Horizon thermodynamics

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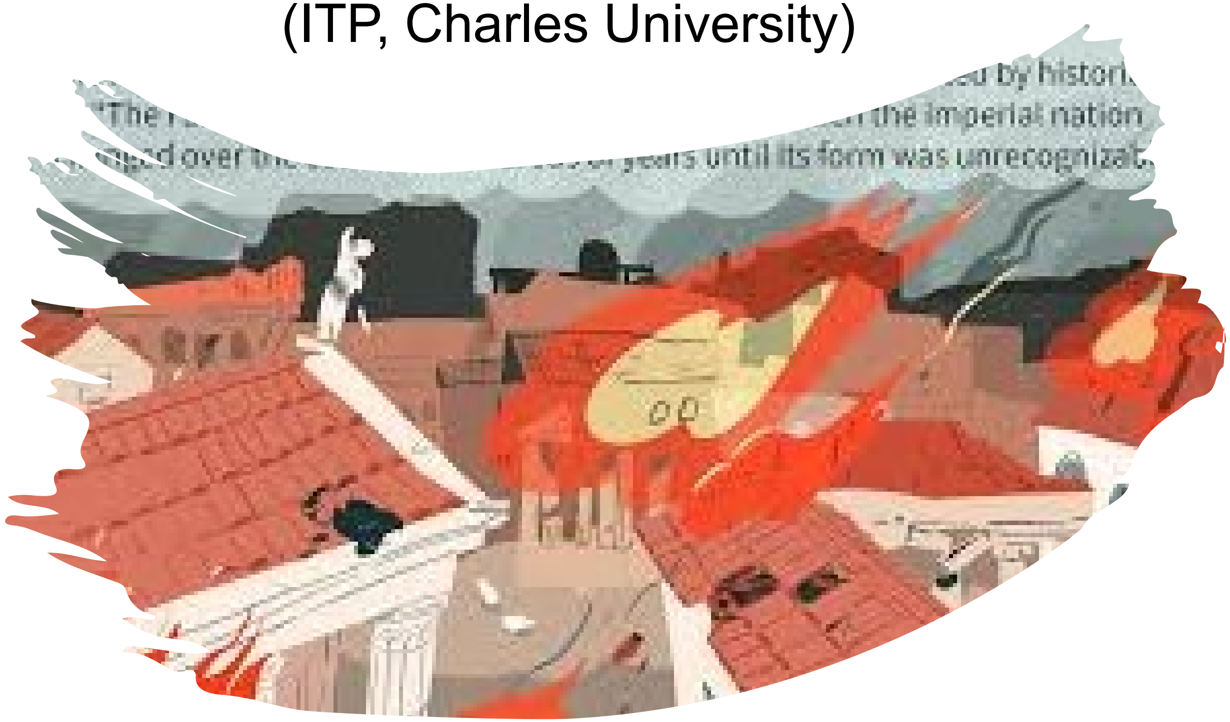

SIGRAV International School 2025
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Plan for the course

Lecture 1: Introduction to BH thermodynamics

BH thermodynamics, Euclidean magic, thermodynamics of BH-Rindler-dS horizons, observer is crucial!

Lecture 2: AdS black holes & black hole chemistry

AdS black holes, thermodynamics with variable cosmological constant, a few words about holography

Lecture 3: Mysterious topological density & BH TDs

2 more ways to study BH thermodynamics, Gauss-Bonnet topological density: 4d GB gravity & modified entropy, shall the area law prevail?

Lecture 1: Introduction to black hole thermodynamics

Plan for Lecture 1

I. **Black holes as thermodynamic objects**
   I. From BH mechanics to BH TDs
   II. Black hole evaporation

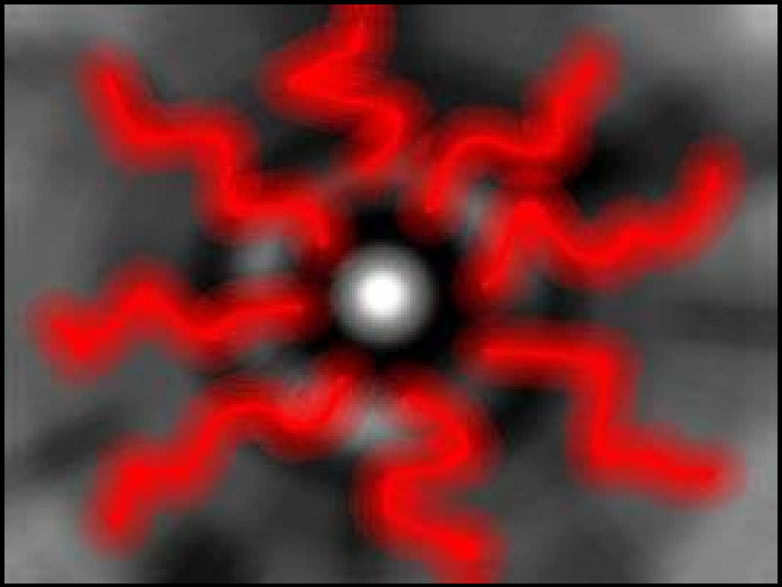
II. **Euclidean magic**
   I. Temperature from regularity
   II. Entropy from gravitational action

III. **Horizon TDs: examples**
   I. Schwarzschild BH
   II. Rindler
   III. De Sitter

IV. **What is going on?** A few words about QFT in CS

V. **Summary**

I. Black Holes as Thermodynamic Objects



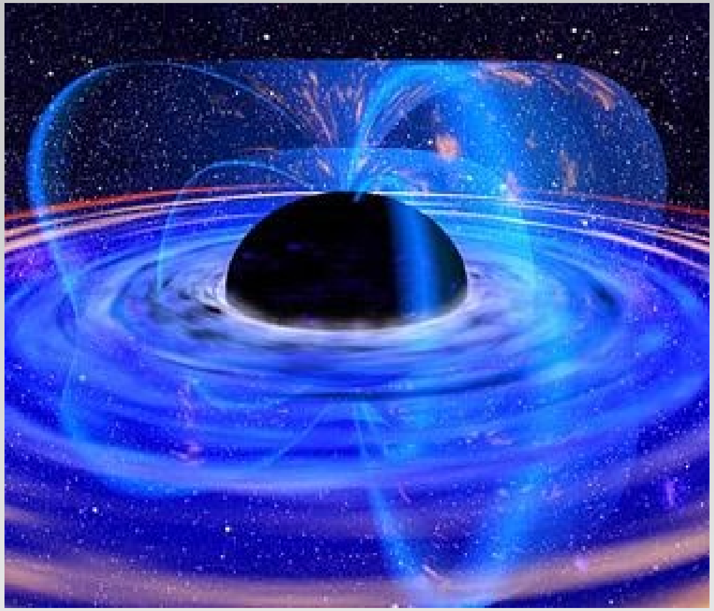
Black holes as thermodynamic objects

If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations-then so much the worse for Maxwell's equations. If it is found to be contradicted by observation-well these experimentalists do bungle things sometimes. But if your theory is found to be against the second law of thermodynamics I can give you no hope; there is nothing for it but to collapse in deepest humiliation.

