{\frac{1}{4}}$$

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Renormalized SET

$$\langle \hat{T}_{\lambda\rho}\rangle_{\rm ren} = -w_{\lambda\rho} + \frac{1}{3}\nabla_\lambda\nabla_\rho w - \frac{1}{12}g_{\lambda\rho}\nabla_\kappa\nabla^\kappa w + \frac{1}{6}R_{\lambda\rho}w - g_{\lambda\rho}v_1$$

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[Page *PRD* **25** 1499 (1982); Allen, Folacci & Gibbons *PLB* **189** 304 (1987)]

Hadamard renormalization

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Homework

Develop a practical framework for Hadamard renormalization on black hole space-times

Renormalized stress-energy tensor

Elizabeth Winstanley

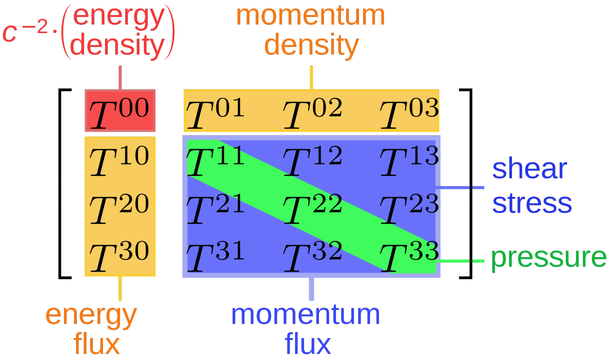
School of Mathematical and Physical Sciences
The University of Sheffield

University of Sheffield

Stress-energy tensor (SET) expectation value

Semi-classical Einstein equations

$$G_{\lambda\rho} + \Lambda g_{\lambda\rho} = 8\pi \langle \hat{T}_{\lambda\rho} \rangle$$



Quantum scalar field $\hat{\Phi}$

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$$\langle \hat{\Phi}^2(x) \rangle_{\rm ren} = \lim_{x' \to x} \left\{ -{\rm i} \left[G_{\rm F}(x,x') - G_{\rm S}(x,x') \right] \right\}$$

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Develop a practical framework for Hadamard renormalization on black hole space-times

Black holes

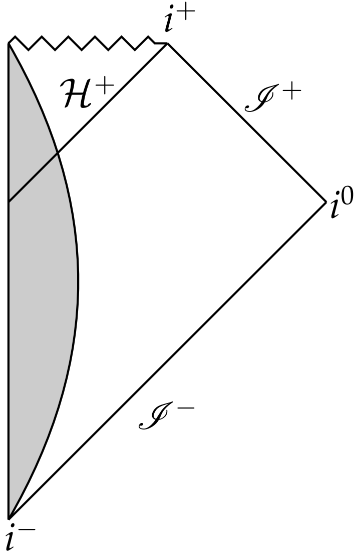
Hawking *CMP* **43** 199 (1975)
Boulware *PRD* **11** 1404 (1975)
Unruh *PRD* **14** 870 (1976)
Hartle & Hawking *PRD* **13** 2188 (1976)
Israel *PLA* **57** 107 (1976)
Candelas *PRD* **21** 2185 (1980)

Hawking radiation

[Hawking *CMP* **43** 199 (1975)]

Hawking radiation

- Black hole formed by gravitational collapse



[Hawking *CMP* **43** 199 (1975)]

Hawking radiation

- Black hole formed by gravitational collapse
- Vacuum state at $\mathscr{I}^-$

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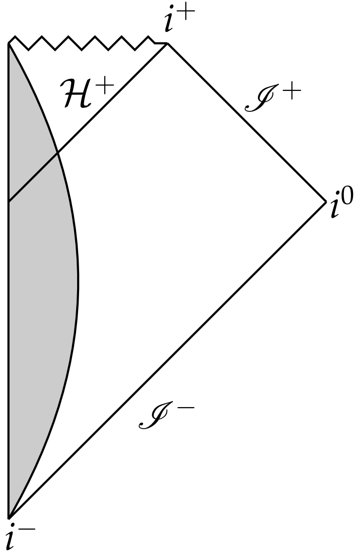


Hawking radiation

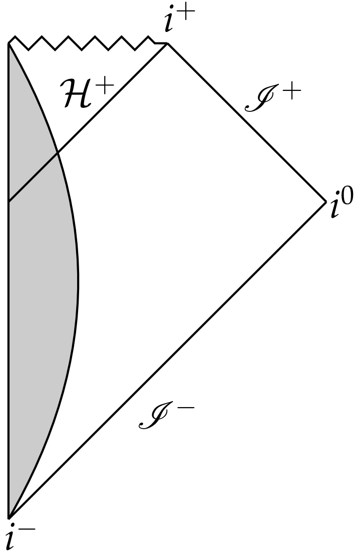
- Black hole formed by gravitational collapse
- Vacuum state at $\mathscr{I}^-$

Thermal radiation at $\mathscr{I}^+$

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi}$$

κ – surface gravity

[Hawking *CMP* **43** 199 (1975)]



Schwarzschild black hole

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$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2 \qquad f(r) = 1 - \frac{2M}{r}$$

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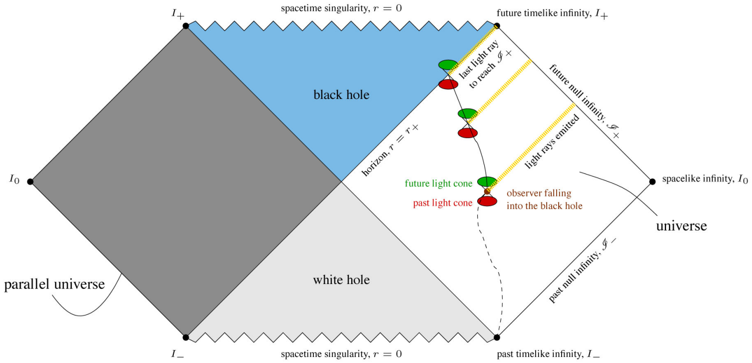

[Figure: Ambrosetti, Charbonneau & Weinfurtner 0810.2631]

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Scalar field modes

$$\phi_{\omega\ell m}(t,r,\theta,\varphi) = \frac{1}{\mathcal{N}r} e^{-i\omega t} Y_{\ell m}(\theta,\varphi)\psi_{\omega\ell}(r)$$

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$Y_{\ell m}(\theta,\varphi)$ – spherical harmonics

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Radial equation

$$\left[-\frac{d^2}{dr_*{}^2} + V_\ell(r_*)\right]\psi_{\omega\ell}(r) = \omega^2\psi_{\omega\ell}(r) \qquad \frac{dr_*}{dr} = \frac{1}{f(r)}$$

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Tortoise coordinate

$$r_* \to -\infty \quad \text{as } r \to 2M \qquad\qquad r_* \to +\infty \quad \text{as } r \to \infty$$

Radial potential

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r_*^2} + V_\ell(r_*) \right] \psi_{\omega\ell}(r) = \omega^2 \psi_{\omega\ell}(r)$$

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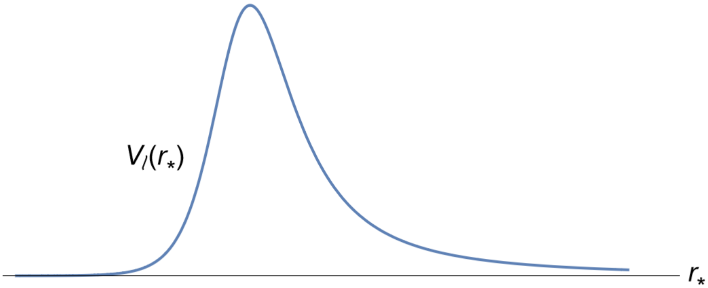

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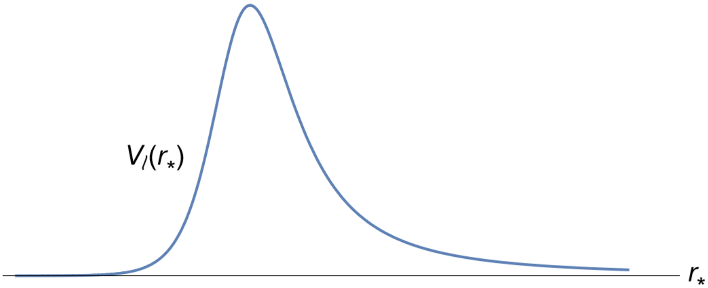
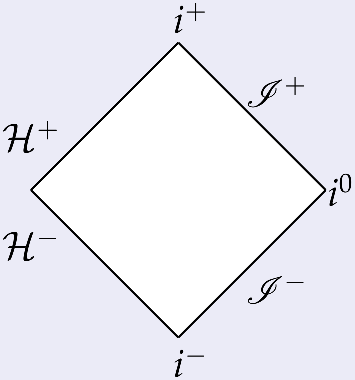


$$V_\ell(r_*) \to 0 \qquad r_* \to \pm\infty$$

"In" modes $\psi^{\text{in}}_{\omega\ell}$

$$\begin{cases} & r_* \to -\infty \\ & r_* \to \infty \end{cases}$$



“In” modes $\psi^{\text{in}}_{\omega\ell}$

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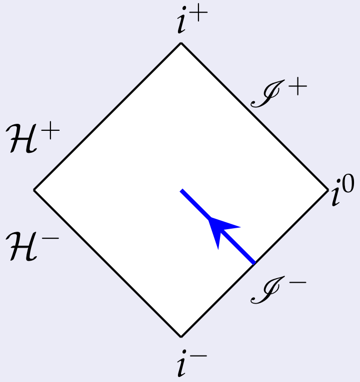

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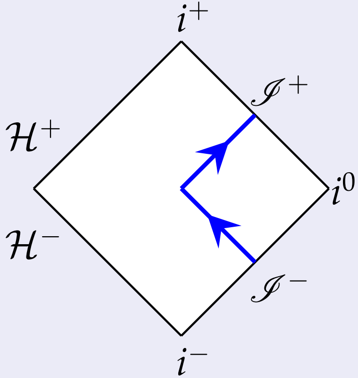

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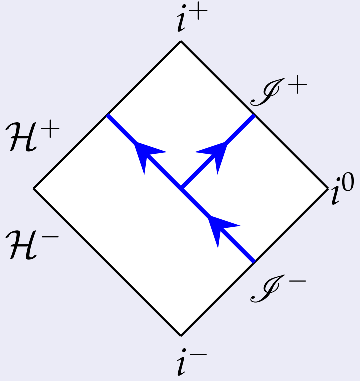

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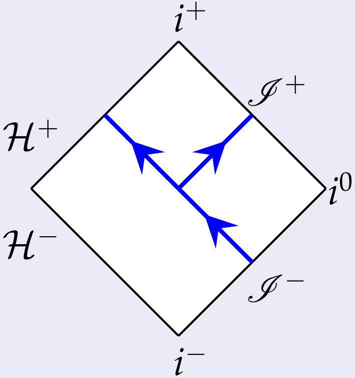
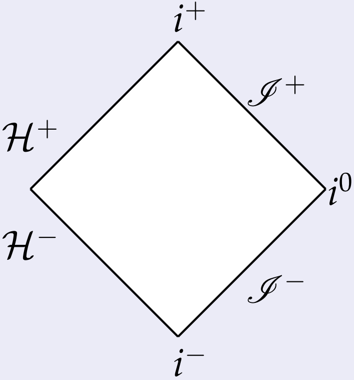


"Up" modes $\psi^{\text{up}}_{\omega\ell}$

$$\begin{cases} & r_* \to -\infty \\ & r_* \to \infty \end{cases}$$



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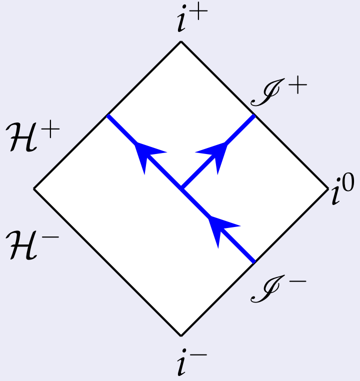
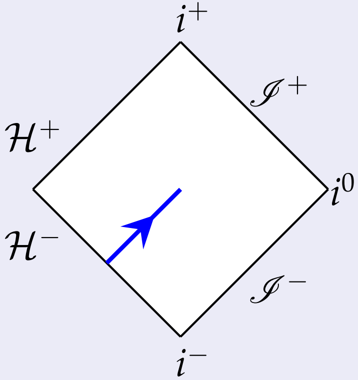


“Up” modes $\psi^{\mathrm{up}}_{\omega\ell}$

$$\begin{cases} e^{i\omega r_*} & r_* \to -\infty \\ & r_* \to \infty \end{cases}$$



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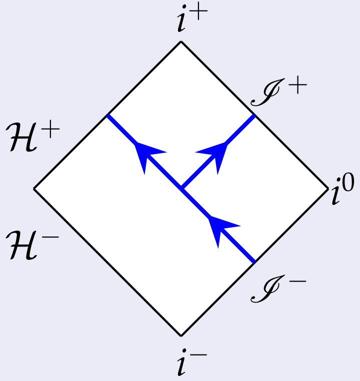

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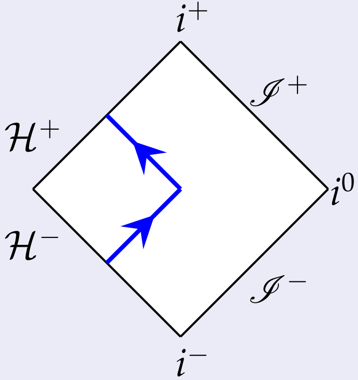
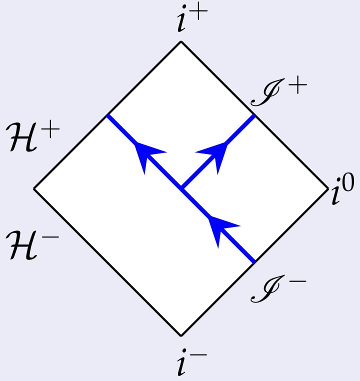


"In" modes $\psi^{\rm in}_{\omega\ell}$

$$\begin{cases} B^{\rm in}_{\omega\ell} e^{-i\omega r_*} & r_* \to -\infty \\ e^{-i\omega r_*} + A^{\rm in}_{\omega\ell} e^{i\omega r_*} & r_* \to \infty \end{cases}$$



"Up" modes $\psi^{\rm up}_{\omega\ell}$

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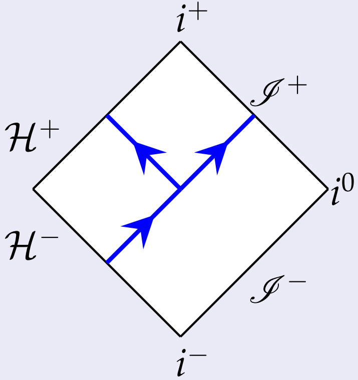
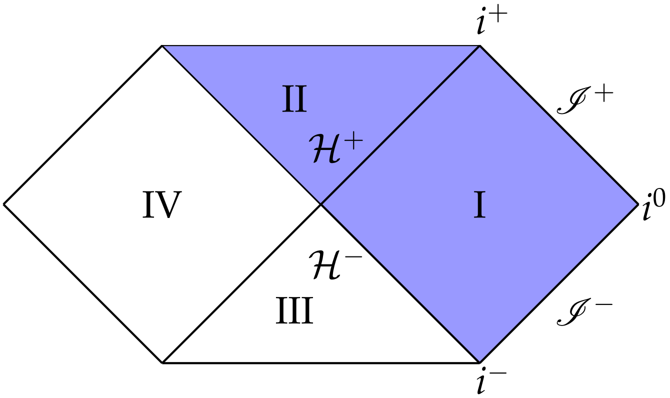


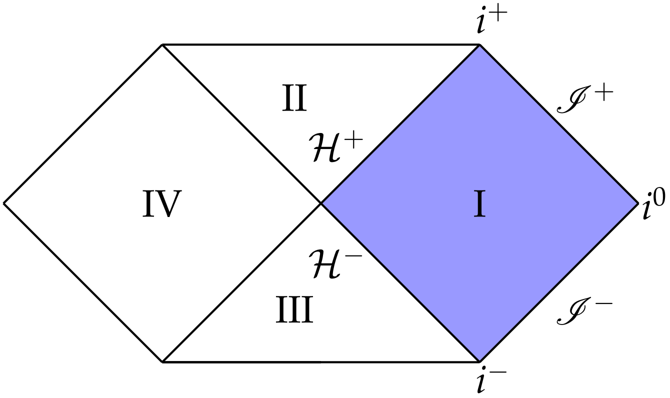
Unruh state $|U\rangle$ [Unruh *PRD* **14** 870 (1976)]

- Hawking radiation at $\mathscr{I}^+$
- Regular at $\mathcal{H}^+$



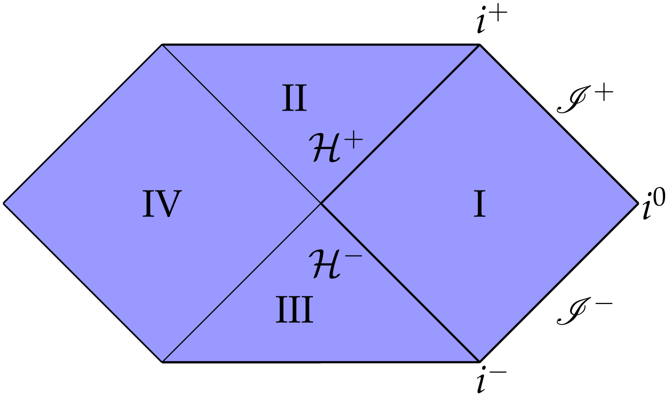
Boulware state $|B\rangle$ [Boulware *PRD* **11** 1404 (1975)]

- State which is as empty as possible at infinity
- Diverges on the event horizon



Hartle-Hawking state $|H\rangle$ [Hartle & Hawking *PRD* **13** 2188 (1976), Israel *PLA* **57** 107 (1976)]

- Represents a black hole in thermal equilibrium with a heat bath
- Regular on and outside event horizon



Feynman Green functions

Feynman Green functions

Boulware state

Feynman Green functions

Boulware state

$$-\mathrm{i}G_{\mathrm{F}}^{\mathrm{B}}(t,r,\theta,\varphi;t',r,\theta',\varphi') = \int_0^\infty \mathrm{d}\omega \, \frac{\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{|\mathcal{N}|^2 r^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi) Y_{\ell m}(\theta',\varphi') \times \left[\left| \psi_{\omega\ell}^{\mathrm{in}}(r) \right|^2 + \left| \psi_{\omega\ell}^{\mathrm{up}}(r) \right|^2 \right]$$

