Lecture 2: AdS black holes & black Hole Chemistry

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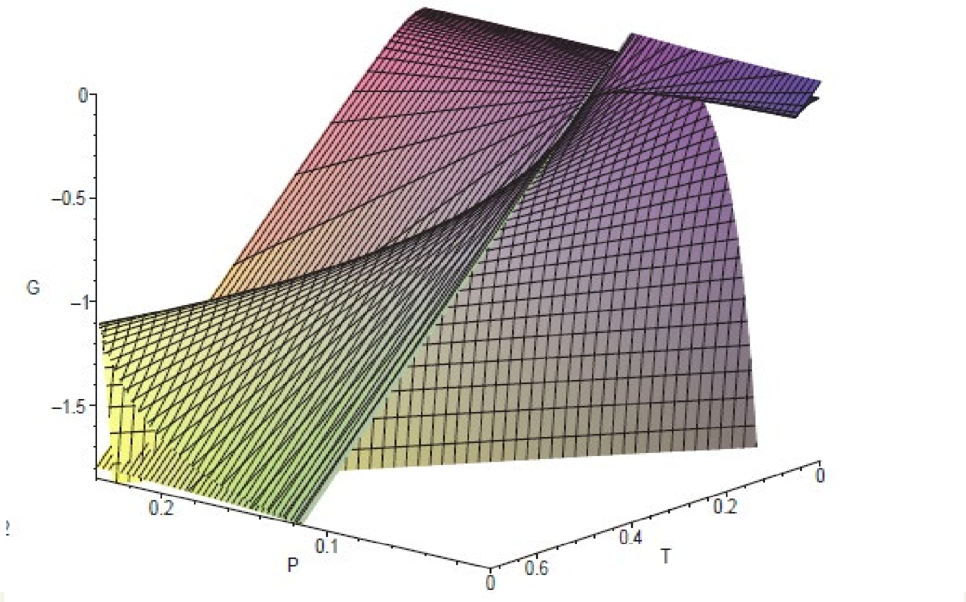


SIGRAV International School 2025
Vietri Sul Mare, Italy
Feb 17-21, 2025

Prelude: Van der Waals fluid & critical phenomena

Johannes Diderik van der Waals (1837-1923)