Sir Arthur Stanley Eddington

Gifford Lectures (1927), *The Nature of the Physical World* (1928), 74.

Black holes and their characteristics



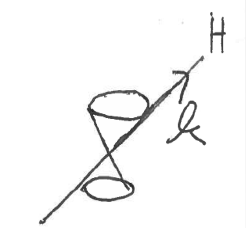
Schwarzschild black hole:

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \qquad f = 1 - \frac{2m}{r}$$

- asymptotic mass (total energy)

$$M = -\frac{1}{8\pi}\int_{S^2_\infty} *dk = m \qquad k = \partial_t$$

- black hole horizon: (radius $r_+ = 2M$)



surface gravity $$(k^b \nabla_b k^a)_{|H} = \kappa k^a_{|H}$$

$$\Rightarrow \quad \kappa = \frac{f'(r_+)}{2} = \frac{M}{r_+^2} = \frac{M}{(2M)^2} = \frac{1}{4M} = \frac{1}{2r_+}$$

horizon area: $$A = \int \sqrt{\det \gamma}\, d\theta d\varphi = \int r_+^2 \sin\theta\, d\theta d\varphi = 4\pi r_+^2$$

Schwarzschild characteristics: summary

Horizon: $$r_+ = 2M$$

Mass: $$M$$

Surface gravity: $$\kappa = \frac{f'(r_+)}{2} = \frac{1}{2r_+}$$

Horizon area: $$A = \int \sqrt{\det \gamma}\, d\theta d\varphi = 4\pi r_+^2 \,.$$

Good idea:

$$dM = \frac{dr_+}{2}\,, \qquad dA = 8\pi r_+ dr_+$$

1st law of black hole mechanics:

$$dM = \frac{\kappa}{2\pi}\frac{dA}{4}$$

Laws of black hole mechanics

- Bardeen, Carter, Hawking (1973)

- **Zeroth law:** The surface gravity κ is constant on the black hole horizon.
- **First law:**

$$dM = \frac{\kappa}{2\pi}\frac{dA}{4} + \underbrace{\Omega dJ + \Phi dQ}_{\text{work terms}} . \tag{5.8}$$

Here, Ω is the angular velocity of the black hole horizon, and Φ is its 'electrostatic potential'.

- **Second law:** Classically, the area of the horizon never decreases (provided the null energy condition holds).

$$dA \geq 0 . \tag{5.9}$$

- **Third law:** It is impossible to reduce κ to zero in a finite number of steps.

- Essentially equivalent to **gravitational dynamics**
- Despite the resemblance with laws of TDs, **classical** BHs **are black**

Black hole thermodynamics?

$$dM = \frac{\kappa}{2\pi}\frac{dA}{4}$$

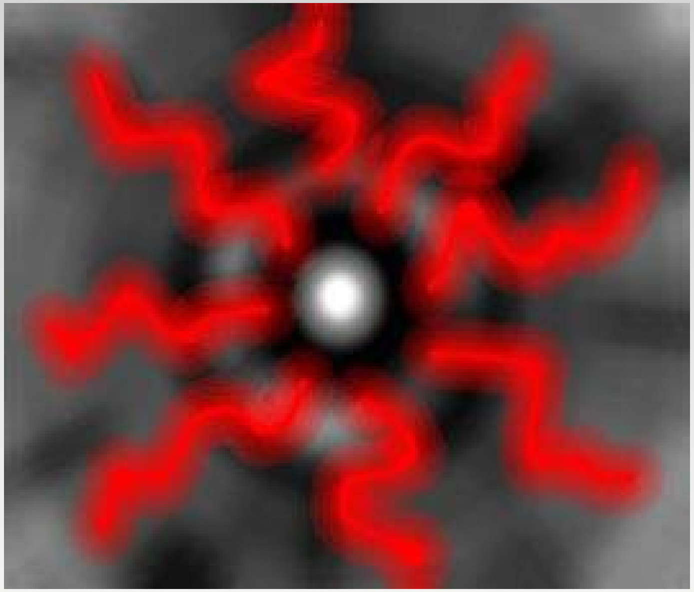
Bekenstein?

$$dE = TdS$$

Hawking (1974):

$$T = \frac{\kappa}{2\pi} \quad \Rightarrow \quad S = \frac{A}{4}$$

derived using QFT in curved spacetime



Other approaches: Euclidean path integral approach (Gibbons & Hawking-1977), tunnelling, LQG, string theory,…

Classical laws of black hole mechanics become laws of "normal" thermodynamics

Black hole entropy

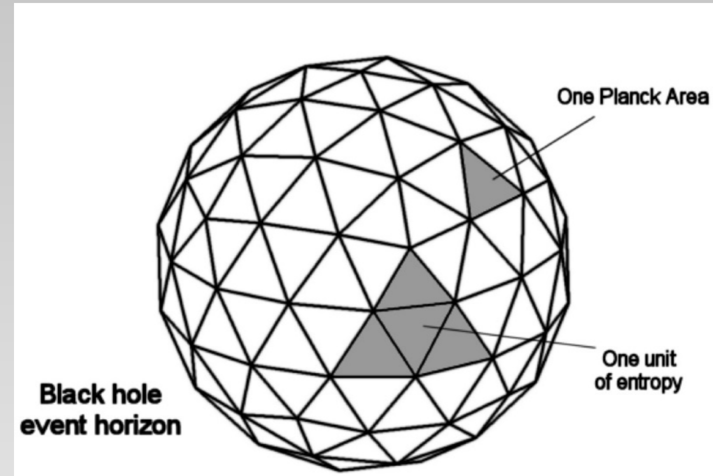
relativity

Stat. mech

$$S = \frac{A}{4}\frac{c^3 k_B}{\hbar G_N}$$

gravity

QM



- Is huge: $$S = \frac{k_B}{4}\frac{A}{l_P^2}\,, \quad l_P = \sqrt{\frac{G\hbar}{c^3}}$$
- Is **holographic**: $$S \propto A$$
- Bekenstein's (universal) bound: $$S \leq \frac{A}{4}$$