Feynman Green functions

Boulware state

$$-\mathrm{i}G_{\mathrm{F}}^{\mathrm{B}}(t,r,\theta,\varphi;t',r,\theta',\varphi') = \int_0^\infty \mathrm{d}\omega \, \frac{\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{|\mathcal{N}|^2 r^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi) Y_{\ell m}(\theta',\varphi') \times \left[\left| \psi_{\omega\ell}^{\mathrm{in}}(r) \right|^2 + \left| \psi_{\omega\ell}^{\mathrm{up}}(r) \right|^2 \right]$$

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$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\left(\varphi - \varphi'\right)$$

Feynman Green functions

Boulware state

$$-\mathrm{i}G_{\mathrm{F}}^{\mathrm{B}}(x,x') = \int_0^\infty \mathrm{d}\omega \, \frac{\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{4\pi|\mathcal{N}|^2 r^2} \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\gamma) \left[\left| \psi_{\omega\ell}^{\mathrm{in}}(r) \right|^2 + \left| \psi_{\omega\ell}^{\mathrm{up}}(r) \right|^2 \right]$$

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Unruh state

Feynman Green functions

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$$-\mathrm{i}G_{\mathrm{F}}^{\mathrm{U}}(x,x') = \int_0^\infty \mathrm{d}\omega \, \frac{\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{4\pi|\mathcal{N}|^2 r^2} \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\gamma) \left[\left|\psi_{\omega\ell}^{\mathrm{in}}(r)\right|^2 + \left|\psi_{\omega\ell}^{\mathrm{up}}(r)\right|^2 \coth\left(\frac{\pi\omega}{\kappa}\right) \right]$$

Feynman Green functions

Boulware state

$$-\mathrm{i}G_{\mathrm{F}}^{\mathrm{B}}(x,x') = \int_0^\infty \mathrm{d}\omega \, \frac{\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{4\pi|\mathcal{N}|^2 r^2} \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\gamma) \left[\left|\psi_{\omega\ell}^{\mathrm{in}}(r)\right|^2 + \left|\psi_{\omega\ell}^{\mathrm{up}}(r)\right|^2 \right]$$

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Hartle-Hawking state

$$-\mathrm{i}G_{\mathrm{F}}^{\mathrm{H}}(x,x') = \int_0^\infty \mathrm{d}\omega \, \frac{\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{4\pi|\mathcal{N}|^2 r^2} \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\gamma) \left[\left|\psi_{\omega\ell}^{\mathrm{in}}(r)\right|^2 + \left|\psi_{\omega\ell}^{\mathrm{up}}(r)\right|^2 \right] \coth\left(\frac{\pi\omega}{\kappa}\right)$$

Feynman Green functions

Boulware state

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Unruh state

$$-\mathrm{i}G_{\mathrm{F}}^{\mathrm{U}}(x,x') = \int_0^\infty \mathrm{d}\omega \, \frac{\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{4\pi|\mathcal{N}|^2 r^2} \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\gamma) \left[\left| \psi_{\omega\ell}^{\mathrm{in}}(r) \right|^2 + \left| \psi_{\omega\ell}^{\mathrm{up}}(r) \right|^2 \coth\left(\frac{\pi\omega}{\kappa}\right) \right]$$

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Renormalized expectation values

$$\langle \hat{T}_{\lambda\rho}(x) \rangle_{\text{ren}} = \lim_{x' \to x} \left[\mathcal{T}_{\lambda\rho} \left\{ -\text{i} \left[G_{\text{F}}(x, x') - G_{\text{S}}(x, x') \right] \right\} \right] - g_{\lambda\rho} v_1$$

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Feynman Green function $G_{\mathrm{F}}(x,x')$

- Mode sum over separable solutions of the Klein-Gordon equation
- Modes typically found numerically

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Feynman Green function $G_{\rm F}(x,x')$

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Hadamard parametrix $G_{\rm S}(x,x')$

- Purely geometric
- Taylor series expansions for x' close to x

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- Mode sum over separable solutions of the Klein-Gordon equation
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Hadamard parametrix $G_{\mathrm{S}}(x,x')$

- Purely geometric
- Taylor series expansions for x' close to x

Homework

Devise a method to subtract $G_{\mathrm{S}}(x,x')$ from $G_{\mathrm{F}}(x,x')$ so that the answer can be computed

WKB-based implementation

Candelas & Howard *PRD* **29** 1618 (1984)
Howard & Candelas *PRL* **53** 403 (1984)
Howard *PRD* **30** 2532 (1984)
Anderson, Hiscock & Samuel *PRD* **51** 4337 (1995)
EW & Young *PRD* **77** 024008 (2008)
Flachi & Tanaka *PRD* **78** 064011 (2008)
Breen & Ottewill *PRD* **82** 084019 (2010)
Breen & Ottewill *PRD* **85** 084029 (2012)

WKB-based implementation

$$\langle \hat{\Phi}^2(x) \rangle_{\text{ren}} = \lim_{x' \to x} \left\{ -\text{i} \left[G_{\text{F}}(x, x') - G_{\text{S}}(x, x') \right] \right\}$$

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point splitting

$$-{\rm i} G_{\rm F}^{\rm B}(x,x') = \int_0^\infty {\rm d}\omega \, \frac{{\rm e}^{-{\rm i}\omega(t-t')}}{4\pi |\mathcal{N}|^2 r^2} \sum_{\ell=0}^{\infty} (2\ell+1) \, P_\ell(\cos\gamma) \left[\left| \psi_{\omega\ell}^{\rm in}(r) \right|^2 + \left| \psi_{\omega\ell}^{\rm up}(r) \right|^2 \right]$$

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Time-like point splitting

$$-\mathrm{i}G_{\mathrm{F}}^{\mathrm{B}}(x,x') = \int_0^{\infty} \mathrm{d}\omega \, \frac{\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{4\pi |\mathcal{N}|^2 r^2} \sum_{\ell=0}^{\infty} (2\ell+1) \, P_{\ell}(\cos\gamma) \left[\left| \psi_{\omega\ell}^{\mathrm{in}}(r) \right|^2 + \left| \psi_{\omega\ell}^{\mathrm{up}}(r) \right|^2 \right]$$

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Time-like point splitting

$$-{\rm i}G_{\rm F}^{\rm B}(x,x') = \int_0^\infty {\rm d}\omega \, \frac{{\rm e}^{-{\rm i}\omega\epsilon}}{4\pi |\mathcal{N}|^2 r^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left[\left| \psi_{\omega\ell}^{\rm in}(r) \right|^2 + \left| \psi_{\omega\ell}^{\rm up}(r) \right|^2 \right]$$

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$$\langle \hat{\Phi}^2(x) \rangle_{\rm ren} = \lim_{\epsilon \to 0} \left\{ \left[\int_0^\infty d\omega \, \frac{e^{-{\rm i}\omega\epsilon}}{4\pi |\mathcal{N}|^2 r^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left[\left| \psi_{\omega\ell}^{\rm in}(r) \right|^2 + \left| \psi_{\omega\ell}^{\rm up}(r) \right|^2 \right] \right. \right.$$
$$\left. \left. \qquad \right] + \left[\qquad + {\rm i} G_{\rm S}(\epsilon) \right] \right\}$$

WKB-based implementation

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$$\langle \hat{\Phi}^2(x) \rangle_{\text{ren}} = \lim_{\epsilon \to 0} \Bigg\{ \Bigg[\int_0^\infty \mathrm{d}\omega \, \frac{\mathrm{e}^{-\mathrm{i}\omega\epsilon}}{4\pi |\mathcal{N}|^2 r^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left[\left| \psi_{\omega\ell}^{\mathrm{in}}(r) \right|^2 + \left| \psi_{\omega\ell}^{\mathrm{up}}(r) \right|^2 \right] - \int_{\omega=0}^{\infty} \mathrm{d}\omega \sum_{\ell=0}^{\infty} \mathrm{WKB} \Bigg] + \Bigg[\int_{\omega=0}^{\infty} \mathrm{d}\omega \sum_{\ell=0}^{\infty} \mathrm{WKB} + \mathrm{i} G_{\mathrm{S}}(\epsilon) \Bigg] \Bigg\}$$

WKB approximation

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$$-\frac{\mathrm{d}^2\psi_{\omega\ell}}{\mathrm{d}r_*^2} + V_\ell(r_*)\psi_{\omega\ell} = \omega^2\psi_{\omega\ell}$$

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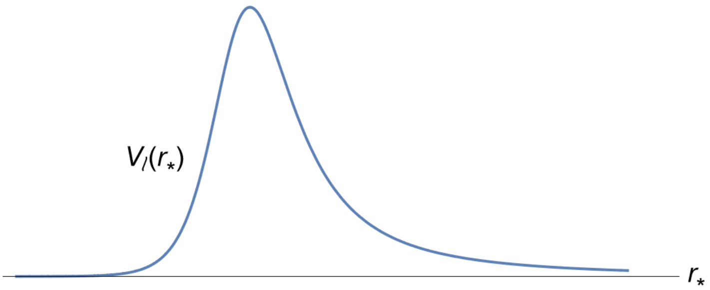

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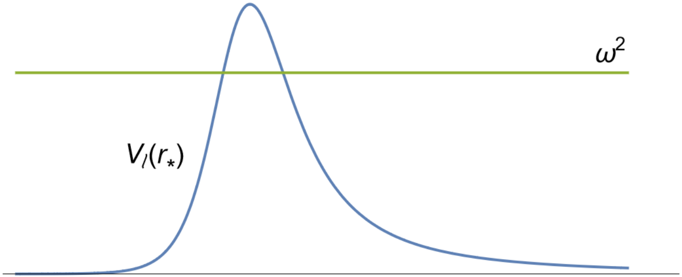

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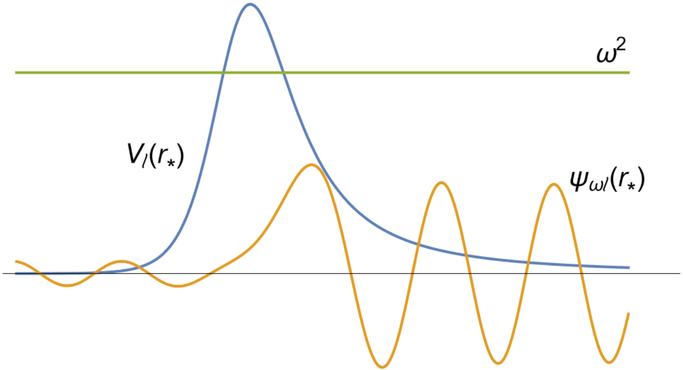

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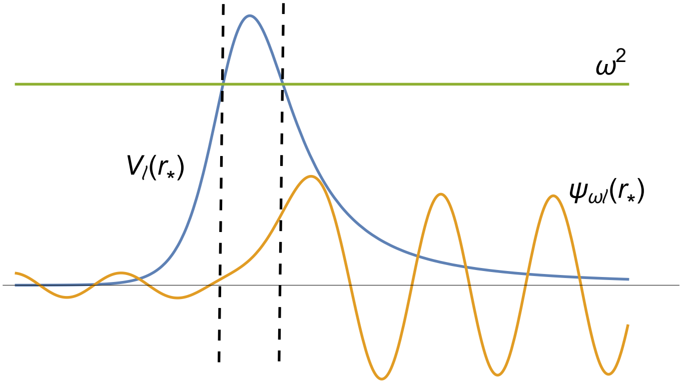

Euclideanization

Euclideanization

- Wick rotation $t \to -\mathrm{i}\tau$, $\omega \to \mathrm{i}\omega$

Euclideanization

- Wick rotation $t \to -\mathrm{i}\tau$, $\omega \to \mathrm{i}\omega$
- Radial equation

$$-\frac{\mathrm{d}^2\psi_{\omega\ell}}{\mathrm{d}r_*^2} + V_\ell(r_*)\psi_{\omega\ell} = -\omega^2\psi_{\omega\ell}$$

Euclideanization

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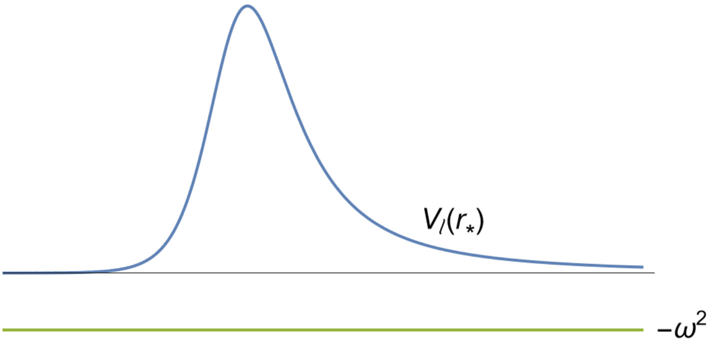

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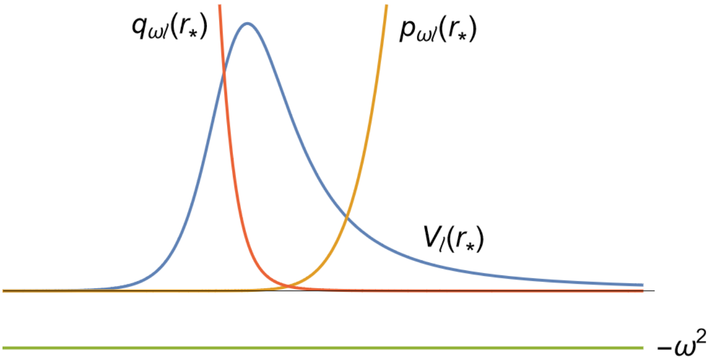

Euclidean geometry

Euclidean geometry

Nonrotating, spherically symmetric black hole

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2 \qquad f(r_h) = 0$$

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$$ds^2 = f(r)\,d\tau^2 + f(r)^{-1}\,dr^2 \qquad f(r_h) = 0$$

Wick rotation

$$t \to -i\tau$$

Euclidean geometry

Nonrotating, spherically symmetric black hole

$$ds^2 = f(r)\, d\tau^2 + f(r)^{-1}\, dr^2 \qquad\qquad f(r_h) = 0$$

Near-horizon metric

$$ds^2 = 2\kappa\,(r - r_h)\; d\tau^2 + \left[2\kappa\,(r - r_h)\right]^{-1}\; dr^2 \qquad\qquad \kappa = \frac{f'(r_h)}{2}$$

Euclidean geometry

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$$ds^2 = \kappa^2 x^2\,d\tau^2 + dx^2 \qquad x = \sqrt{\frac{2\,(r - r_h)}{\kappa}}$$

Euclidean geometry

Nonrotating, spherically symmetric black hole

$$ds^2 = f(r)\,d\tau^2 + f(r)^{-1}\,dr^2 \qquad f(r_h) = 0$$

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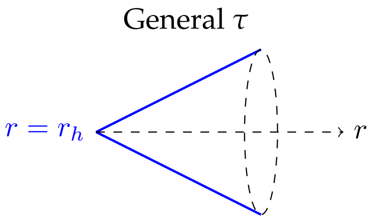

Euclidean geometry

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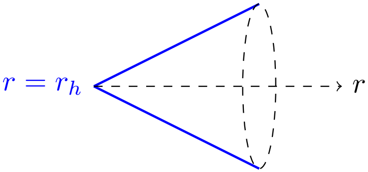
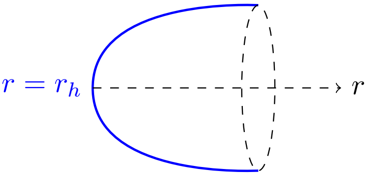
Near-horizon metric

$$ds^2 = \kappa^2 x^2\,d\tau^2 + dx^2$$

General τ



τ period $2\pi/\kappa$



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$$ds^2 = f(r)\,d\tau^2 + f(r)^{-1}\,dr^2 \qquad\qquad f(r_h) = 0$$

Near-horizon metric

$$ds^2 = \kappa^2 x^2\,d\tau^2 + dx^2$$

- Euclidean time coordinate periodic $\tau \to \tau + 2\pi/\kappa$

Euclidean geometry

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- Euclidean time coordinate periodic $\tau \to \tau + 2\pi/\kappa$
- Thermal state at temperature $\kappa/2\pi = T_{\mathrm{H}}$

Euclidean geometry

Nonrotating, spherically symmetric black hole

$$\mathrm{d}s^2 = f(r)\,\mathrm{d}\tau^2 + f(r)^{-1}\,\mathrm{d}r^2 \qquad f(r_h) = 0$$

Near-horizon metric

$$\mathrm{d}s^2 = \kappa^2 x^2\,\mathrm{d}\tau^2 + \mathrm{d}x^2$$

- Euclidean time coordinate periodic $\tau \to \tau + 2\pi/\kappa$
- Thermal state at temperature $\kappa/2\pi = T_{\mathrm{H}}$
- Hartle-Hawking state

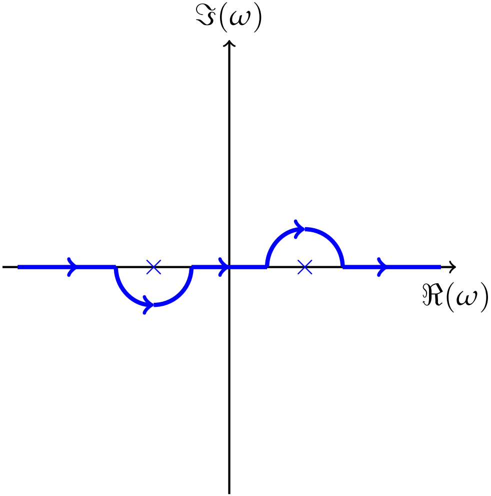
Euclidean Green function

[Fulling & Ruijsenaars *Phys. Rept.* **152** 135 (1987)]

Euclidean Green function

Minkowski space-time

$$-\mathrm{i}G_{\mathrm{F}}(x,x') = -\frac{\mathrm{i}}{16\pi^4}\int \mathrm{d}\omega\,\mathrm{d}^3\mathbf{p}\,\frac{\mathrm{e}^{-\mathrm{i}\omega|t-t'|}\mathrm{e}^{\mathrm{i}\mathbf{p}.(\mathbf{x}-\mathbf{x}')}}{-\omega^2+|\mathbf{p}.\mathbf{p}|+\mu^2}$$



[Fulling & Ruijsenaars *Phys. Rept.* **152** 135 (1987)]

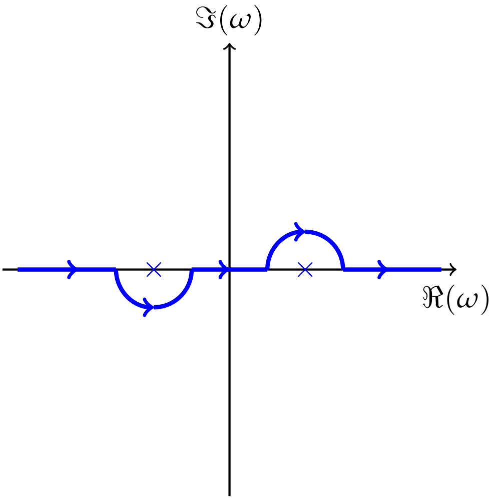
Euclidean Green function

Minkowski space-time

$$-\mathrm{i}G_{\mathrm{F}}(x,x') = -\frac{\mathrm{i}}{16\pi^4}\int d\omega\, d^3\mathbf{p}\, \frac{e^{-\mathrm{i}\omega|t-t'|}e^{\mathrm{i}\mathbf{p}.(\mathbf{x}-\mathbf{x}')}}{-\omega^2+|\mathbf{p}.\mathbf{p}|+\mu^2}$$

Wick rotation

$$t \to -\mathrm{i}\tau \qquad \omega \to \mathrm{i}\omega$$



[Fulling & Ruijsenaars *Phys. Rept.* **152** 135 (1987)]

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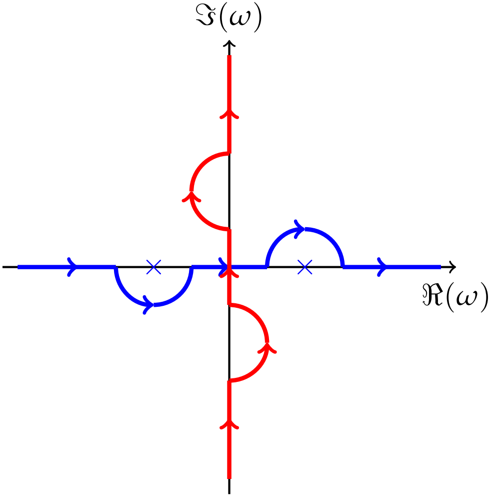

[Fulling & Ruijsenaars *Phys. Rept.* **152** 135 (1987)]

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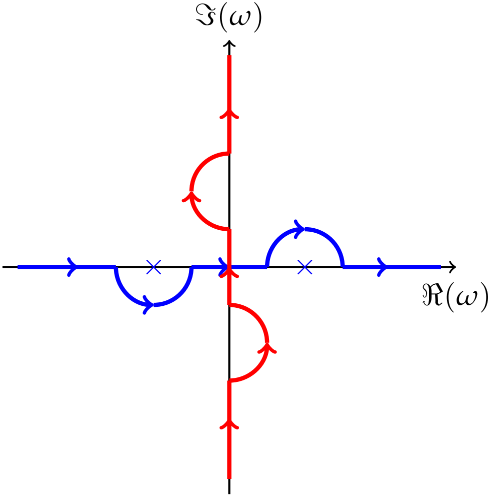
$$-\mathrm{i}G_{\mathrm{F}}(x,x') \to \frac{1}{16\pi^4}\int \mathrm{d}\omega\,\mathrm{d}^3\mathbf{p}\,\frac{\mathrm{e}^{-\mathrm{i}\omega|\tau-\tau'|}\mathrm{e}^{\mathrm{i}\mathbf{p}.(\mathbf{x}-\mathbf{x}')}}{\omega^2+|\mathbf{p}.\mathbf{p}|+\mu^2}$$



[Fulling & Ruijsenaars *Phys. Rept.* **152** 135 (1987)]

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Minkowski space-time

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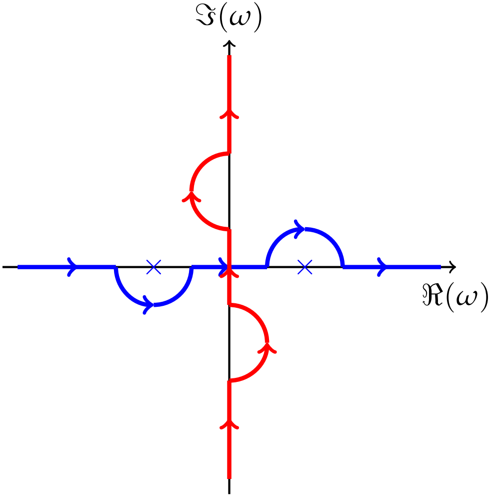
Wick rotation

$$t \to -\mathrm{i}\tau \qquad \omega \to \mathrm{i}\omega$$

$$-\mathrm{i}G_{\mathrm{F}}(x,x') \to \frac{1}{16\pi^4}\int \mathrm{d}\omega\,\mathrm{d}^3\mathbf{p}\,\frac{e^{-\mathrm{i}\omega|\tau-\tau'|}e^{\mathrm{i}\mathbf{p}.(\mathbf{x}-\mathbf{x}')}}{\omega^2+|\mathbf{p}.\mathbf{p}|+\mu^2} = G_{\mathrm{E}}(x,x')$$



[Fulling & Ruijsenaars *Phys. Rept.* **152** 135 (1987)]

Euclidean Green function

Euclidean Green function

Mode sum representation

$$G_{\mathrm{E}}(x,x') = \frac{\kappa}{4\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{in\kappa\Delta\tau} (2\ell+1)\, P_\ell(\cos\gamma) p_{n\ell}(r_<) q_{n\ell}(r_>)$$

Euclidean Green function

Mode sum representation

$$G_{\mathrm{E}}(x,x') = \frac{\kappa}{4\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{in\kappa\Delta\tau} (2\ell+1)\, P_\ell(\cos\gamma) p_{n\ell}(r_<) q_{n\ell}(r_>)$$

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Euclidean Green function

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- Periodic in τ
- Legendre polynomials $P_\ell(\cos\gamma)$

Euclidean Green function

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- Radial functions $p_{n\ell}$, $q_{n\ell}$ computed numerically
  - $r_< = \min\{r, r'\}$, $r_> = \max\{r, r'\}$
  - $p_{n\ell}$ regular at event horizon, $q_{n\ell}$ regular at infinity

Euclidean Green function

Mode sum representation

$$G_E(x,x') = \frac{\kappa}{4\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} e^{in\kappa\epsilon} (2\ell+1) p_{n\ell}(r) q_{n\ell}(r)$$

- Periodic in τ
- Legendre polynomials $P_\ell(\cos\gamma)$
- Radial functions $p_{n\ell}$, $q_{n\ell}$ computed numerically
  - $r_< = \min\{r, r'\}$, $r_> = \max\{r, r'\}$
  - $p_{n\ell}$ regular at event horizon, $q_{n\ell}$ regular at infinity
- Time-like point-splitting $\Delta\tau = \epsilon$, $\Delta r = 0$, $\gamma = 0$

Hadamard parametrix $\Delta\tau = \epsilon$

$$G_S(x,x') = \frac{U(x,x')}{\sigma(x,x')} + V(x,x') \log\left[\frac{\sigma(x,x')}{L^2}\right]$$

[Anderson, Hiscock & Samuel *PRD* **51** 4337 (1995)]

Hadamard parametrix $\Delta\tau = \epsilon$

$$
\begin{aligned}
G_{\mathrm{S}}(x,x') &= \frac{U(x,x')}{\sigma(x,x')} + V(x,x') \log\left[\frac{\sigma(x,x')}{L^2}\right] \\
&= \frac{1}{4\pi^2 f\epsilon^2} + \frac{1}{8\pi^2}\left[\mu^2 - \left(\xi - \frac{1}{6}\right)R\right]\left[\mathsf{C} + \frac{1}{2}\log\left(\frac{f\epsilon^2}{4L^2}\right)\right] \\
&\quad - \frac{\mu^2}{16\pi^2} + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r} + \ldots
\end{aligned}
$$

[Anderson, Hiscock & Samuel *PRD* **51** 4337 (1995)]

Hadamard parametrix $\Delta\tau = \epsilon$

$$
\begin{aligned}
G_{\mathrm{S}}(x,x') &= \frac{U(x,x')}{\sigma(x,x')} + V(x,x') \log\left[\frac{\sigma(x,x')}{L^2}\right] \\
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\end{aligned}
$$

Distributional identities

[Anderson, Hiscock & Samuel *PRD* **51** 4337 (1995)]

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$$= \frac{1}{4\pi^2 f \epsilon^2} + \frac{1}{8\pi^2}\left[\mu^2 - \left(\xi - \frac{1}{6}\right)R\right]\left[C + \frac{1}{2}\log\left(\frac{f\epsilon^2}{4L^2}\right)\right]$$
$$- \frac{\mu^2}{16\pi^2} + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r} + \ldots$$

Distributional identities

$$\frac{1}{\epsilon^2} = -\kappa^2 \sum_{n=1}^{\infty} n\cos(n\kappa\epsilon) - \frac{\kappa^2}{12} + \ldots$$

[Anderson, Hiscock & Samuel *PRD* **51** 4337 (1995)]

Hadamard parametrix $\Delta\tau = \epsilon$

$$
\begin{aligned}
G_{\mathrm{S}}(x,x') &= \frac{U(x,x')}{\sigma(x,x')} + V(x,x')\log\left[\frac{\sigma(x,x')}{L^2}\right] \\
&= \frac{1}{4\pi^2 f\epsilon^2} + \frac{1}{8\pi^2}\left[\mu^2 - \left(\xi - \frac{1}{6}\right)R\right]\left[\mathsf{C} + \frac{1}{2}\log\left(\frac{f\epsilon^2}{4L^2}\right)\right] \\
&\quad - \frac{\mu^2}{16\pi^2} + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r} + \ldots
\end{aligned}
$$

Distributional identities

$$
\frac{1}{\epsilon^2} = -\kappa^2\sum_{n=1}^{\infty} n\cos(n\kappa\epsilon) - \frac{\kappa^2}{12} + \ldots
$$

$$
-\frac{1}{2}\log\left(\kappa^2\epsilon^2\right) = \sum_{n=1}^{\infty}\frac{\cos(n\kappa\epsilon)}{n} + \ldots
$$

[Anderson, Hiscock & Samuel *PRD* **51** 4337 (1995)]

Renormalized VP

$$\langle \hat{\Phi}^2 \rangle_{\text{ren}} =$$

Renormalized VP

$$\langle \hat{\Phi}^2 \rangle_{\text{ren}} = \lim_{\epsilon \to 0} \left[G_{\text{E}}(x, x') - G_{\text{S}}(x, x') \right]$$

Renormalized VP

$$\langle \hat{\Phi}^2 \rangle_{\text{ren}} = \lim_{\epsilon \to 0} \left[G_{\text{E}}(x, x') - G_{\text{S}}(x, x') \right] = \langle \hat{\Phi}^2 \rangle_{\text{analytic}} + \langle \hat{\Phi}^2 \rangle_{\text{numeric}}$$

$$\langle \hat{\Phi}^2 \rangle_{\text{analytic}} =$$

$$\langle \hat{\Phi}^2 \rangle_{\text{numeric}} =$$

Renormalized VP

$$\langle \hat{\Phi}^2 \rangle_{\rm ren} = \lim_{\epsilon \to 0} \left[G_{\rm E}(x,x') - G_{\rm S}(x,x') \right] = \langle \hat{\Phi}^2 \rangle_{\rm analytic} + \langle \hat{\Phi}^2 \rangle_{\rm numeric}$$

$$\langle \hat{\Phi}^2 \rangle_{\rm analytic} =$$

$$\langle \hat{\Phi}^2 \rangle_{\rm numeric} = \frac{\kappa}{4\pi^2} \sum_{n=1}^{\infty} \left\{ \sum_{\ell=0}^{\infty} (2\ell+1)\, p_{n\ell}(r) q_{n\ell}(r) \right. \\ + \frac{\kappa}{8\pi^2} \sum_{\ell=0}^{\infty} (2\ell+1)\, p_{0\ell}(r) q_{0\ell}(r)$$

Renormalized VP

$$\langle \hat{\Phi}^2 \rangle_{\text{ren}} = \lim_{\epsilon \to 0} \left[G_{\text{E}}(x,x') - G_{\text{S}}(x,x') \right] = \langle \hat{\Phi}^2 \rangle_{\text{analytic}} + \langle \hat{\Phi}^2 \rangle_{\text{numeric}}$$

$$\begin{aligned}\langle \hat{\Phi}^2 \rangle_{\text{analytic}} = & -\frac{1}{8\pi^2}\left[\mu^2 - \left(\xi - \frac{1}{6}\right) R\right]\left[\mathsf{C} + \frac{1}{2}\log\left(\frac{f\kappa^2}{4L^2}\right)\right] \\ & + \frac{\mu^2}{16\pi^2} - \frac{f'^2}{192\pi^2 f} + \frac{f''}{96\pi^2} + \frac{f'}{48\pi^2 r} + \frac{\kappa^2}{48\pi^2 f}\end{aligned}$$

$$\begin{aligned}\langle \hat{\Phi}^2 \rangle_{\text{numeric}} = & \frac{\kappa}{4\pi^2} \sum_{n=1}^{\infty} \left\{ \sum_{\ell=0}^{\infty} (2\ell+1)\, p_{n\ell}(r) q_{n\ell}(r) + \frac{n\kappa}{f} + \frac{1}{2n\kappa}\left[\mu^2 + \left(\xi - \frac{1}{6}R\right)\right]\right\} \\ & + \frac{\kappa}{8\pi^2} \sum_{\ell=0}^{\infty} (2\ell+1)\, p_{0\ell}(r) q_{0\ell}(r)\end{aligned}$$

Numeric part

$$
\begin{aligned}
\langle\hat{\Phi}^2\rangle_{\text{numeric}} = \frac{\kappa}{4\pi^2}\sum_{n=1}^{\infty}\Bigg\{\sum_{\ell=0}^{\infty}\Big[(2\ell+1)\,p_{n\ell}(r)q_{n\ell}(r) \qquad\qquad \Big] \\
+\frac{n\kappa}{f}+\frac{1}{2n\kappa}\left[\mu^2+\left(\xi-\frac{1}{6}R\right)\right]\Bigg\} \\
+\frac{\kappa}{8\pi^2}\sum_{\ell=0}^{\infty}\Big[(2\ell+1)\,p_{0\ell}(r)q_{0\ell}(r) \qquad\qquad \Big]
\end{aligned}
$$

Numeric part

$$\langle \hat{\Phi}^2 \rangle_{\text{numeric}} = \frac{\kappa}{4\pi^2} \sum_{n=1}^{\infty} \left\{ \sum_{\ell=0}^{\infty} \left[(2\ell+1)\, p_{n\ell}(r) q_{n\ell}(r) - \text{WKB approximation} \right] + \text{WKB approximation} + \frac{n\kappa}{f} + \frac{1}{2n\kappa} \left[\mu^2 + \left(\xi - \frac{1}{6} R \right) \right] \right\} + \frac{\kappa}{8\pi^2} \sum_{\ell=0}^{\infty} \left[(2\ell+1)\, p_{0\ell}(r) q_{0\ell}(r) - \text{WKB approximation} \right]$$

Numeric part

$$
\begin{aligned}
\langle \hat{\Phi}^2 \rangle_{\text{numeric}} = {} & \frac{\kappa}{4\pi^2} \sum_{n=1}^{\infty} \left\{ \sum_{\ell=0}^{\infty} \left[(2\ell+1)\, p_{n\ell}(r) q_{n\ell}(r) - \text{WKB approximation} \right] \right. \\
& \left. + \text{WKB approximation} + \frac{n\kappa}{f} + \frac{1}{2n\kappa} \left[\mu^2 + \left(\xi - \frac{1}{6} R \right) \right] \right\} \\
& + \frac{\kappa}{8\pi^2} \sum_{\ell=0}^{\infty} \left[(2\ell+1)\, p_{0\ell}(r) q_{0\ell}(r) - \text{WKB approximation} \right]
\end{aligned}
$$

- Numerical mode sums

Numeric part

$$\langle \hat{\Phi}^2 \rangle_{\text{numeric}} = \frac{\kappa}{4\pi^2} \sum_{n=1}^{\infty} \Bigg\{ \sum_{\ell=0}^{\infty} \left[(2\ell+1)\, p_{n\ell}(r) q_{n\ell}(r) - \text{WKB approximation} \right] + \text{WKB approximation} + \frac{n\kappa}{f} + \frac{1}{2n\kappa} \left[\mu^2 + \left(\xi - \frac{1}{6} R \right) \right] \Bigg\} + \frac{\kappa}{8\pi^2} \sum_{\ell=0}^{\infty} \left[(2\ell+1)\, p_{0\ell}(r) q_{0\ell}(r) - \text{WKB approximation} \right]$$

- Numerical mode sums
- Semianalytic sums and integrals

RSET on Schwarzschild

$$ds^2 = f(r)\,d\tau^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2 \qquad f(r) = 1 - \frac{2M}{r}$$

RSET on Schwarzschild

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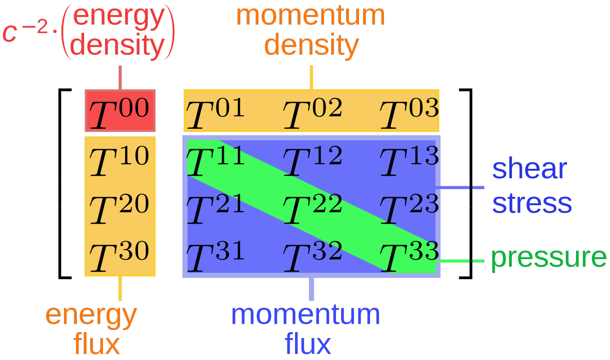
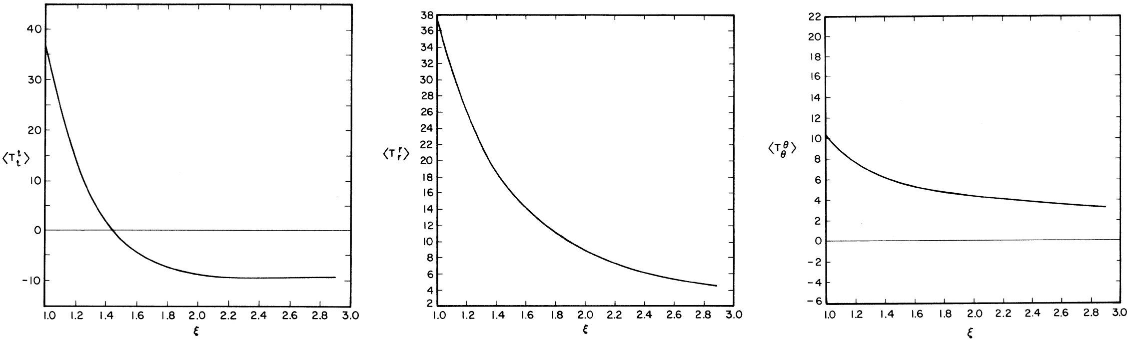