Black hole evaporation

- Hawking temperature for Schwarzschild

$$T = \frac{\hbar c^3}{8\pi k_B G M} \propto \frac{1}{M}$$

$$\sim 6 \times 10^{-8} \frac{M_\odot}{M} K$$

- Effective Stefan-Boltzmann law:

$$\frac{dM}{dt} \propto -\sigma T^4 A \propto -\frac{1}{M^2}$$

- BH completely evaporates

$$t_{\rm evap} \approx \left(\frac{M}{M_\odot}\right)^3 \times 10^{71}\ s$$

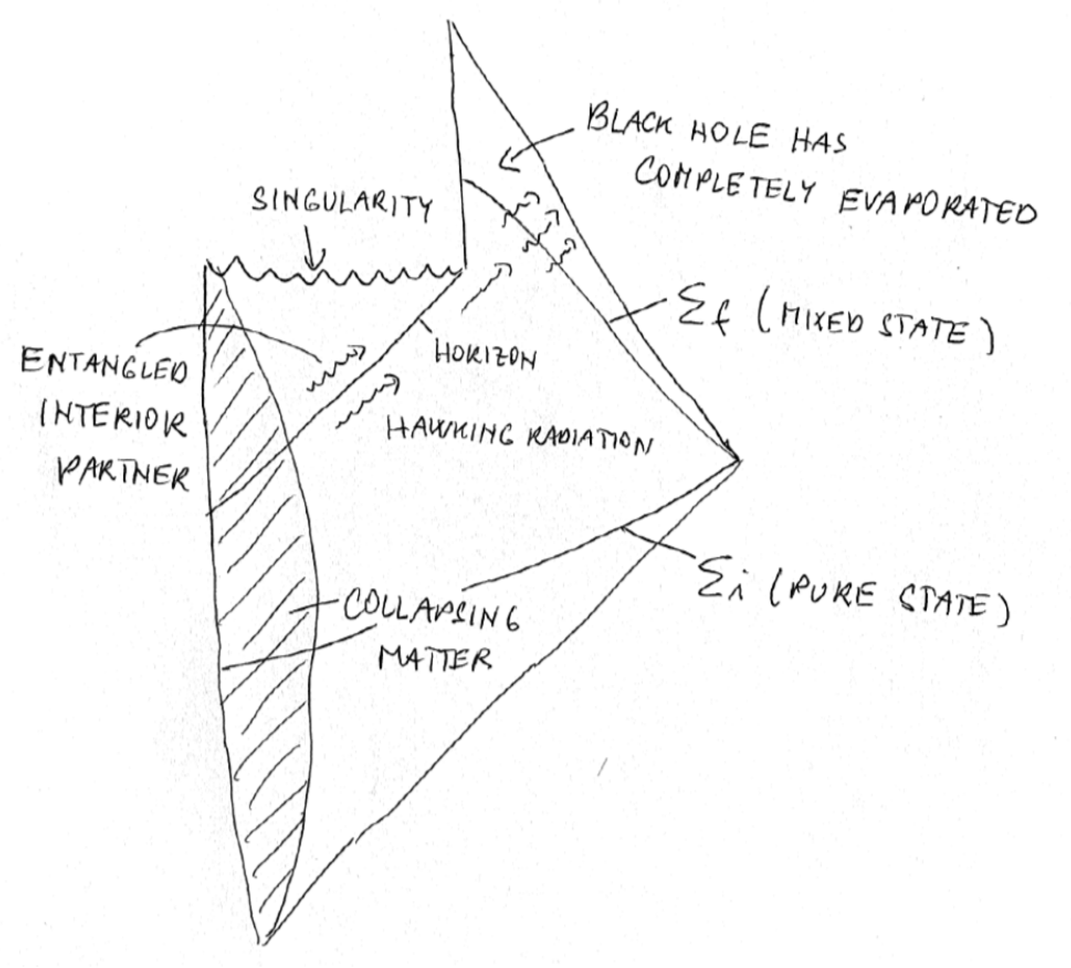
- Note also that:
  i) negative **specific heat**

$$C = T\frac{\partial S}{\partial T} = -\frac{1}{8\pi T^2}$$

ii) **Generalized 2nd law:**

$$S_{\rm TOT} = S_{\rm BH} + S_{\rm outside} \geq 0$$

Black hole info paradox (Hawking 1976)



- Thermal Hawking radiation leads to black hole evaporation.
- If BH completely evaporates, we violated unitary evolution of QM (evolved from the pure state in the beginning to a mixed state at the end) – **info loss** (see Erik Curiel's lecture).
- Contradicts the intuition from **AdS/CFT correspondence**

$$Z = \int Dg e^{-S_E[g]} \approx e^{-S_E(g_c)}$$

II. Euclidean magic

- G. Gibbons, S. Hawking*, Action integrals and partitions functions in quantum gravity*, Phys. Rev D 15, 2752, 1977.
- G. Gibbons and S. Hawking, *Cosmological event horizons, thermodynamics, and particle creation*, Phys. Rev D 15, 2738, 1977.

Euclidean trick (Gibbons & Hawking 1977)

• **Thermal Green functions** have periodicity in **Euclidean** time

$$\tau = it$$

$$G(\tau) = G(\tau + \beta)\,, \quad \beta = 1/T\,.$$

(Conversely, periodicity of G defines a thermal state. A thermometer interacting with the given field for a long time will register this temperature.)

• **Quantum fields** in the vicinity of black holes have this property (as seen by distant static observers).