RSET on Schwarzschild

$$ds^2 = f(r)\,d\tau^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2 \qquad f(r) = 1 - \frac{2M}{r}$$



[Howard & Candelas *PRL* **53** 403 (1984)]

WKB-based implementation

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Advantages

- First practical implementation
- WKB approximation can be found algebraically

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- WKB approximation is nonuniform in r
- Numerical issues near the horizon

Homework

Devise a better Euclidean method of performing renormalization

Extended coordinates method

Taylor & Breen *PRD* **94** 125024 (2016)
Taylor & Breen *PRD* **96** 105020 (2017)
Morley, Taylor & EW *CQG* **35** 235010 (2018)
Breen & Taylor *PRD* **98** 105006 (2018)
Morley, Taylor & EW *PRD* **103** 045007 (2021)
Taylor, Breen & Ottewill *PRD* **106** 065023 (2022)
Arrechea, Breen, Ottewill & Taylor *PRD* **108** 125004 (2023)
Arrechea, Breen, Ottewill, Pisani & Taylor `arXiv:2409.04528`

Renormalization strategy

Renormalization strategy

Mode sum representations

$$G_{\mathrm{E}}(x,x') = \frac{\kappa}{8\pi^2}\sum_{n=-\infty}^{\infty}\sum_{\ell=0}^{\infty} \mathrm{e}^{\mathrm{i}n\kappa\Delta\tau}(2\ell+1)\,P_\ell(\cos\gamma)p_{n\ell}(r_<)q_{n\ell}(r_>)$$

Renormalization strategy

Mode sum representations

$$G_{\rm E}(x,x') = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} e^{in\kappa\Delta\tau} (2\ell+1)\, P_\ell(\cos\gamma) p_{n\ell}(r_<) q_{n\ell}(r_>)$$

Renormalization strategy

Mode sum representations

Set $\Delta r = 0$

$$G_{\rm E}(x,x') = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} e^{in\kappa\Delta\tau} (2\ell+1)\, P_\ell(\cos\gamma) p_{n\ell}(r) q_{n\ell}(r)$$

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Set $\Delta r = 0$

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$$G_{\mathrm{S}}(x,x') = \frac{\kappa}{8\pi^2}\sum_{n=-\infty}^{\infty}\sum_{\ell=0}^{\infty} e^{in\kappa\Delta\tau}(2\ell+1)\,P_\ell(\cos\gamma)\Gamma_{n\ell}(r)$$

Renormalization strategy

Mode sum representations

Set $\Delta r = 0$

$$
\begin{aligned}
G_{\mathrm{E}}(x,x') &= \frac{\kappa}{8\pi^2}\sum_{n=-\infty}^{\infty}\sum_{\ell=0}^{\infty} \mathrm{e}^{in\kappa\Delta\tau}(2\ell+1)\,P_\ell(\cos\gamma)p_{n\ell}(r)q_{n\ell}(r) \\
G_{\mathrm{S}}(x,x') &= \frac{\kappa}{8\pi^2}\sum_{n=-\infty}^{\infty}\sum_{\ell=0}^{\infty} \mathrm{e}^{in\kappa\Delta\tau}(2\ell+1)\,P_\ell(\cos\gamma)\Gamma_{n\ell}(r)
\end{aligned}
$$

Regularized Green function

$$G_{\mathrm{R}}(x,x')$$

Renormalization strategy

Mode sum representations

Set $\Delta r = 0$

$$
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G_{\mathrm{E}}(x,x') &= \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{in\kappa\Delta\tau} (2\ell+1)\, P_\ell(\cos\gamma) p_{n\ell}(r) q_{n\ell}(r) \\
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\end{aligned}
$$

Regularized Green function

$$G_{\mathrm{R}}(x,x') = G_{\mathrm{E}}(x,x') - G_{\mathrm{S}}(x,x')$$

Renormalization strategy

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Set $\Delta r = 0$

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\begin{aligned}
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G_{\mathrm{S}}(x,x') &= \frac{\kappa}{8\pi^2}\sum_{n=-\infty}^{\infty}\sum_{\ell=0}^{\infty} e^{in\kappa\Delta\tau}(2\ell+1)\,P_\ell(\cos\gamma)\Gamma_{n\ell}(r)
\end{aligned}
$$

Regularized Green function

$$
\begin{aligned}
G_{\mathrm{R}}(x,x') &= G_{\mathrm{E}}(x,x') - G_{\mathrm{S}}(x,x') \\
&= \frac{\kappa}{8\pi^2}\sum_{n=-\infty}^{\infty}\sum_{\ell=0}^{\infty} e^{in\kappa\Delta\tau}(2\ell+1)\,P_\ell(\cos\gamma)\left[p_{n\ell}(r)q_{n\ell}(r) - \Gamma_{n\ell}(r)\right]
\end{aligned}
$$

Extended coordinates

[Taylor & Breen *PRD* **94** 125024 (2016), Taylor & Breen *PRD* **96** 105020 (2017)]

Extended coordinates

$$\varpi^2 = \frac{2}{\kappa^2}\left[1 - \cos\left(\kappa\Delta\tau\right)\right]$$

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Extended coordinates

$$\varpi^2 = \frac{2}{\kappa^2}\left[1 - \cos\left(\kappa \Delta \tau\right)\right]$$
$$s^2 = f(r)\varpi^2 + 2r^2\left(1 - \cos\gamma\right)$$

[Taylor & Breen *PRD* **94** 125024 (2016), Taylor & Breen *PRD* **96** 105020 (2017)]

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Hadamard parametrix $G_S(x, x')$

[Taylor & Breen *PRD* **94** 125024 (2016), Taylor & Breen *PRD* **96** 105020 (2017)]

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Hadamard parametrix $G_S(x, x')$

$$\sum_{a=0}^{c}\sum_{b=-a}^{a}\mathcal{U}_{ab}(r)\frac{\varpi^{2a+2b}}{s^{2b+2}}$$

[Taylor & Breen *PRD* **94** 125024 (2016), Taylor & Breen *PRD* **96** 105020 (2017)]

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Hadamard parametrix $G_S(x, x')$

$$\sum_{a=0}^{c}\sum_{b=-a}^{a}\mathcal{U}_{ab}(r)\frac{\varpi^{2a+2b}}{s^{2b+2}} + \sum_{a=0}^{c-1}\sum_{b=0}^{a}\mathcal{V}_{ab}(r)s^{2a-2b}\varpi^{2b}\log\left(\frac{s^2}{L^2}\right)$$

[Taylor & Breen *PRD* **94** 125024 (2016), Taylor & Breen *PRD* **96** 105020 (2017)]

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$$+ \text{ polynomial in } \varpi^2, s^2$$

[Taylor & Breen *PRD* **94** 125024 (2016), Taylor & Breen *PRD* **96** 105020 (2017)]

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[Taylor & Breen *PRD* **94** 125024 (2016), Taylor & Breen *PRD* **96** 105020 (2017)]

Mode sum representation of $G_S(x, x')$

[Taylor & Breen *PRD* **94** 125024 (2016), Taylor & Breen *PRD* **96** 105020 (2017)]

Mode sum representation of $G_S(x, x')$

$U(x, x')$ part

[Taylor & Breen *PRD* **94** 125024 (2016), Taylor & Breen *PRD* **96** 105020 (2017)]

Mode sum representation of $G_S(x, x')$

$U(x, x')$ part

$$\frac{\varpi^{2a+2b}}{s^{2b+2}} = \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{in\kappa\Delta\tau} \, (2\ell + 1) \, P_\ell(\cos\gamma) \, {}^{U}\Psi_{n\ell ab}(r)$$

[Taylor & Breen *PRD* **94** 125024 (2016), Taylor & Breen *PRD* **96** 105020 (2017)]

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$${}^{U}\Psi_{n\ell ab}(r) = \frac{\kappa}{4\pi} \int_{\Delta\tau=0}^{\frac{2\pi}{\kappa}} \int_{\cos\gamma=-1}^{1} \frac{\varpi^{2a+2b}}{s^{2b+2}} e^{-in\kappa\Delta\tau} P_\ell(\cos\gamma)\, d(\cos\gamma)\, d(\Delta\tau)$$

[Taylor & Breen *PRD* **94** 125024 (2016), Taylor & Breen *PRD* **96** 105020 (2017)]

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$V(x,x')$ part

[Taylor & Breen *PRD* **94** 125024 (2016), Taylor & Breen *PRD* **96** 105020 (2017)]

Mode sum representation of $G_S(x, x')$

$U(x, x')$ part

$$\frac{\varpi^{2a+2b}}{s^{2b+2}} = \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} e^{in\kappa\Delta\tau} (2\ell+1) P_\ell(\cos\gamma) \, {}^U\Psi_{n\ell ab}(r)$$

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$V(x, x')$ part

$$s^{2a-2b} \varpi^{2b} \log\left(\frac{s^2}{L^2}\right) = \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} e^{in\kappa\Delta\tau} (2\ell+1) P_\ell(\cos\gamma) \, {}^V\Psi_{n\ell ab}(r)$$

[Taylor & Breen *PRD* **94** 125024 (2016), Taylor & Breen *PRD* **96** 105020 (2017)]

Mode sum representation of $G_S(x,x')$

$U(x,x')$ part

$$\frac{\varpi^{2a+2b}}{s^{2b+2}} = \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{in\kappa\Delta\tau} \left(2\ell+1\right) P_\ell(\cos\gamma) \, {}^{U}\Psi_{n\ell ab}(r)$$

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$V(x,x')$ part

$$s^{2a-2b} \varpi^{2b} \log\left(\frac{s^2}{L^2}\right) = \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{in\kappa\Delta\tau} \left(2\ell+1\right) P_\ell(\cos\gamma) \, {}^{V}\Psi_{n\ell ab}(r)$$

${}^{U/V}\Psi_{n\ell ab}(r)$ given in terms of associated Legendre functions

[Taylor & Breen *PRD* **94** 125024 (2016), Taylor & Breen *PRD* **96** 105020 (2017)]

Mode sum representation of $G_S(x, x')$

Mode sum representation of $G_{\rm S}(x,x')$

$$G_{\rm S}(x,x') = \sum_{a=0}^{c}\sum_{b=-a}^{a} \mathcal{U}_{ab}(r)\frac{\varpi^{2a+2b}}{s^{2b+2}} + \sum_{a=0}^{c-1}\sum_{b=0}^{a} \mathcal{V}_{ab}(r) s^{2a-2b}\varpi^{2b}\log\left(\frac{s^2}{L^2}\right)$$
$$+ \ldots$$

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$$= \sum_{n=-\infty}^{\infty}\sum_{\ell=0}^{\infty} e^{in\kappa\Delta\tau}(2\ell+1)P_\ell(\cos\gamma) \times \left\{\sum_{a=0}^{c}\sum_{b=-a}^{a}\mathcal{U}_{ab}(r)\,{}^{\mathcal{U}}\Psi_{n\ell ab}(r) + \sum_{a=0}^{c-1}\sum_{b=0}^{a}\mathcal{V}_{ab}(r)\,{}^{\mathcal{V}}\Psi_{n\ell ab}(r)\right\} + \ldots$$

Mode sum representation of $G_S(x, x')$

$$
\begin{aligned}
G_S(x,x') &= \sum_{a=0}^{c} \sum_{b=-a}^{a} \mathcal{U}_{ab}(r) \frac{\varpi^{2a+2b}}{s^{2b+2}} + \sum_{a=0}^{c-1} \sum_{b=0}^{a} \mathcal{V}_{ab}(r) s^{2a-2b} \varpi^{2b} \log\left(\frac{s^2}{L^2}\right) \\
&\quad + \ldots \\
&= \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{in\kappa\Delta\tau} (2\ell+1) P_\ell(\cos\gamma) \\
&\quad \times \left\{ \sum_{a=0}^{c} \sum_{b=-a}^{a} \mathcal{U}_{ab}(r) {}^{U}\Psi_{n\ell ab}(r) + \sum_{a=0}^{c-1} \sum_{b=0}^{a} \mathcal{V}_{ab}(r) {}^{V}\Psi_{n\ell ab}(r) \right\} \\
&\quad + \ldots \\
&= \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{in\kappa\Delta\tau} (2\ell+1) P_\ell(\cos\gamma) \Gamma_{n\ell}(r) \\
&\quad + \ldots
\end{aligned}
$$

Schwarzschild black hole: Hartle-Hawking state

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2 \qquad f(r) = 1 - \frac{2M}{r}$$

[Howard & Candelas *PRL* **53** 403 (1984); Taylor, Breen & Ottewill *PRD* **106** 065023 (2022)]

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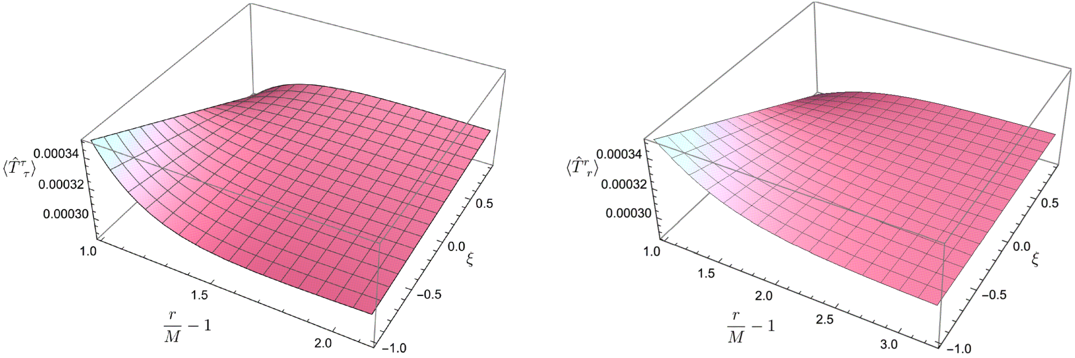
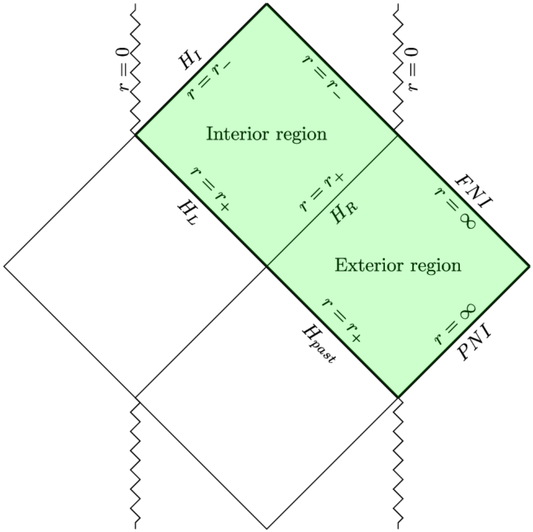


[Howard & Candelas *PRL* **53** 403 (1984); Taylor, Breen & Ottewill *PRD* **106** 065023 (2022)]

Reissner-Nordström black hole



[Figure: Anempodistov *PRD* **103** 105008 (2021)]

Reissner-Nordström black hole

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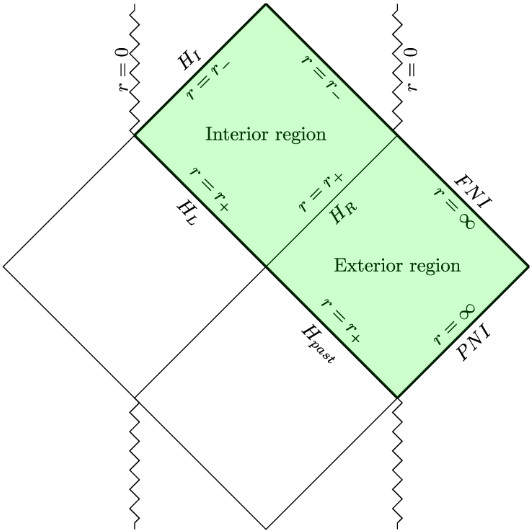
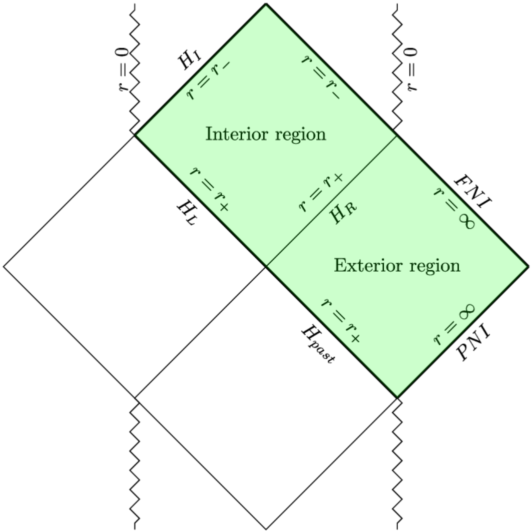


[Figure: Anempodistov *PRD* **103** 105008 (2021)]

Reissner-Nordström black hole

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2$$

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$



[Figure: Anempodistov *PRD* **103** 105008 (2021)]

Reissner-Nordström black hole

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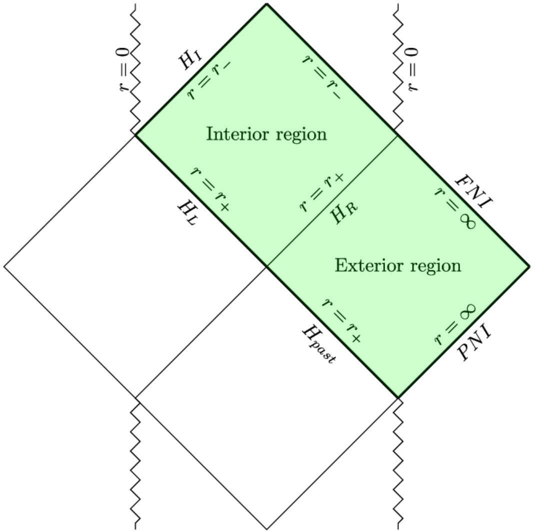
Horizons $f(r_\pm) = 0$

$$r_\pm = M \pm \sqrt{M^2 - Q^2}$$



[Figure: Anempodistov *PRD* **103** 105008 (2021)]

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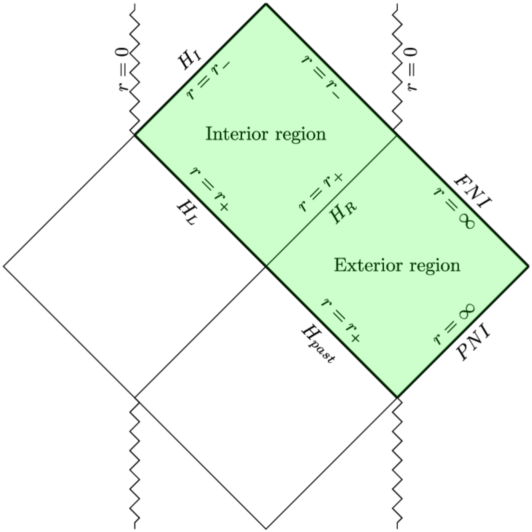
- r_+ event horizon



[Figure: Anempodistov *PRD* **103** 105008 (2021)]

Reissner-Nordström black hole

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- r_+ event horizon
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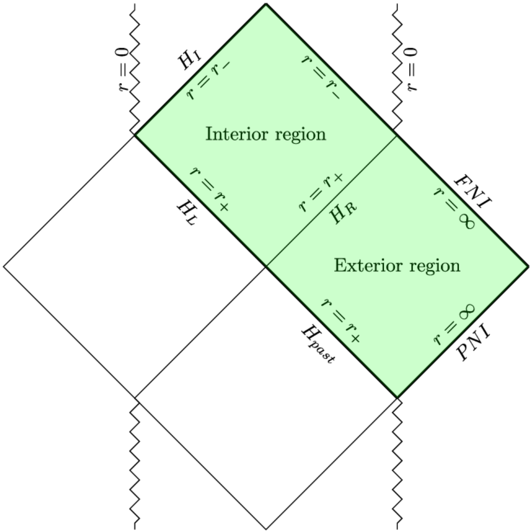
[Figure: Anempodistov *PRD* **103** 105008 (2021)]



Reissner-Nordström black hole

[Arrechea, Breen, Ottewill & Taylor *PRD* **108** 125004 (2023)]

Reissner-Nordström black hole

- Extended coordinates for HH

[Arrechea, Breen, Ottewill & Taylor *PRD* **108** 125004 (2023)]

Reissner-Nordström black hole

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[Arrechea, Breen, Ottewill & Taylor *PRD* **108** 125004 (2023)]

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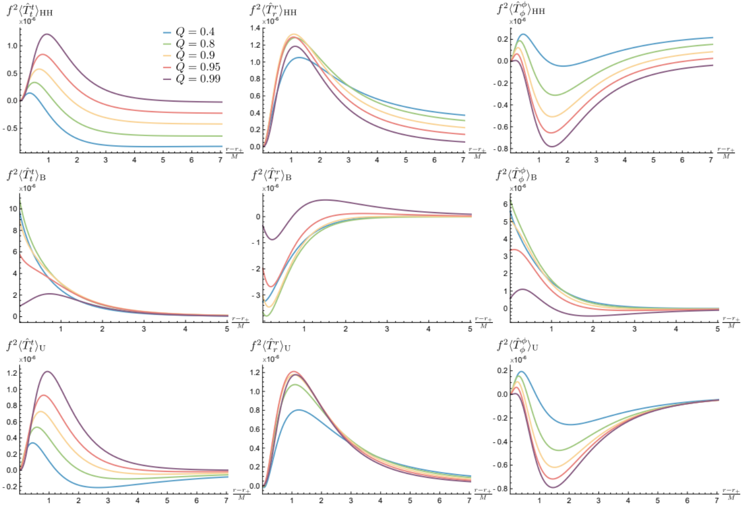
[Arrechea, Breen, Ottewill & Taylor *PRD* **108** 125004 (2023)]



Reissner-Nordström black hole

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\varphi^2$$

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[Figure: Shipley & Dolan *CQG* **33** 175001 (2016)]

Extremal Reissner-Nordström black hole

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2$$

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Extremal Reissner-Nordström black hole

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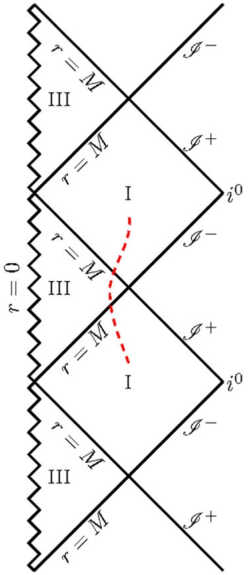

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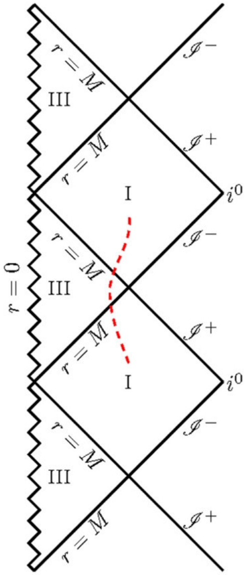
- $T_{\mathrm{H}} = 0$

[Figure: Shipley & Dolan *CQG* **33** 175001 (2016)]



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$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2$$

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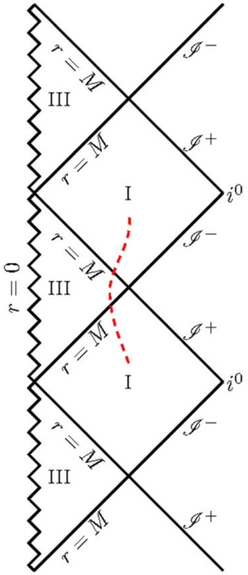
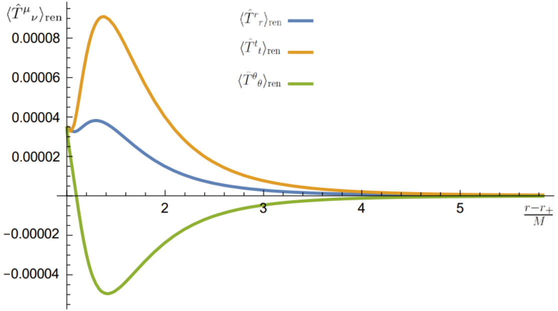
- $T_H = 0$
- HH state = Boulware state

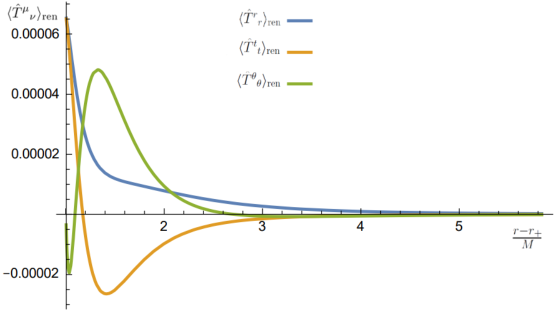
[Figure: Shipley & Dolan *CQG* **33** 175001 (2016)]



Extremal Reissner-Nordström black hole

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2 \qquad f(r) = \left(1 - \frac{Q}{r}\right)^2$$





[Arrechea, Breen, Ottewill, Pisani & Taylor `arXiv:2409.04528`]

Schwarzschild-adS black holes

Schwarzschild-adS black holes

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\Omega_k^2 \qquad f(r) = k - \frac{2M}{r} - \frac{\Lambda r^2}{3} \qquad \Lambda < 0$$

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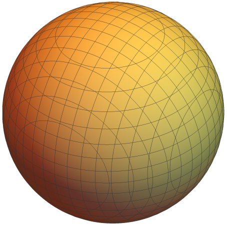
$k = 1$



$$\mathrm{d}\Omega_1^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$$

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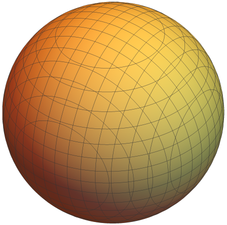
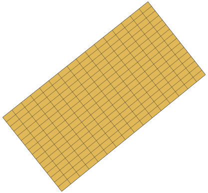


$k = 0$



$$d\Omega_1^2 = d\theta^2 + \sin^2\theta\,d\varphi^2 \qquad d\Omega_0^2 = d\theta^2 + \theta^2\,d\varphi^2$$

Schwarzschild-adS black holes

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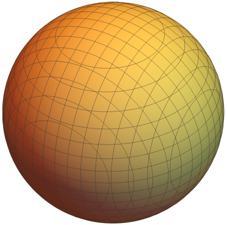

$$\mathrm{d}\Omega_1^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$$

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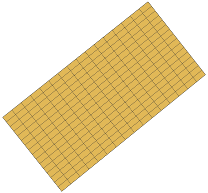
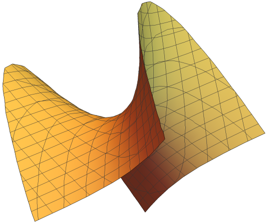


$$\mathrm{d}\Omega_0^2 = \mathrm{d}\theta^2 + \theta^2\,\mathrm{d}\varphi^2$$

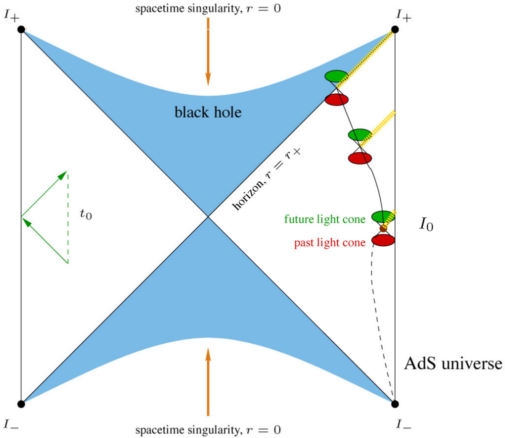
$k = -1$



$$\mathrm{d}\Omega_{-1}^2 = \mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\varphi^2$$

Schwarzschild-adS black holes

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\Omega_k^2 \qquad f(r) = k - \frac{2M}{r} - \frac{\Lambda r^2}{3} \qquad \Lambda < 0$$



[Figure: Ambrosetti, Charbonneau & Weinfurtner 0810.2631]

Schwarzschild-adS black holes

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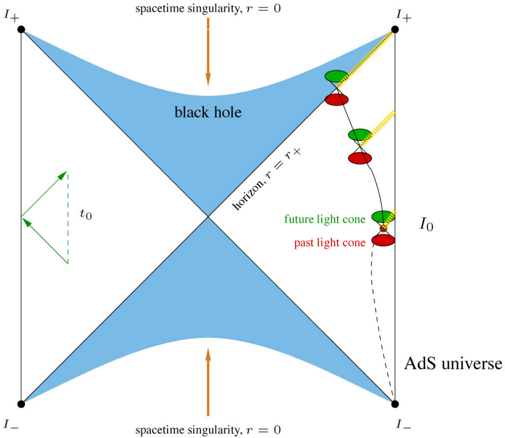
Boundary conditions at $r \to \infty$

[Figure: Ambrosetti, Charbonneau & Weinfurtner 0810.2631]



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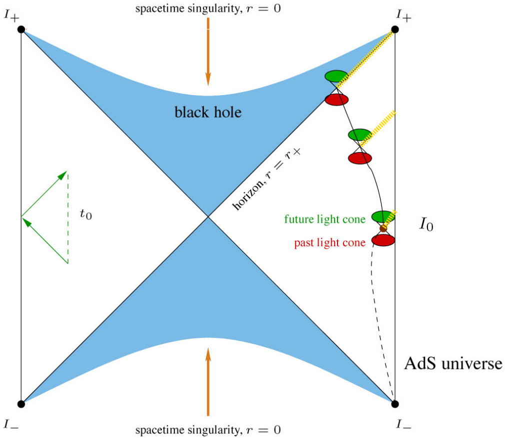
Boundary conditions at $r \to \infty$

- Dirichlet

$$\Phi = 0$$

[Figure: Ambrosetti, Charbonneau & Weinfurtner 0810.2631]



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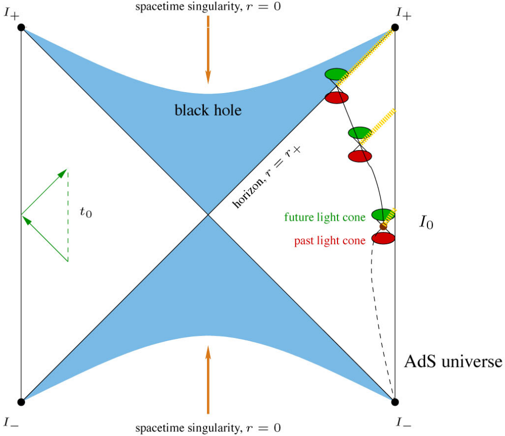
Boundary conditions at $r \to \infty$

- Dirichlet

$$\Phi = 0$$

- Neumann

$$\frac{\partial \Phi}{\partial n} = 0$$



[Figure: Ambrosetti, Charbonneau & Weinfurtner `0810.2631`]

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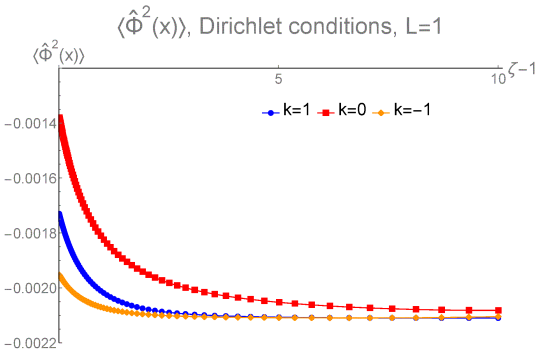
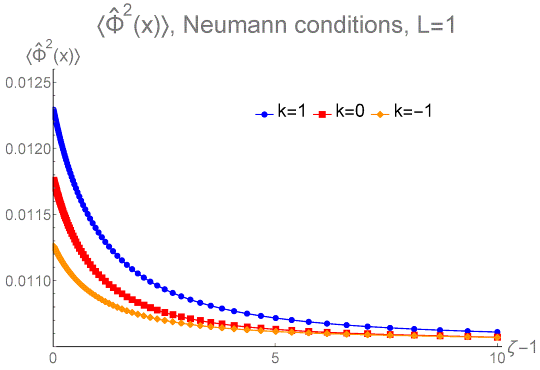




[Flachi & Tanaka *PRD* **78** 064011 (2008); Morley, Taylor & EW *CQG* **35** 235010 (2018), *PRD* **103** 045007 (2021)]

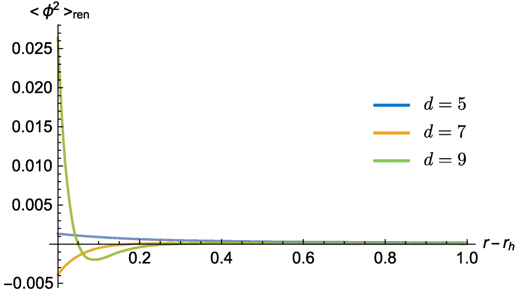
Schwarzschild-Tangherlini black holes

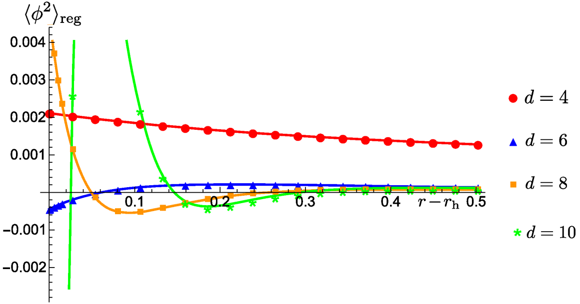
Schwarzschild-Tangherlini black holes

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\Omega_{d-2}^2 \qquad f(r) = 1 - \left(\frac{r_h}{r}\right)^{d-3}$$

Schwarzschild-Tangherlini black holes

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\Omega^2_{d-2} \qquad f(r) = 1 - \left(\frac{r_h}{r}\right)^{d-3}$$





[Taylor & Breen *PRD* **94** 125024 (2016); Taylor & Breen *PRD* **96** 105020 (2017)]

Extended coordinates implementation

Extended coordinates implementation

Advantages

- Mode-by-mode renormalization
- Topological black holes
- Higher-dimensional black holes

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- Euclidean technique
- Hartle-Hawking state
- Boulware state

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Homework

Extended the "extended coordinates" implementation to

- Rotating black holes
- SET on higher-dimensional black holes

Pragmatic mode sum implementation

Levi & Ori *PRD* **91** 104028 (2015)
Levi & Ori *PRD* **94** 044054 (2016)
Levi & Ori *PRL* **117** 231101 (2016)
Levi, Eilon, Ori & van de Meent *PRL* **118** 141102 (2017)
Levi *PRD* **95** 025007 (2017)
Lanir, Levi, Ori & Sela *PRD* **97** 024033 (2018)
Lanir, Levi & Ori *PRD* **98** 084017 (2018)
Lanir, Ori, Zilberman, Sela, Maline & Levi *PRD* **99** 061502 (2019)
Zilberman, Levi & Ori *PRL* **124** 171302 (2020)
Zilberman & Ori *PRD* **104** 024066 (2021)

VP in Boulware state

VP in Boulware state

$$\langle \hat{\Phi}^2(x) \rangle_{\mathrm{ren}} = \lim_{x' \to x} \left\{ -\mathrm{i} \left[G_{\mathrm{F}}(x,x') - G_{\mathrm{S}}(x,x') \right] \right\}$$

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Green function

$$-{\rm i}G_{\rm F}^{\rm B}(x,x') = \int_0^\infty {\rm d}\omega \, \frac{{\rm e}^{{\rm i}\omega(t-t')}}{4\pi|\mathcal{N}|^2 r^2} \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\gamma) \left[\left| \psi_{\omega\ell}^{\rm in}(r) \right|^2 + \left| \psi_{\omega\ell}^{\rm up}(r) \right|^2 \right]$$

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Time-like point-splitting $t - t' = \epsilon$,

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Time-like point-splitting $t - t' = \epsilon, \;\gamma = 0$

$$-\mathrm{i}G_\mathrm{F}^\mathrm{B}(x, x') = \int_0^\infty \mathrm{d}\omega \, \frac{\mathrm{e}^{\mathrm{i}\omega\epsilon}}{4\pi|\mathcal{N}|^2 r^2}$$