• What about the **gravitational field** itself? Consider **Euclideanized Schwarzschild**:

$$ds^2 = f\,d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2$$

Euclidean trick (Gibbons & Hawking 1977)

- **Near horizon** expand:

$$f = \underbrace{f(r_+)}_{0} + \underbrace{(r - r_+)}_{\Delta r} \underbrace{f'(r_+)}_{2\kappa} + \cdots = 2\kappa \Delta r$$

$$ds^2 = 2\kappa \Delta r d\tau^2 + \frac{dr^2}{2\kappa \Delta r} + r_+^2 d\Omega^2$$

- Change variables:

$$d\rho^2 = \frac{dr^2}{2\kappa \Delta r} \quad \Leftrightarrow \quad d\rho = \frac{dr}{\sqrt{2\kappa \Delta r}} \quad \Leftrightarrow \quad \Delta r = \frac{\kappa}{2}\rho^2$$

$$ds^2 = \kappa^2 \rho^2 d\tau^2 + d\rho^2 + r_+^2 d\Omega^2 = \rho^2 d\varphi^2 + d\rho^2 + \ldots .$$

$\varphi = \kappa \tau$ … looks like **flat space** in polar provided:

φ has a period 2π.

(otherwise conical singularity exists at ρ=0)

Euclidean trick (Gibbons & Hawking 1977)

- **Original manifold** non-singular:

$$\varphi \sim \varphi + 2\pi \quad \Leftrightarrow \quad \tau \sim \tau + \underbrace{2\pi/\kappa}_{\beta} \quad \Leftrightarrow \quad \boxed{T = \frac{\kappa}{2\pi}\,,}$$

… which is the **Hawking's temperature**.

Gravitational partition function

$$\boxed{Z = \int Dg e^{-S_E[g]} \approx e^{-S_E(g_c)}}$$

(using WKB approximation)

- **Free energy**:

$$F = -\frac{1}{\beta}\log Z \approx \frac{S_E}{\beta} \quad \Rightarrow \quad S = -\frac{\partial F}{\partial T}$$

Euclidean trick (Gibbons & Hawking 1977)

- **Gravitational action**:

$$S_E = \int_\Omega \frac{d^4x\sqrt{g}R}{16\pi G} + \int_{\partial\Omega} \frac{d^3x\epsilon\sqrt{h}K}{8\pi G} + \text{counter terms}$$

Einstein-Hilbert action
(gives Einstein equations)

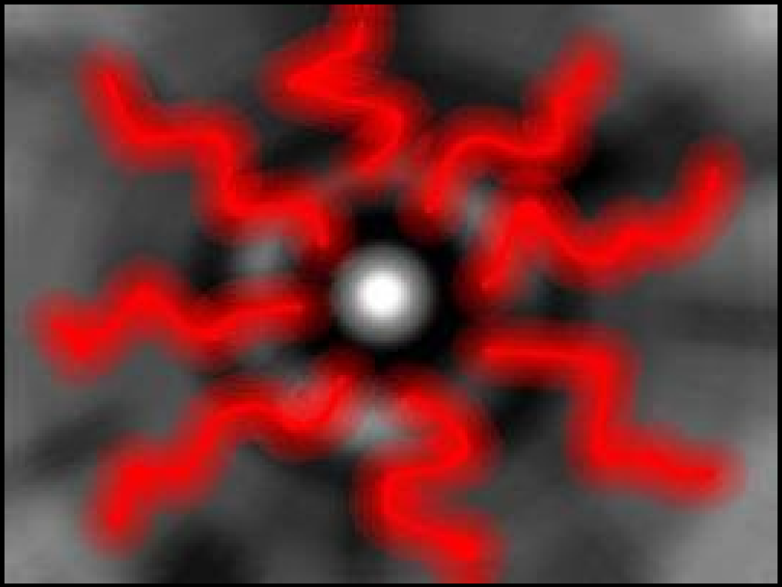
York-Gibbons-Hawking term
(yields well posed variational principle with Dirichlet BCs)

Counter terms: “renormalize” the value of the action
(In AdS given covariantly by *holographic renormalization*. In flat space no covariant prescription exists!)

- The prescription confirms **Bekenstein’s area law!**

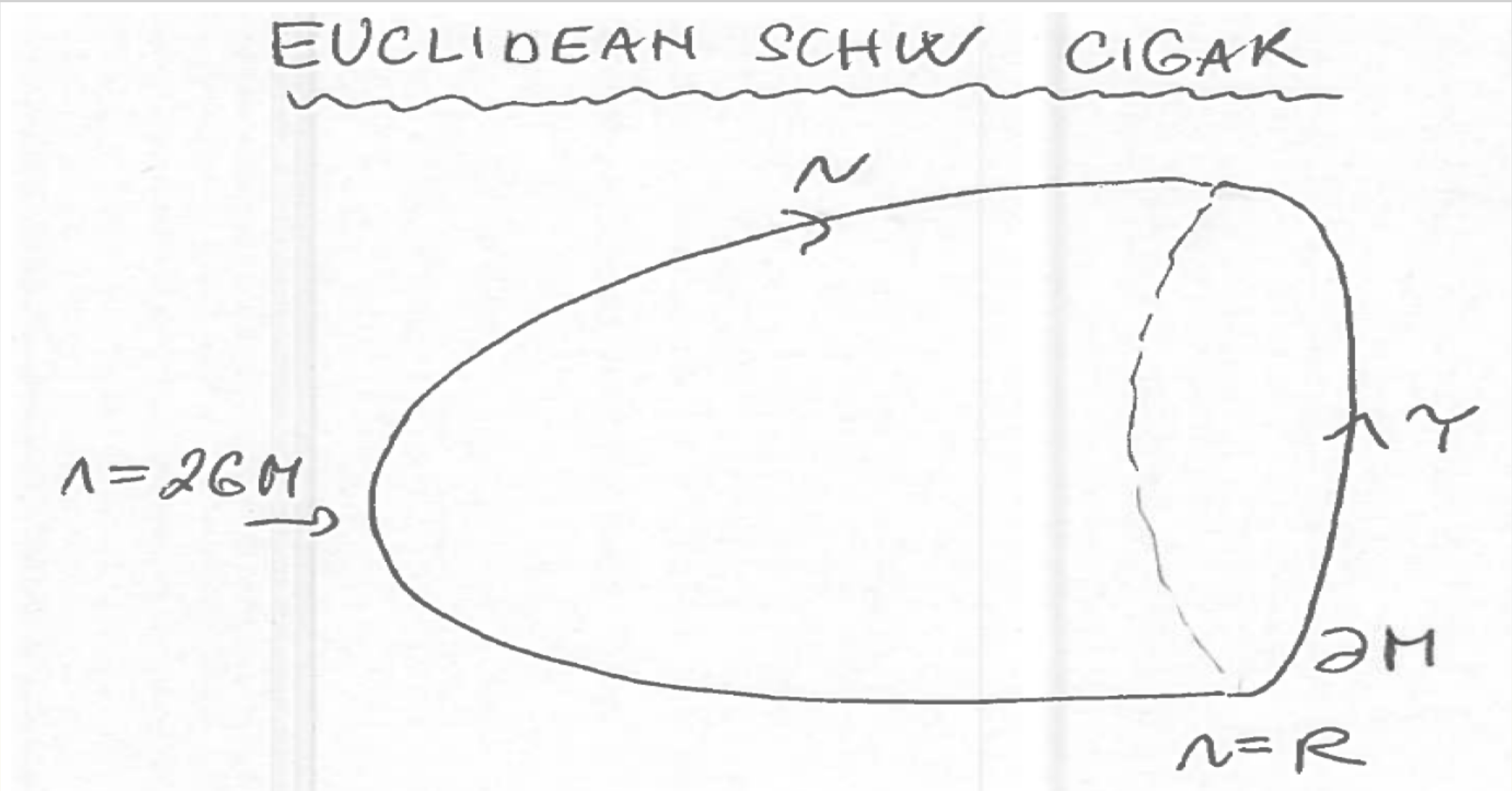
$$S = -\frac{\partial F}{\partial T} = \frac{A}{4}$$