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Time-like point-splitting $t - t' = \epsilon,\ \gamma = 0$

$$-\mathrm{i}G_{\mathrm{F}}^{\mathrm{B}}(x,x') = \int_0^\infty \mathrm{d}\omega\,\frac{\mathrm{e}^{\mathrm{i}\omega\epsilon}}{4\pi|\mathcal{N}|^2r^2}\sum_{\ell=0}^\infty(2\ell+1)\left[\left|\psi^{\mathrm{in}}_{\omega\ell}(r)\right|^2 + \left|\psi^{\mathrm{up}}_{\omega\ell}(r)\right|^2\right]$$

$$= \int_0^\infty \mathrm{d}\omega\,\frac{\mathrm{e}^{\mathrm{i}\omega\epsilon}}{4\pi|\mathcal{N}|^2r^2}F(\omega)$$

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Time-like point-splitting $t - t' = \epsilon, \; \gamma = 0$

$$\begin{aligned} -\mathrm{i}G_{\mathrm{F}}^{\mathrm{B}}(x,x') &= \int_0^\infty \mathrm{d}\omega \, \frac{\mathrm{e}^{\mathrm{i}\omega\epsilon}}{4\pi|\mathcal{N}|^2 r^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left[\left| \psi_{\omega\ell}^{\mathrm{in}}(r) \right|^2 + \left| \psi_{\omega\ell}^{\mathrm{up}}(r) \right|^2 \right] \\ &= \int_0^\infty \mathrm{d}\omega \, \frac{\mathrm{e}^{\mathrm{i}\omega\epsilon}}{4\pi|\mathcal{N}|^2 r^2} F(\omega) \end{aligned}$$

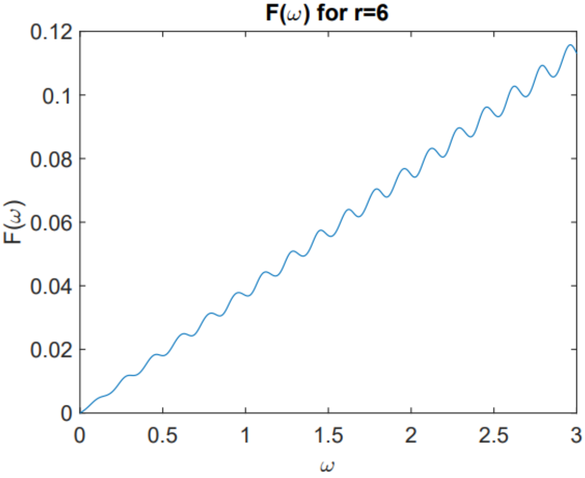
$F(\omega)$

$F(\omega)$

$$\sum_{\ell=0}^{\infty} (2\ell+1) \left[\left| \psi^{\text{in}}_{\omega\ell}(r) \right|^2 + \left| \psi^{\text{up}}_{\omega\ell}(r) \right|^2 \right]$$

- Diverges linearly as $\omega \to \infty$

[Figure: Levi & Ori *PRD* **91** 104028 (2015)]



Hadamard parametrix $t - t' = \epsilon$

$$-\mathrm{i}G_{\mathrm{S}}(x,x') = \frac{U(x,x')}{\sigma(x,x')} + V(x,x')\log\left[\frac{\sigma(x,x')}{L^2}\right]$$

Hadamard parametrix $t - t' = \epsilon$

$$-\mathrm{i}G_{\mathrm{S}}(x,x') = \frac{1}{4\pi^2 f \epsilon^2} + \frac{1}{8\pi^2}\left[\mu^2 - \left(\xi - \frac{1}{6}\right) R\right]\left[\mathsf{C} + \frac{1}{2}\log\left(\frac{f\epsilon^2}{4L^2}\right)\right]$$
$$- \frac{\mu^2}{16\pi^2} + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r} + \ldots$$

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Integral representation of singular terms

Hadamard parametrix $t - t' = \epsilon$

$$-\mathrm{i}G_{\mathrm{S}}(x,x') = \frac{1}{4\pi^2 f \epsilon^2} + \frac{1}{8\pi^2}\left[\mu^2 - \left(\xi - \frac{1}{6}\right) R\right]\left[\mathsf{C} + \frac{1}{2}\log\left(\frac{f\epsilon^2}{4L^2}\right)\right] - \frac{\mu^2}{16\pi^2} + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r} + \dots$$

Integral representation of singular terms

$$\epsilon^{-2} = -\int_{\omega=0}^{\infty} \omega\, e^{\mathrm{i}\omega\epsilon}\, \mathrm{d}\omega$$

Hadamard parametrix $t - t' = \epsilon$

$$-\mathrm{i}G_S(x,x') = \frac{1}{4\pi^2 f \epsilon^2} + \frac{1}{8\pi^2}\left[\mu^2 - \left(\xi - \frac{1}{6}\right) R\right]\left[\mathsf{C} + \frac{1}{2}\log\left(\frac{f\epsilon^2}{4L^2}\right)\right]$$
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Integral representation of singular terms

$$\epsilon^{-2} = -\int_{\omega=0}^{\infty} \omega\, \mathrm{e}^{\mathrm{i}\omega\epsilon}\, \mathrm{d}\omega$$
$$\log(\epsilon\alpha) = -\int_{\omega=0}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega\epsilon}}{\omega + \alpha}\, \mathrm{d}\omega + \left(\frac{\mathrm{i}\pi}{2} - \mathsf{C}\right) + \dots$$

Hadamard parametrix $t - t' = \epsilon$

$$-\mathrm{i}G_{\mathrm{S}}(x,x') = \frac{1}{4\pi^2 f\epsilon^2} + \frac{1}{8\pi^2}\left[\mu^2 - \left(\xi - \frac{1}{6}\right)R\right]\left[\mathsf{C} + \frac{1}{2}\log\left(\frac{f\epsilon^2}{4L^2}\right)\right] - \frac{\mu^2}{16\pi^2} + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r} + \ldots$$

Integral representation of singular terms

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$$\log(\epsilon\alpha) = -\int_{\omega=0}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega\epsilon}}{\omega+\alpha}\, \mathrm{d}\omega + \left(\frac{\mathrm{i}\pi}{2} - \mathsf{C}\right) + \ldots$$

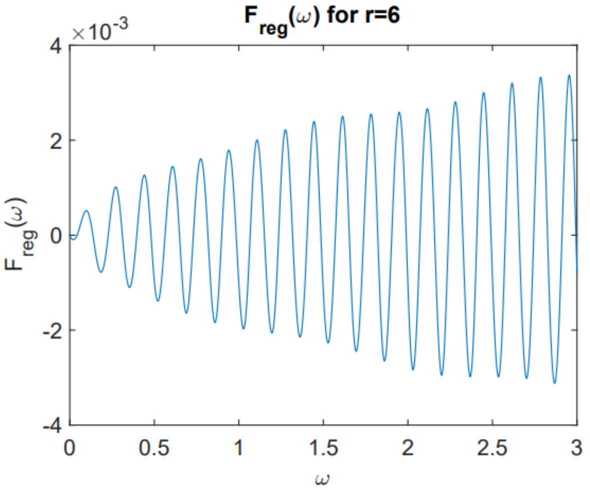
$$-\mathrm{i}G_{\mathrm{S}}(x,x') = \int_{\omega=0}^{\infty}\left[\mathcal{A}(r)\omega + \frac{\mathcal{B}(r)}{\omega + \sqrt{f}/2L}\right]\mathrm{e}^{\mathrm{i}\omega\epsilon}\, \mathrm{d}\omega + \mathcal{C}(r) + \ldots$$

$F_{\text{reg}}(\omega)$

$F_{\text{reg}}(\omega)$

$$F_{\text{reg}}(\omega) = F(\omega) - \left[\mathcal{A}(r)\omega + \frac{\mathcal{B}(r)}{\omega + \sqrt{f}/2L} \right]$$

- Oscillates as $\omega \to \infty$



[Figure: Levi & Ori *PRD* **91** 104028 (2015)]

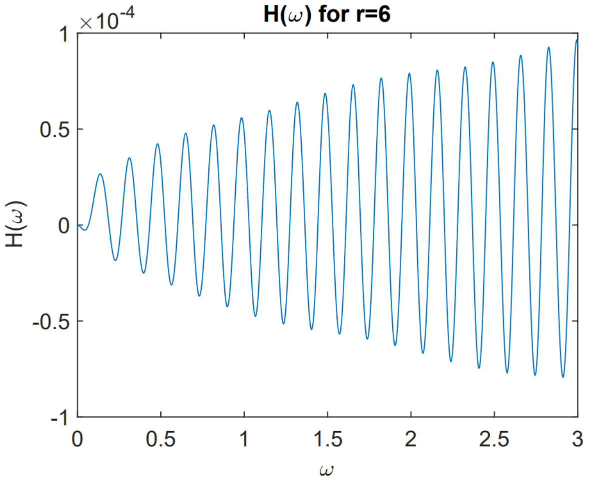
$H(\omega)$

$H(\omega)$

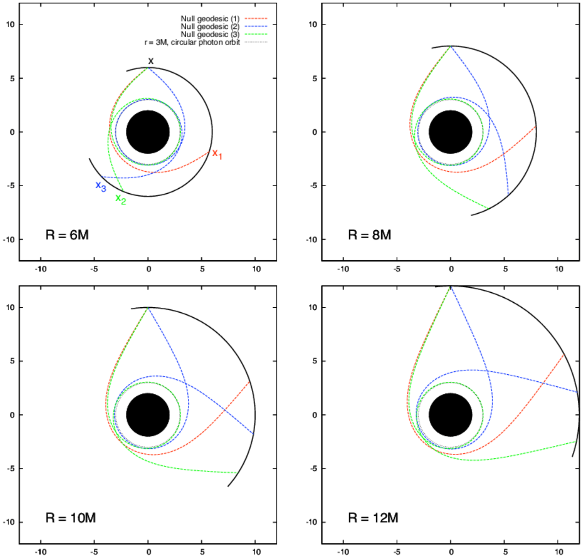
$$H(\omega) = \int_0^{\omega} F_{\mathrm{reg}}(\omega')\, \mathrm{d}\omega'$$

- Oscillates as $\omega \to \infty$
- No limit as $\omega \to \infty$

[Figure: Levi & Ori *PRD* **91** 104028 (2015)]



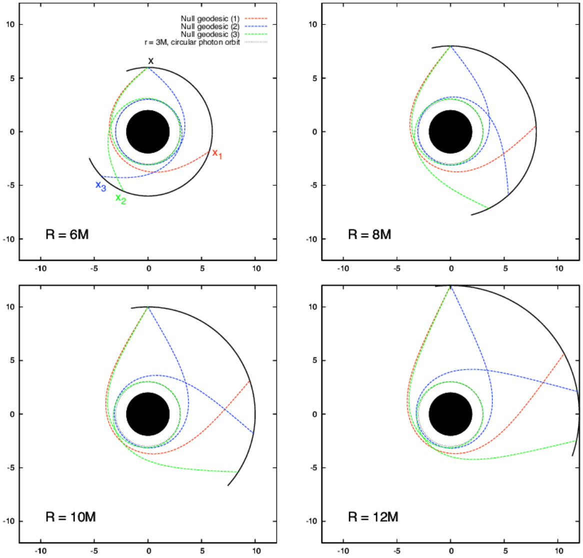
Null geodesics



[Figure: Casals, Dolan, Ottewill & Wardell *PRD* **79** 124043 (2009)]

Null geodesics

- Null geodesics orbit black hole

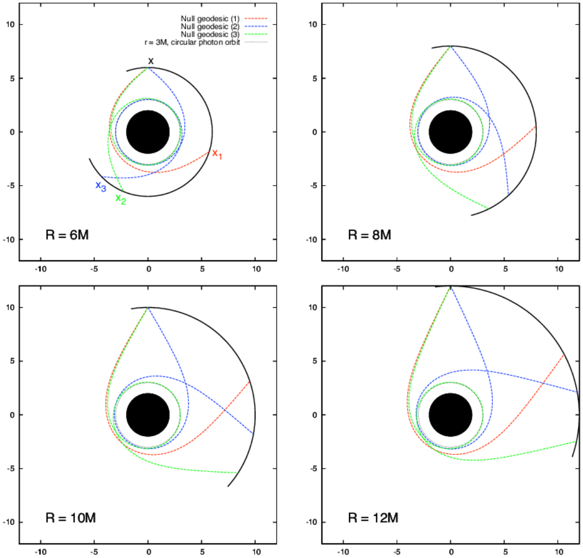


[Figure: Casals, Dolan, Ottewill & Wardell *PRD* **79** 124043 (2009)]

Null geodesics

- Null geodesics orbit black hole
- Return to same spatial point at different time

[Figure: Casals, Dolan, Ottewill & Wardell *PRD* **79** 124043 (2009)]



Null geodesics

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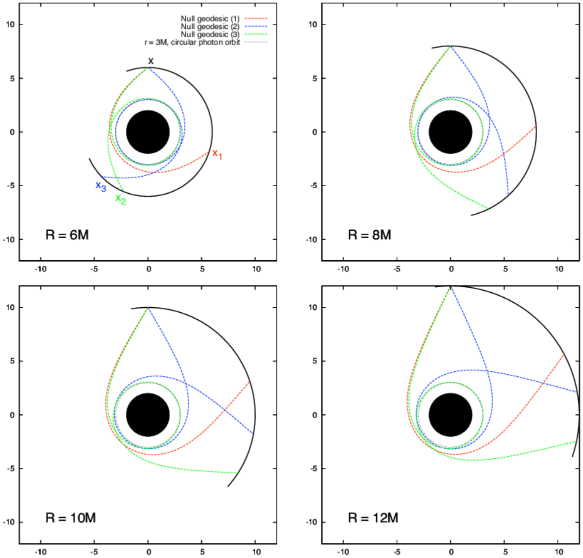
- Nonlocal singularity in Green function



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Null geodesics

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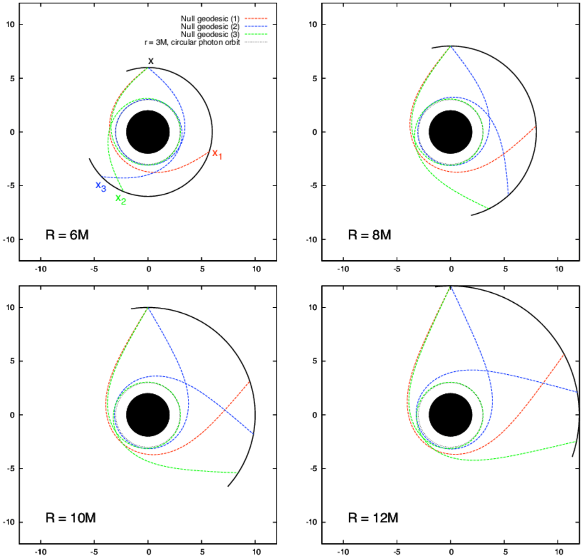
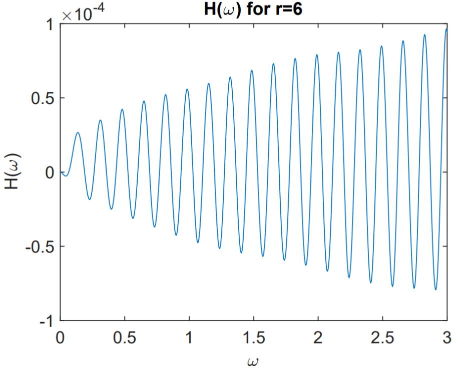
- Return to same spatial point at different time
- Nonlocal singularity in Green function
- Not captured by Hadamard parametrix

[Figure: Casals, Dolan, Ottewill & Wardell *PRD* **79** 124043 (2009)]



Generalized integrals

$$\mathcal{H}(\omega) = \int_{\varpi=0}^{\omega} \mathrm{e}^{\mathrm{i}\varpi\epsilon} \mathcal{G}_{\varpi}(r)\, \mathrm{d}\varpi$$



[Levi & Ori *PRD* **91** 104028 (2015)]

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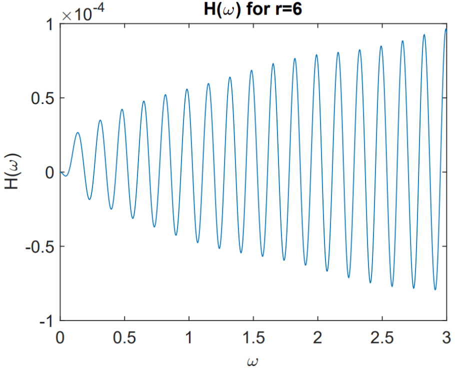

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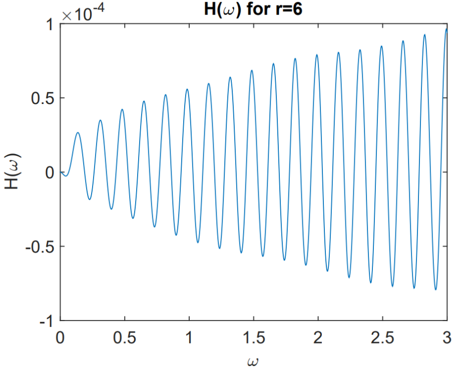

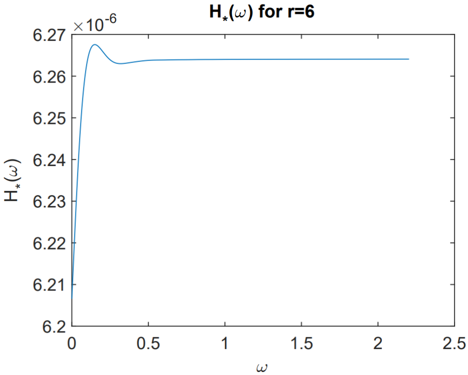

[Levi & Ori *PRD* **91** 104028 (2015)]

SET on Schwarzschild

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2 \qquad f(r) = 1 - \frac{2M}{r}$$

[Levi *PRD* **95** 025007 (2017)]

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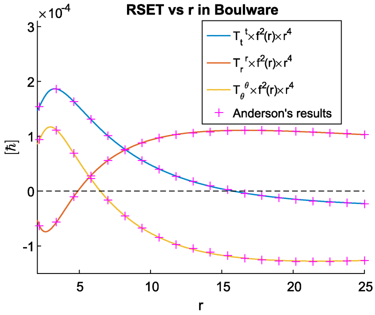

[Levi *PRD* **95** 025007 (2017)]

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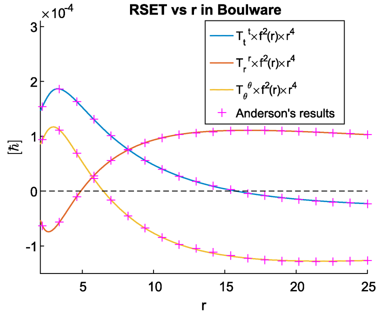

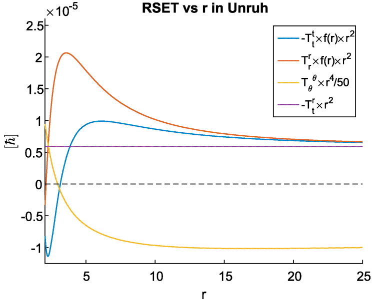

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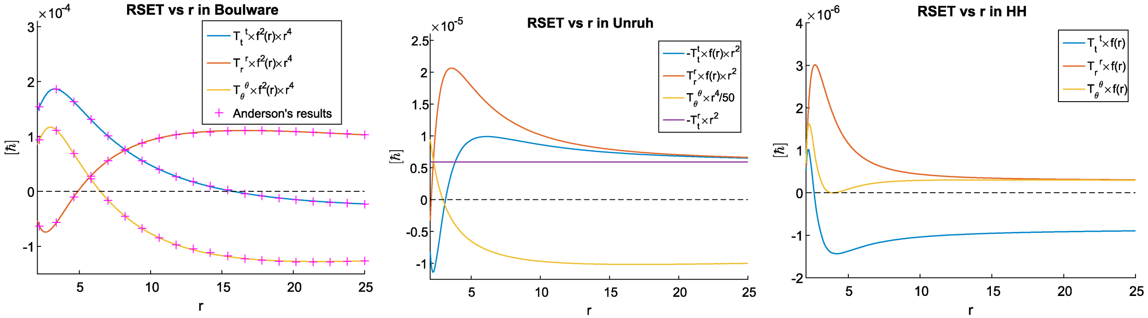
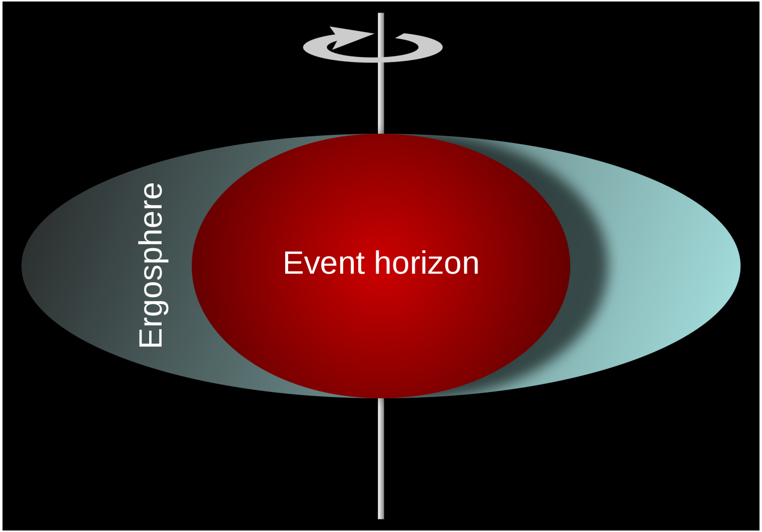


[Levi *PRD* **95** 025007 (2017)]

Rotating black holes



Kerr black hole

Kerr black hole

$$ds^2 = -\Delta\Sigma^{-1}\left[\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi\right]^2 + \Sigma\Delta^{-1}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 + \Sigma^{-1}\sin^2\theta\left[\left(r^2+a^2\right)\mathrm{d}\varphi - a\,\mathrm{d}t\right]^2$$

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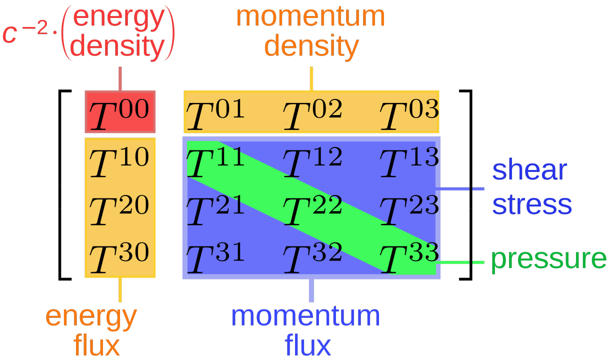

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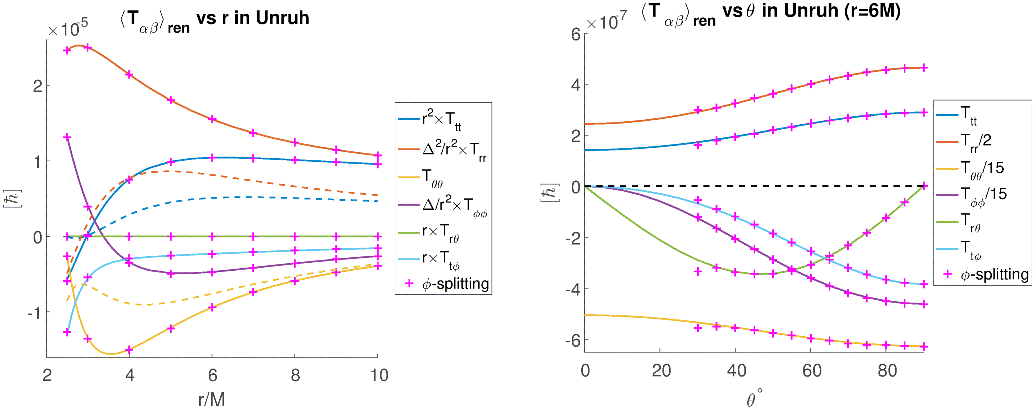

[Levi, Eilon, Ori & van de Meent *PRL* **118** 141102 (2017)]

Pragmatic mode-sum implementation

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Advantages

- Lorentzian space-time
- All quantum states: Boulware, Unruh and Hartle-Hawking
- Rotating and nonrotating black holes

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Homework

Extend pragmatic mode-sum implementation to

- Higher-dimensional black holes

Black hole interiors

Lanir, Levi & Ori *PRD* **98** 084017 (2018)
Lanir, Levi, Ori & Sela *PRD* **97** 024033 (2018)
Zilberman, Levi & Ori *PRL* **124** 171302 (2020)
Hollands, Wald & Zahn *CQG* **37** 115009 (2020)
Hollands, Klein & Zahn *PRD* **102** 085004 (2020)
Zilberman & Ori *PRD* **104** 024066 (2021)
Zilberman, Casals, Ori & Ottewill *PRL* **129** 261102 (2022)
Klein, Soltani, Casals & Hollands *PRL* **132** 121501 (2024)

Zilberman, Casals, Levi, Ori and Ottewill `arXiv:2409.17464`

Inside a Schwarzschild black hole

$$
\begin{aligned}
\mathrm{d}s^2 = & -f(r)\,\mathrm{d}t^2 + f(r)^{-1}\,\mathrm{d}r^2 \\
& + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2
\end{aligned}
$$

$$
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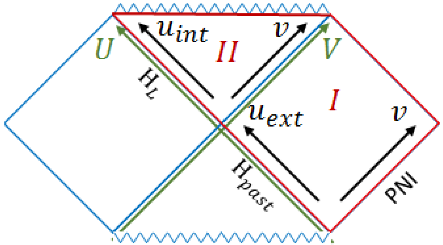
$$



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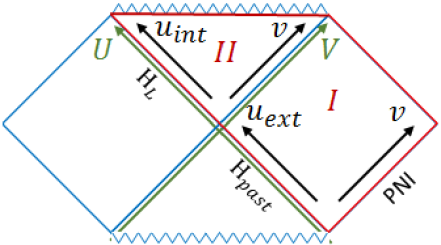
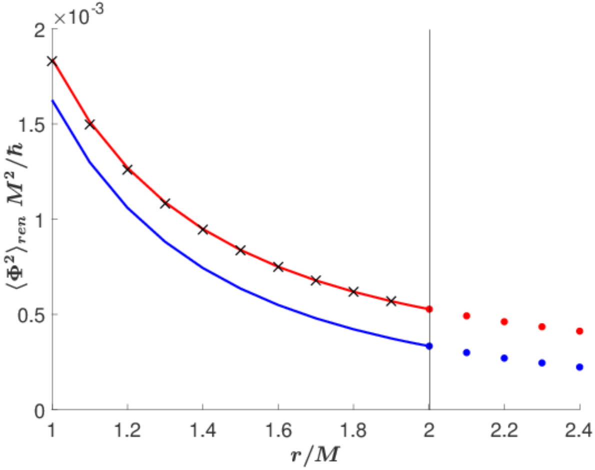
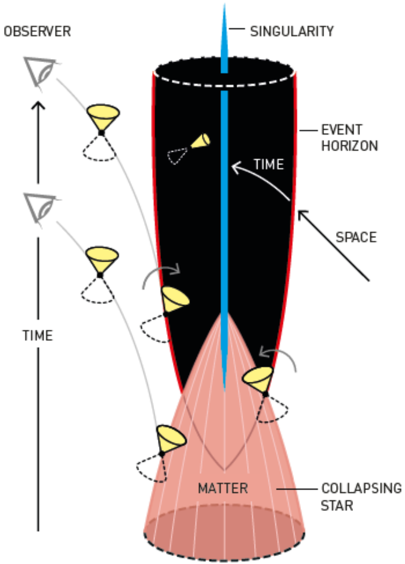




[Lanir, Levi & Ori *PRD* **98** 084017 (2018)]

Cosmic censorship

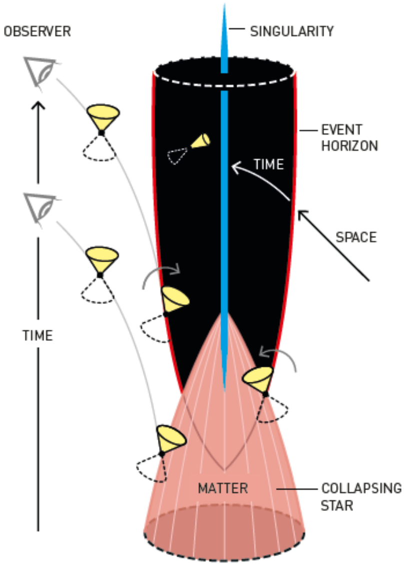


[Figure: Johan Jarnestad
The Royal Swedish Academy of Sciences]

Cosmic censorship

Weak cosmic censorship

- Singularity at the centre of black hole
- Not visible to an observer at infinity



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Cosmic censorship

Weak cosmic censorship

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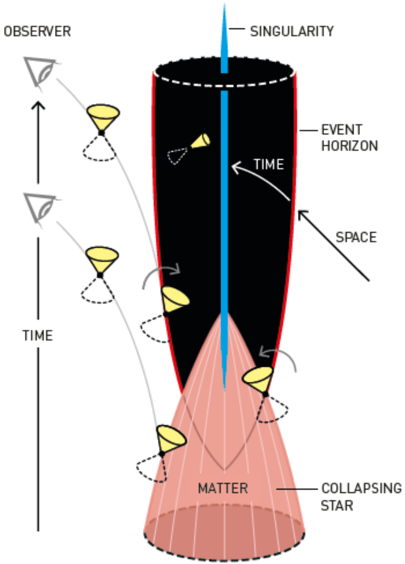
Strong cosmic censorship

- Singularity not visible to an observer inside a black hole
- No breakdown in predictability

[Figure: Johan Jarnestad
The Royal Swedish Academy of Sciences]



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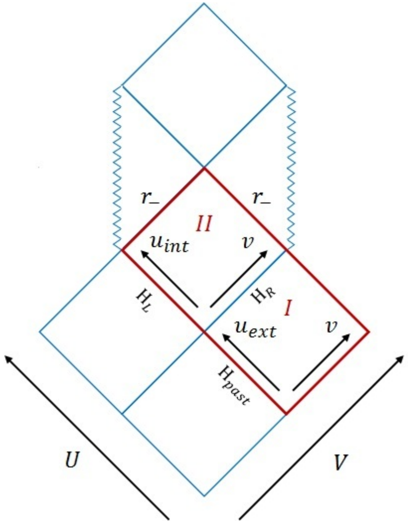

[Figure: Lanir, Levi, Ori & Sela *PRD* **97** 024033 (2018)]

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Horizons $f(r_\pm) = 0$

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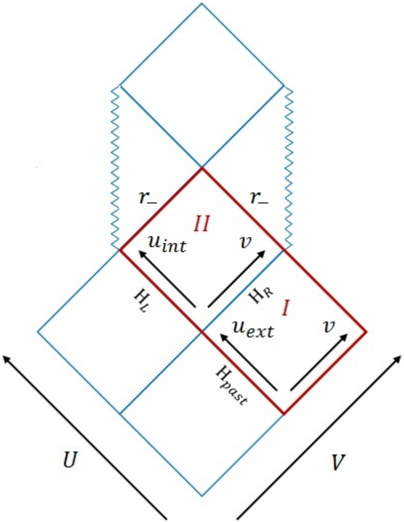

[Figure: Lanir, Levi, Ori & Sela *PRD* **97** 024033 (2018)]

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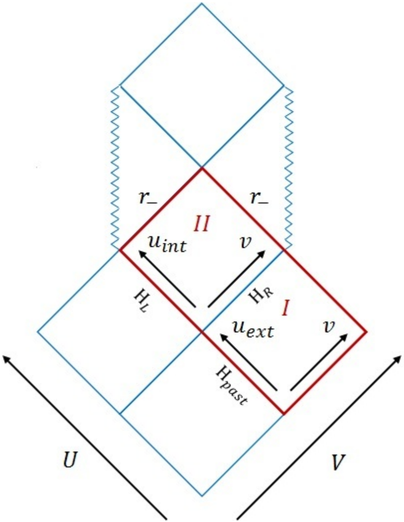

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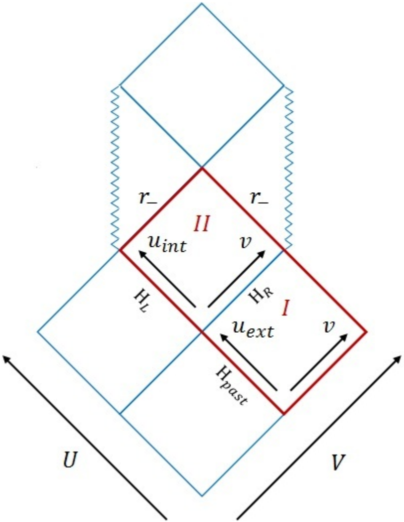

[Figure: Lanir, Levi, Ori & Sela *PRD* **97** 024033 (2018)]

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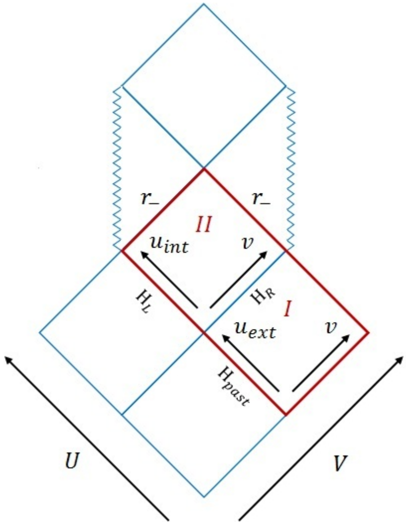

[Figure: Lanir, Levi, Ori & Sela *PRD* **97** 024033 (2018)]

Inside a Reissner-Nordström black hole

Horizons $f(r_\pm) = 0$

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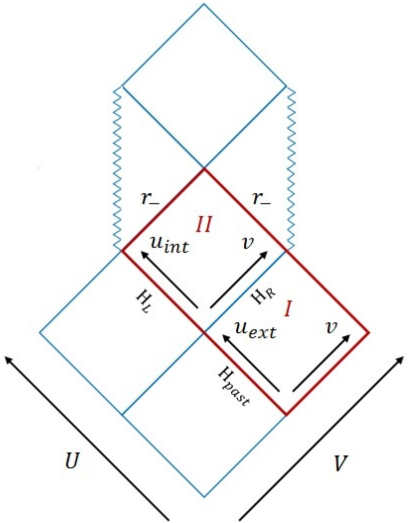
Inner horizon

- Naked singularity inside



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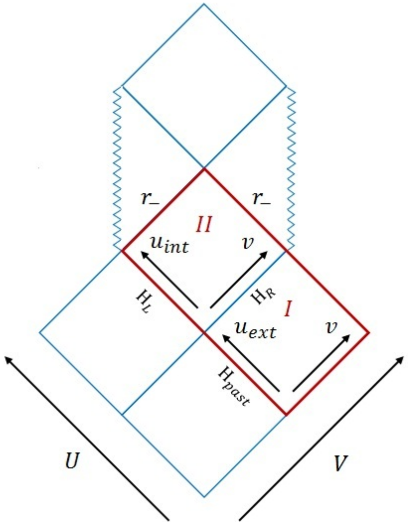
Inner horizon

- Naked singularity inside
- Violates cosmic censorship



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Inner horizon

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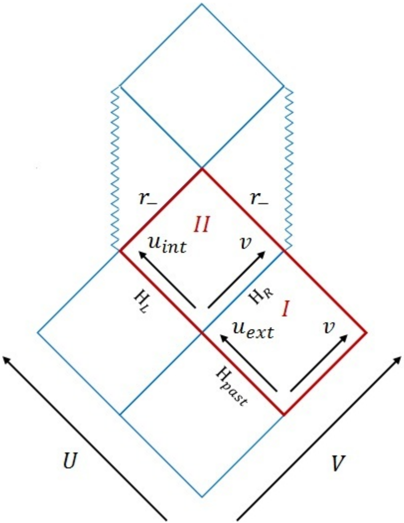
- Violates cosmic censorship
- Classical perturbation diverges
  [Simpson & Penrose *IJTP* **7** 183 (1973)]



[Figure: Lanir, Levi, Ori & Sela *PRD* **97** 024033 (2018)]

Inside a Reissner-Nordström black hole

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At the inner horizon $V \to 0$

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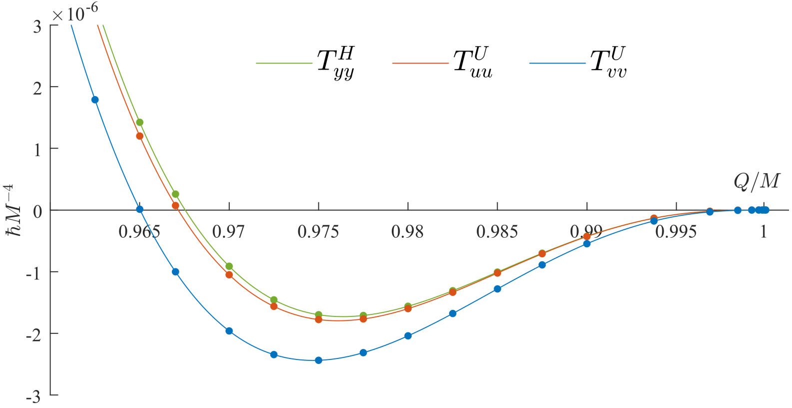