III. Horizon TDs: examples



Example 1: Euclideanized Schwarzschild

$$ds^2 = f d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \qquad f = 1 - \frac{2M}{r}$$



- **Absence** of conical singularity implies:

$$T = \frac{1}{\beta} = \frac{f'(r_+)}{4\pi} = \frac{1}{8\pi M}$$

Example 1: Euclideanized Schwarzschild

$$ds^2 = f d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2 \,, \qquad f = 1 - \frac{2M}{r}$$

- **Background subtraction:**

$$S_E = -\frac{1}{16\pi G}\int_M d^4x\sqrt{g}R - \frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{h}(K - K_0)$$

- Ricci flat: $R_{\mu\nu} = 0 \Rightarrow S_{\text{EH}} = 0$
- Boundary at: $r = R$ $\quad d\gamma^2 = f(R)d\tau^2 + R^2 d\Omega^2$

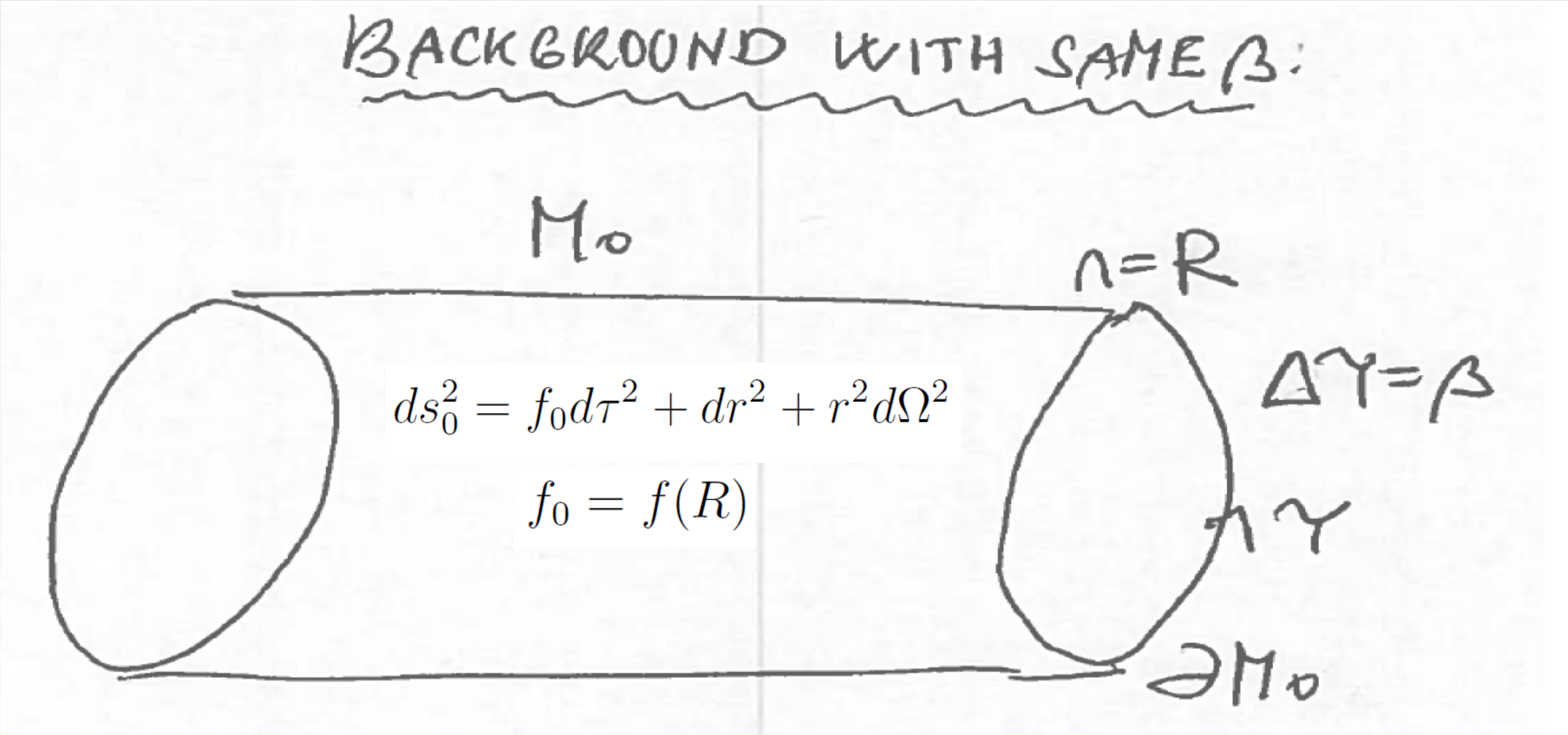
$$K = \nabla_\mu n^\mu = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}n^\mu) \qquad n = \sqrt{f}\partial_r\Big|_{r=R}$$

$$\int K\sqrt{h}d^3x = \underbrace{4\pi\beta}_{\int d\tau d\theta d\phi} R^2\left[\frac{2}{R}f(R) + \frac{1}{2}f'(R)\right] = 4\pi\beta(2R - 3M)$$