[Zilberman, Levi & Ori *PRL* **124** 171302 (2020); Zilberman & Ori *PRD* **104** 024066 (2021)]

Inside a Kerr black hole

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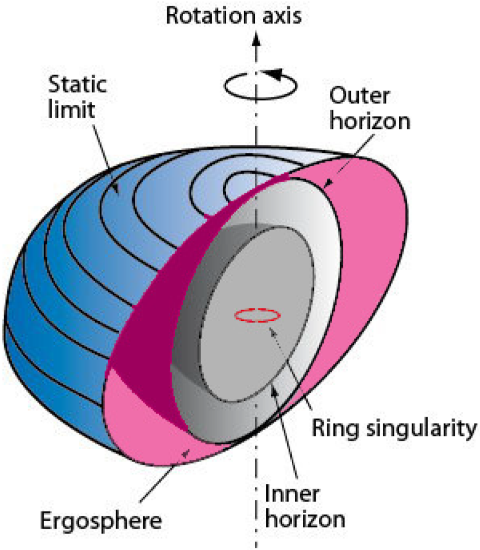

[Figure: Tito & Pavlov *Galaxies* **6** 61 (2018)]

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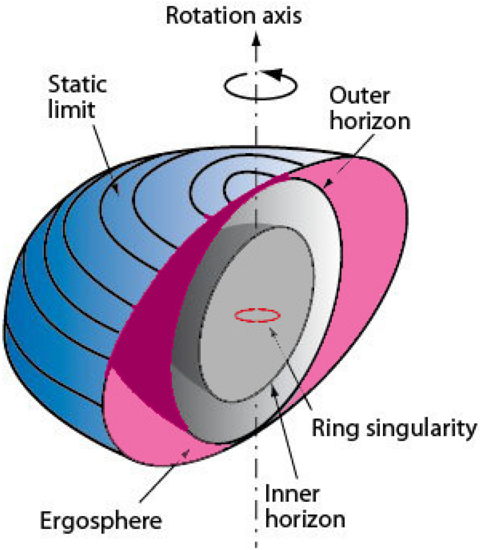
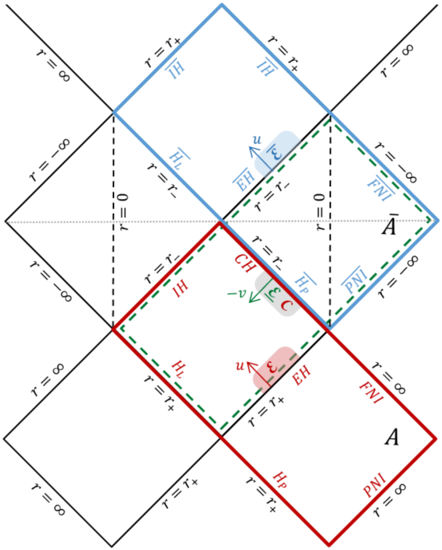


[Figure: Tito & Pavlov *Galaxies* **6** 61 (2018)]

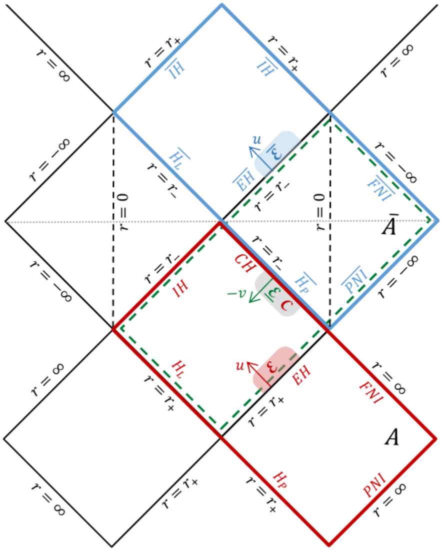
State subtraction



[Figure: Zilberman, Casals, Ori & Ottewill *PRL* **129** 261102 (2022)]

State subtraction

- Hadamard parametrix is independent of the quantum state



[Figure: Zilberman, Casals, Ori & Ottewill *PRL* **129** 261102 (2022)]

State subtraction

- Hadamard parametrix is independent of the quantum state
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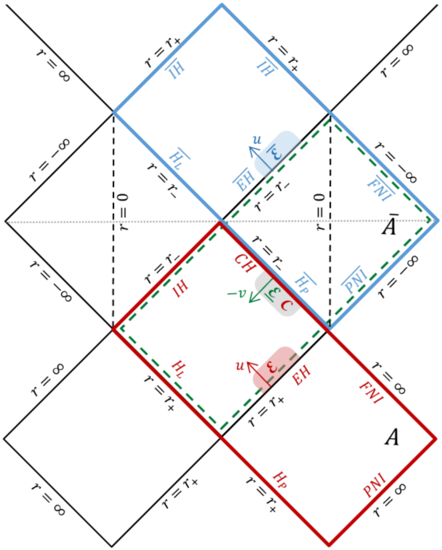
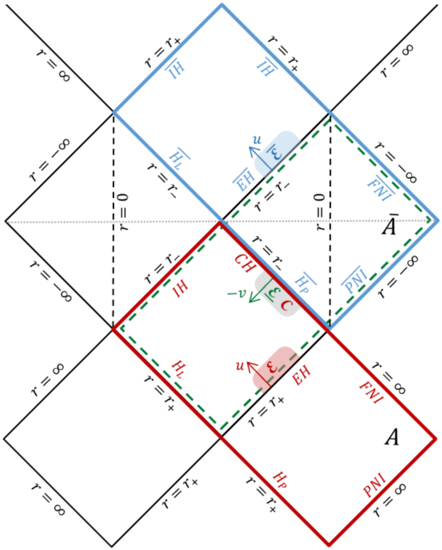
[Figure: Zilberman, Casals, Ori & Ottewill *PRL* **129** 261102 (2022)]



State subtraction

- Hadamard parametrix is independent of the quantum state
- Differences between two quantum states do not require renormalization
- Construct a quantum state regular on the inner horizon

[Figure: Zilberman, Casals, Ori & Ottewill *PRL* **129** 261102 (2022)]



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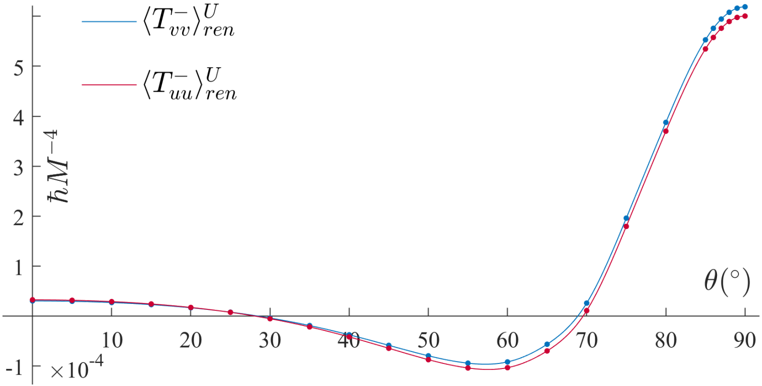

[Zilberman, Casals, Ori & Ottewill *PRL* **129** 261102 (2022)]

Inside a Kerr black hole

Pragmatic mode-sum renormalization

[Zilberman, Casals, Levi,
Ori and Ottewill
`arXiv:2409.17464`]

Inside a Kerr black hole

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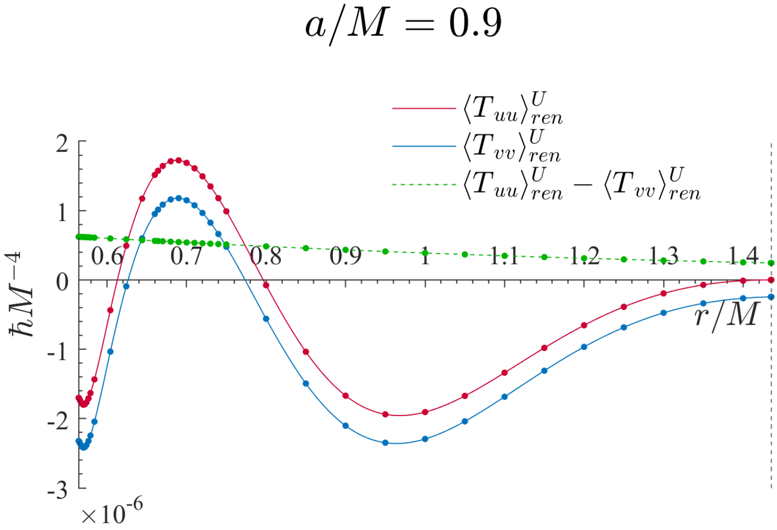

Inside a Reissner-Nordström-de Sitter black hole

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Horizons $f(r) = 0$

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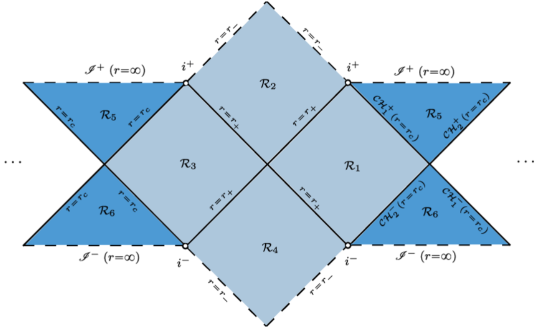

[Figure: Natario & Sasane *Ann. H. Poincaré* **23** 2345 (2022)]

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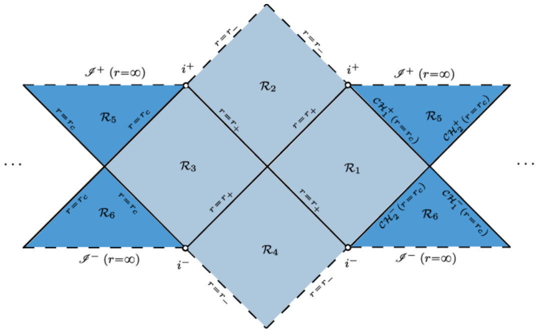

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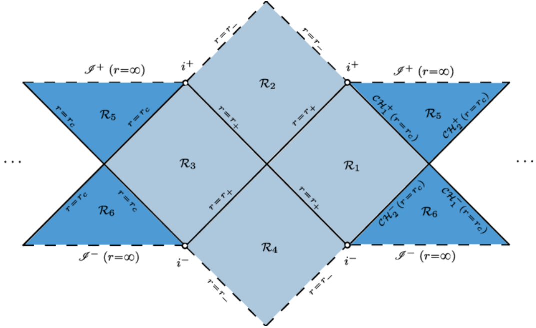
Cosmic censorship fails classically for some RNdS black holes

[Dias et al *JHEP* **10** 001 (2018)
Cardoso et al *PRL* **120** 031103 (2018)
Chrysostomou et al `arXiv:2501.12968`]



[Figure: Natario & Sasane *Ann. H. Poincaré* **23** 2345 (2022)]

Inside a Reissner-Nordström-de Sitter black hole

As $V \to 0$

$$\langle \hat{T}_{VV} \rangle \sim \kappa_-^2 \widetilde{C} \, |V|^{-2} + \ldots$$

[Hollands, Wald & Zahn
CQG **37** 115009 (2020)
Hintz & Klein *CQG* **41** 075006
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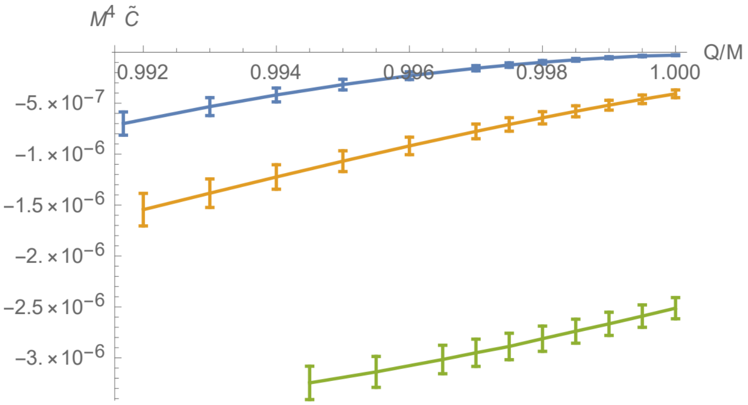
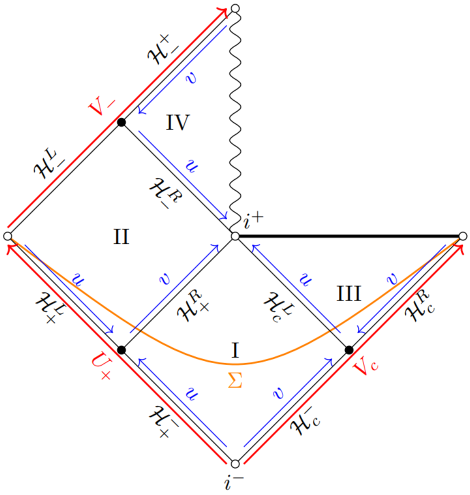


[Hollands, Klein & Zahn *PRD* **102** 085004 (2020)]

Inside a Kerr-de Sitter black hole



[Figure: Klein, Soltani, Casals & Hollands *PRL* **132** 121501 (2024)]

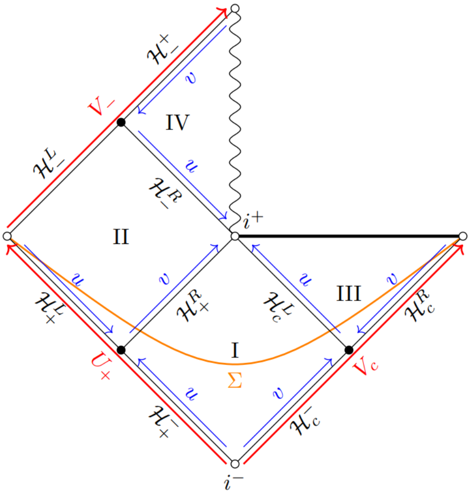
Inside a Kerr-de Sitter black hole

$$
\begin{aligned}
ds^2 = & -\frac{1}{\Sigma\Xi}\left[\Delta_r - a^2\Delta_\theta \sin^2\theta\right] dt^2 \\
& + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2 \\
& + \left[\Delta_\theta \left(r^2+a^2\right)^2 - a^2\Delta_r \sin^2\theta\right] \frac{\sin^2\theta}{\Sigma\Xi} d\varphi^2 \\
& + \frac{2a\sin^2\theta}{\Sigma\Xi}\left[\Delta_r - \Delta_\theta\left(r^2+a^2\right)\right] dt\, d\varphi
\end{aligned}
$$

$$\Delta_r = \left(1 - \tfrac{1}{3}a^2\Lambda\right)\left(r^2+a^2\right) - 2Mr$$

$$\Delta_\theta = 1 + \tfrac{1}{3}a^2\Lambda\cos^2\theta$$

$$\Sigma = r^2 + a^2\cos^2\theta \qquad \Xi = \left(1 + \tfrac{1}{3}a^2\Lambda\right)^2$$



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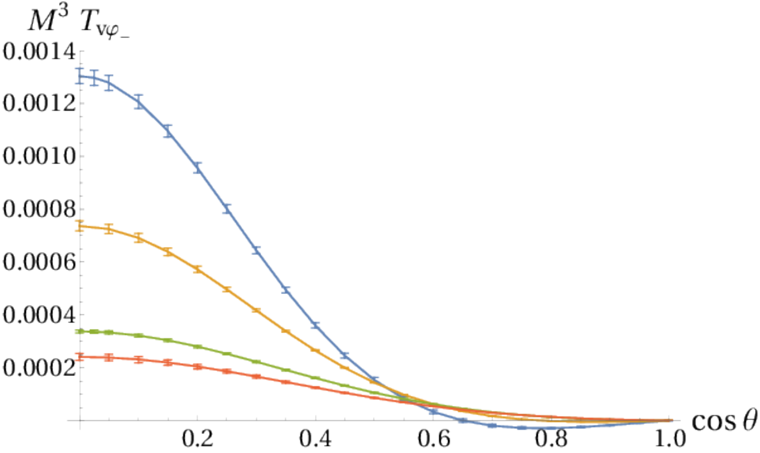

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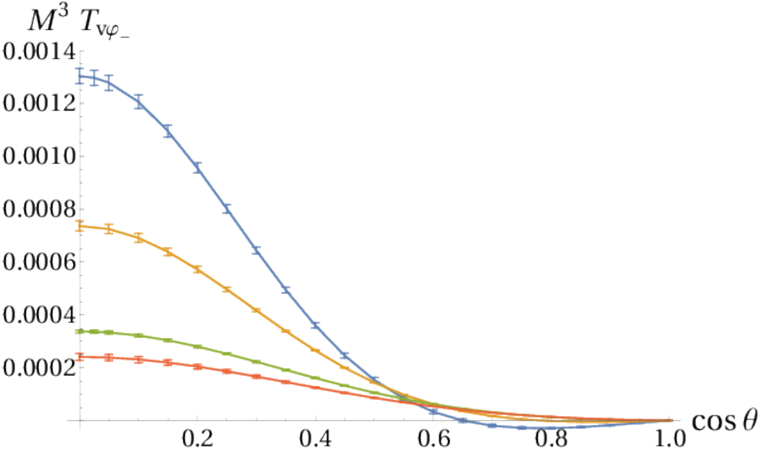

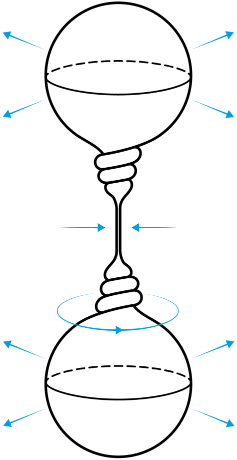

[Klein, Soltani, Casals & Hollands *PRL* **132** 121501 (2024)]

Black hole interiors

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Methods

- State subtraction
- Pragmatic mode sum

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Applications

- Behaviour at inner horizons
- Strong cosmic censorship

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Methods

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Homework

- Extend Euclidean methods to the black hole interior

Renormalized stress-energy tensor

Semi-classical Einstein equations

$$G_{\lambda\rho} + \Lambda g_{\lambda\rho} = 8\pi \langle \hat{T}_{\lambda\rho} \rangle$$

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Homework

Solve the backreaction problem

Questions?