Example 1: Euclideanized Schwarzschild

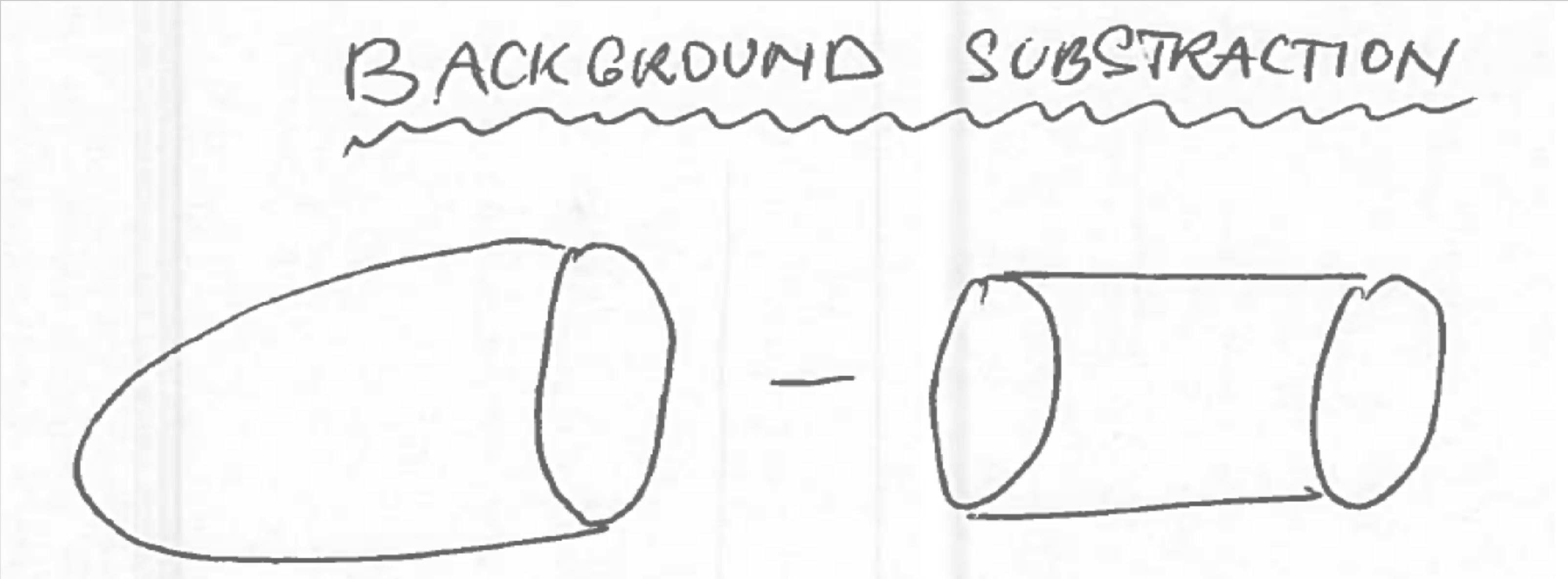
$$ds^2 = f d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \quad f = 1 - \frac{2M}{r}$$

- **Subtract:**



$$\int d^3x \sqrt{h_0} K_0 = 8\pi\beta R\sqrt{f(R)} = 8\pi\beta R\Big(1 - \frac{M}{R} + O(1/R^2)\Big)$$

Example 1: Euclideanized Schwarzschild



$$F = \frac{S_E}{\beta} = \frac{M}{2} = \frac{\beta}{16\pi} = M - TS$$

$$S = -\frac{\partial F}{\partial T} = \frac{\beta^2}{16\pi} = 4\pi M^2 = \pi r_+^2 = \frac{A}{4}$$

… confirmed **Bekenstein's area law!**

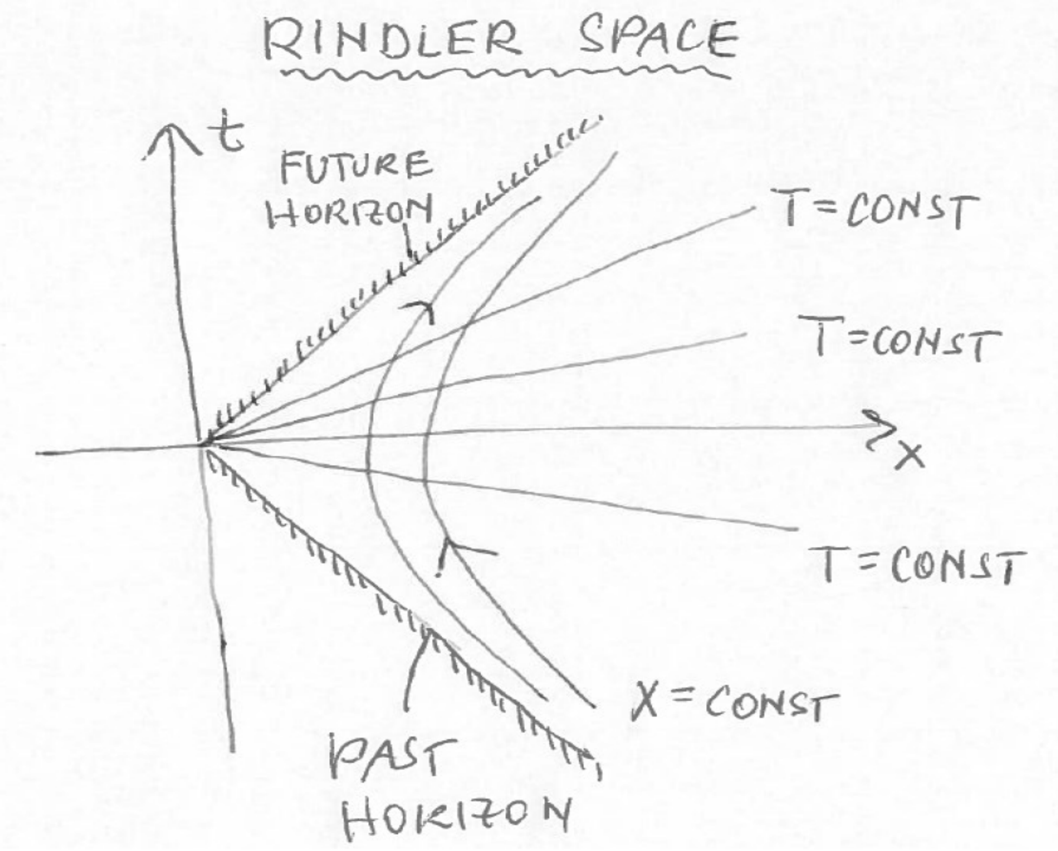
Example 2: Rindler spacetime

- Consider uniformly accelerated observer:

$$a = \sqrt{a_\mu a^\mu} = \text{const}$$

$$t = \frac{1}{a}\sinh(a\tau)\,, \quad x = \frac{1}{a}\cosh(a\tau)$$

- **Rindler frame**:



$$t = \left(\frac{1}{a} + X\right)\sinh(aT)\,,$$

$$x = \left(\frac{1}{a} + X\right)\cosh(aT)$$

Rindler horizon at:

$$X_R = -\frac{1}{a}$$

$$ds^2 = -dt^2 + dx^2 = -(1 + aX)^2 dT^2 + dX^2$$

Example 2: Rindler spacetime

- **Euclidean Rindler**: $\tau = iT$

$$ds_E^2 = (1 + aX)^2 d\tau^2 + dX^2 + dy^2 + dz^2$$

- Zooming on Rindler horizon + change coordinates:

$$\rho = \frac{1 + aX}{a} \quad \Rightarrow \quad d\rho = dX$$

$$ds_E^2 = a^2 \rho^2 d\tau^2 + d\rho^2 + \cdots = \rho^2 d\varphi^2 + d\rho^2$$

$$\varphi \sim \varphi + 2\pi \quad \Leftrightarrow \quad \tau \sim \tau + \underbrace{2\pi/a}_{\beta} \quad \Leftrightarrow \quad \boxed{T = \frac{a}{2\pi},}$$

… which is **Unruh's temperature**.

Example 2: Rindler spacetime

$$ds_E^2 = (1 + aX)^2 d\tau^2 + dX^2 + dy^2 + dz^2$$

- Action calculation: $$S_E = \int_\Omega \frac{d^4x\sqrt{g}R}{16\pi G} + \int_{\partial\Omega} \frac{d^3x\epsilon\sqrt{h}K}{8\pi G}$$

nothing

- Boundary: $X = X_0 = \text{const.}$ $\quad n = \partial_X$

$$K = \nabla_\mu n^\mu = \frac{1}{\sqrt{g}}(\sqrt{g}n^\mu)_{,\mu} = \frac{a}{1 + aX_0}$$

$$\sqrt{h} = 1 + aX_0$$

$$S_E = -\underbrace{\int d\tau}_{\beta} \int \frac{dxdy\sqrt{h}K}{8\pi G} = -\frac{a\beta}{8\pi G}\underbrace{\int dydz}_{A} = -\frac{a\beta A}{8\pi G}$$

Example 2: Rindler spacetime

$$S_E = -\underbrace{\int d\tau}_{\beta} \int \frac{dx dy \sqrt{h} K}{8\pi G} = -\frac{a\beta}{8\pi G} \underbrace{\int dy dz}_{A} = -\frac{a\beta A}{8\pi G}$$

- Free energy: $$F = \frac{S_E}{\beta} = -\frac{a}{2\pi}\frac{A}{4} = -T\frac{A}{4}$$

$$S = -\frac{\partial F}{\partial T} = \frac{A}{4}$$

… which is Bekenstein's result

$$E = \frac{\partial(\beta F)}{\partial \beta} = 0 \quad \Rightarrow \quad F = M - TS = -TS$$

Example 3: de Sitter horizon

- **de Sitter (dS) space** = maximally symmetric solution of EE with positive Lambda:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \qquad \Lambda = \frac{3}{\ell^2}$$

- metric:

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \qquad f = 1 - \frac{r^2}{\ell^2}$$

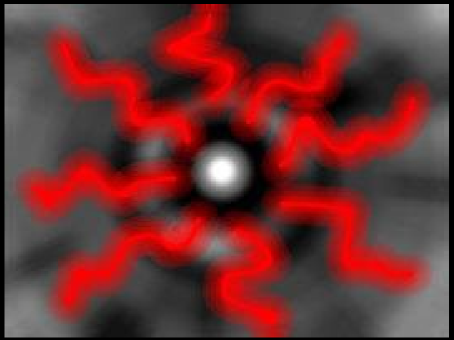
- Cosmological horizon:

$$f(r_c) = 0 \quad \Rightarrow \quad r_c = \ell$$

- repeating the Euclidean trick we get:

$$T = \frac{|f'(r_c)|}{4\pi} = \frac{1}{2\pi\ell}\,.$$

… which is **Gibbons-Hawking temperature**
(note the absolute value!)



IV. What is going on? (A few words about QFT in CS)

Let's quantize time-dependent harmonic oscillator:

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2(t)q^2$$

- EOM & scalar product:

$$\ddot{q} + \omega^2(t)q = 0$$

$$(f_1, f_2) = i[f_1^* \frac{d}{dt} f_2 - (\frac{d}{dt} f_1^*) f_2]_{t=0} = i f_1^* \overleftrightarrow{\partial}_t f_2 \Big|_{t=0}$$

- Given a solution f, f* is independent

$$(f, f) = 1 \Rightarrow (f^*, f^*) = -1, \quad (f, f^*) = 0$$

- Any solution:

$$q(t) = a_f f(t) + a_f^\dagger f^*(t)$$

$$p(t) = \frac{\partial L}{\partial \dot{q}} = \dot{q}(t) = a_f \dot{f}(t) + a_f^\dagger \dot{f}^*(t)$$

- **Quantize**: promote to operators & impose equal time commutation relations

$$\boxed{[q(t), p(t)] = i} \quad \Leftrightarrow \quad \boxed{[a_f, a_f^\dagger] = 1\,, \ [a_f, a_f] = 0 = [a_f^\dagger, a_f^\dagger]\,.}$$

- Define:

$$f-\text{vacuum}: \quad |0_f\rangle: \quad a_f|0_f\rangle = 0\,,$$

$$n-\text{excited state}: \quad |n_f\rangle: \quad |n_f\rangle = \frac{1}{\sqrt{n!}}(a_f^\dagger)^n|0_f\rangle\,.$$

$$\underline{f\text{-number operator}} \quad \boxed{N_f = a_f^\dagger a_f} \quad \Rightarrow \quad N_f|n_f\rangle = n|n_f\rangle$$

- Choose g instead of f:

$$q(t) = a_g g(t) + a_g^\dagger g^*(t)$$

$$[q(t), p(t)] = i \quad \Rightarrow \quad [a_g, a_g^\dagger] = 1\,.$$

$$g-\text{vacuum}: \quad |0_g\rangle: \quad a_g|0_g\rangle = 0\,,$$

$$g-\text{excited state}: \quad |n_g\rangle: \quad |n_g\rangle = \frac{1}{\sqrt{n!}}(a_g^\dagger)^n|0_f\rangle\,,$$

$$g-\text{number operator}: \quad N_g = a_g^\dagger a_g \quad \Rightarrow \quad N_g|n_g\rangle = n|n_g\rangle$$

- We can express 2nd basis in terms of 1st one:

$$g(t) = \alpha f(t) + \beta f^*(t)$$

Bogolubov coefficients

- Bogolubov coefficients $g(t) = \alpha f(t) + \beta f^*(t)$

$$\beta = -(f^*, g)\,, \qquad \alpha = (f, g)$$

- Writing $q(t) = a_f f(t) + a_f^\dagger f^*(t) = a_g g(t) + a_g^\dagger g^*(t)$

$$\Rightarrow \quad a_g = \alpha a_f - \beta^* a_f^\dagger\,, \qquad a_g^\dagger = \alpha^* a_f^\dagger - \beta a_f$$

- **How many particles** g are there in f-vacuum?

$$\begin{aligned}\langle 0_f | N_g | 0_f \rangle &= \langle 0_f | a_g^\dagger a_g | 0_f \rangle = \langle 0_f (\alpha^* a_f^\dagger - \beta a_f)(\alpha a_f - \beta^* a_f^\dagger) | 0_f \rangle \\ &= |\beta|^2 \langle 0_f | a_f a_f^\dagger | 0_f \rangle = |\beta|^2 \,.\end{aligned}$$

Notion of particles is **ambiguous** (depends on the observer): stands behind **Unruh**, **Hawking**, **particle production** in Early Universe, …

Summary of Lecture 1

1) Black holes are **thermodynamic objects.** They can be assigned **Hawking's temperature** and **Bekenstein's entropy**. Obey the standard laws of TDs.

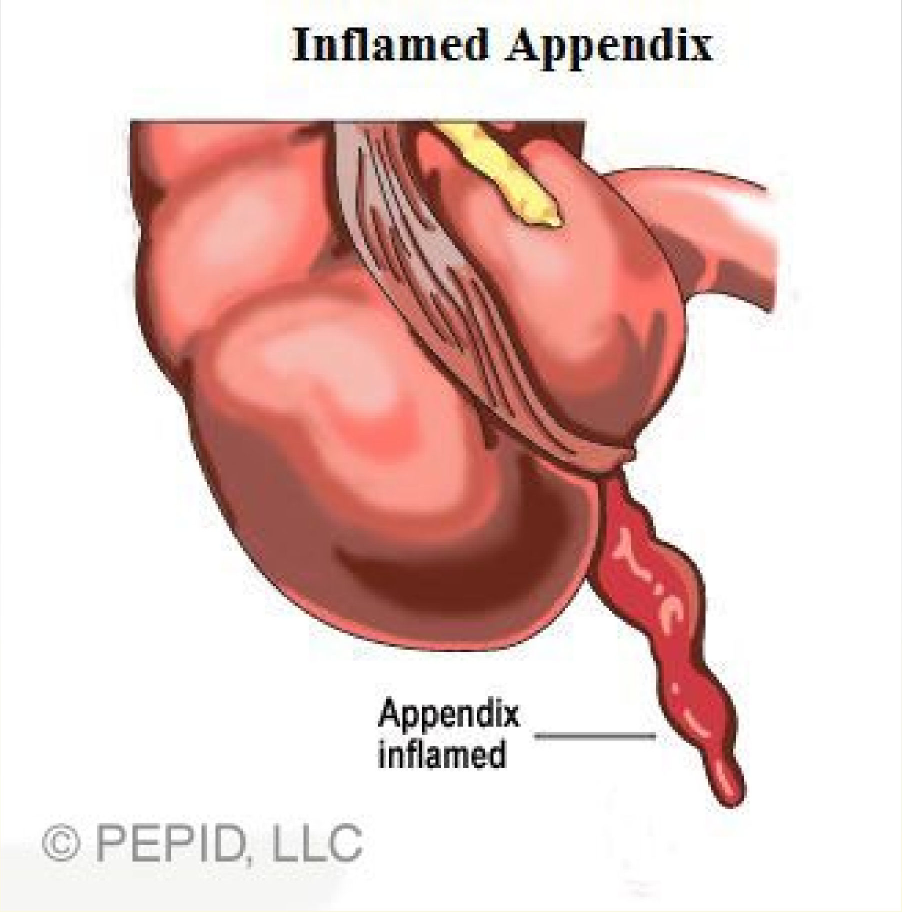
2) **Euclidean magic** predicts thermodynamics of BH, Rindler, or dS horizons. Namely:

   i. **Regularity** of the **Euclideanized manifold** fixes periodicity of the Euclidean time and yields **Hawking temperature** of the black hole.

   ii. G**ravitational partition function** yields free energy, which in WKB approximation recovers the **Bekenstein's area law** (& other conjugate quantities).

3) **Thermal properties** are associated with particular observers (notion of particles is **ambiguous**) .

Appendices



Example 4: temperature of Kerr

- Euclidean Kerr: $t \to i\tau\,, \qquad a \to ib$

$$ds_E^2 = \frac{\Delta}{\Sigma}(d\tau - b\sin^2\theta d\varphi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left((r^2-b^2)d\varphi + bd\tau\right)^2$$

$$\Delta = r^2 - b^2 - 2Mr \qquad \Sigma = r^2 - b^2\cos^2\theta.$$

- When zooming on the horizon: one needs to set: $\Delta(r_+) = 0$

$$d\varphi = -\frac{bd\tau}{r_+^2 - b^2} \quad \Rightarrow \quad d\tau - b\sin^2\theta d\varphi \approx \Sigma_+/(r_+^2 - b^2)d\tau.$$

- Thus $$ds_E^2 \approx \frac{\Sigma_+}{r_+^2-b^2}\left(f d\tau^2 + \frac{dr^2}{f} + \ldots\right), \qquad f = \frac{\Delta}{r_+^2 - b^2}$$

$$T = \frac{\Delta'(r_+)}{4\pi(r_+^2-b^2)} = \frac{r_+ - M}{2\pi(r_+^2 + a^2)}$$