Van der Waals fluid

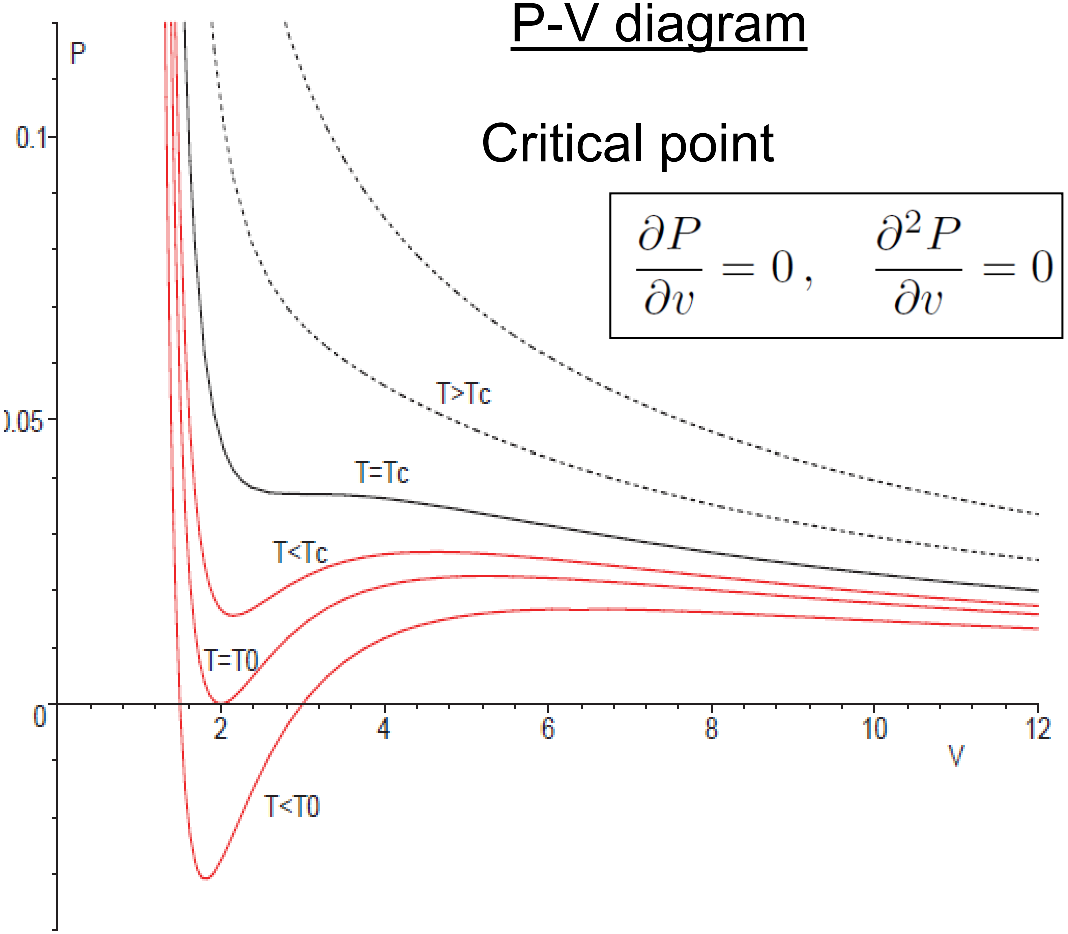
$$\left(P + \frac{a}{v^2}\right)(v - b) = T$$

Parameters a and b characterize the fluid. a measures the **attraction** between particles (a>0) and b corresponds to "**volume of fluid particles**".

P-V diagram

Critical point

$$\frac{\partial P}{\partial v} = 0\,, \qquad \frac{\partial^2 P}{\partial v} = 0$$



Critical point:

$$T_c = \frac{8a}{27b} \qquad v_c = 3b$$

$$P_c = \frac{a}{27b^2}$$

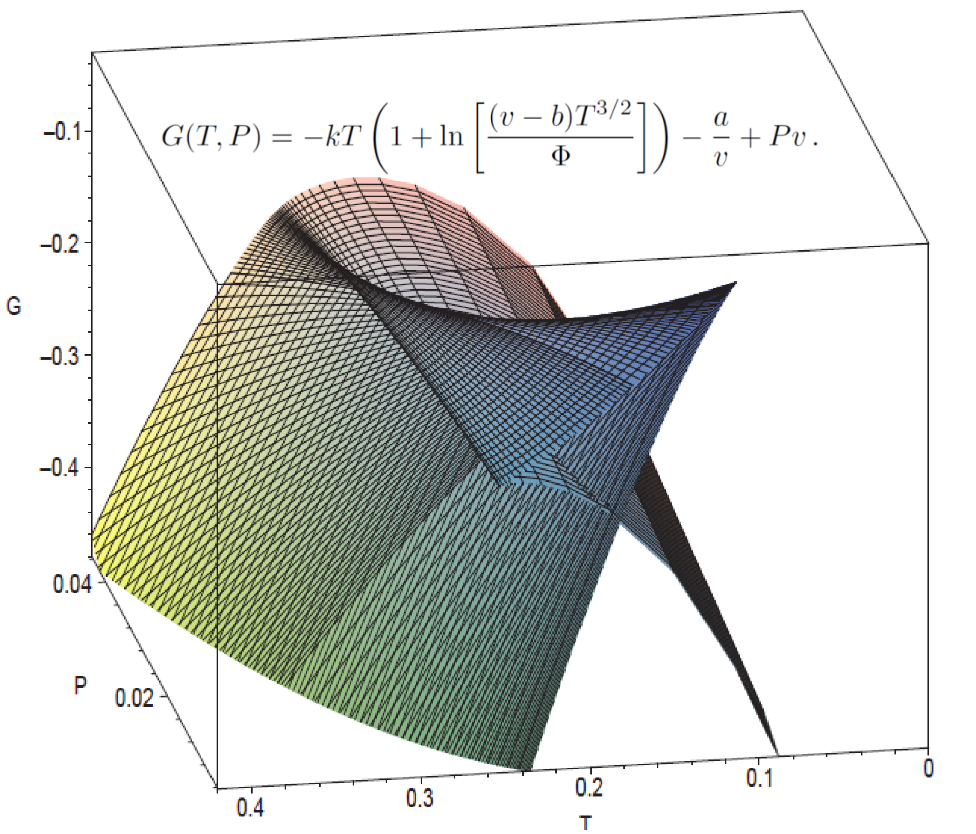
Universal critical ratio

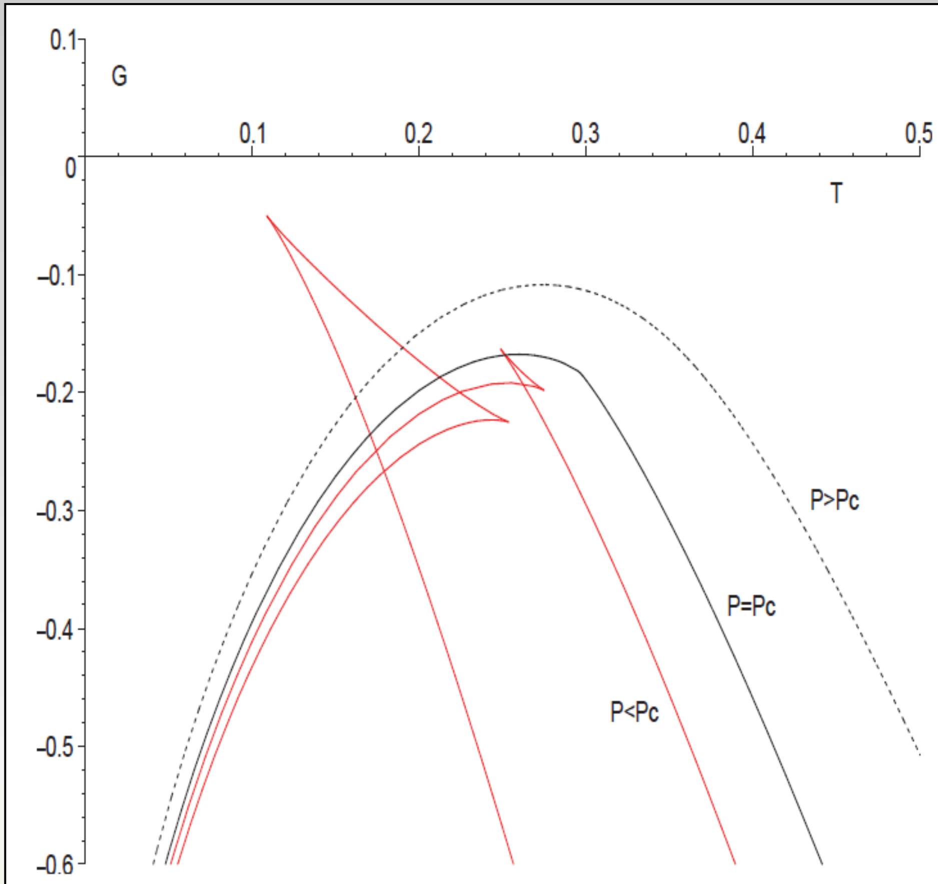
$$\rho_c = \frac{P_c v_c}{T_c} = \frac{3}{8}$$

Gibbs free energy

$$dG = -SdT + vdP$$

Swallow tail behavior





System wants to minimize Gibbs free energy → Phase transitions

Maxwell's equal area law & phase diagrams

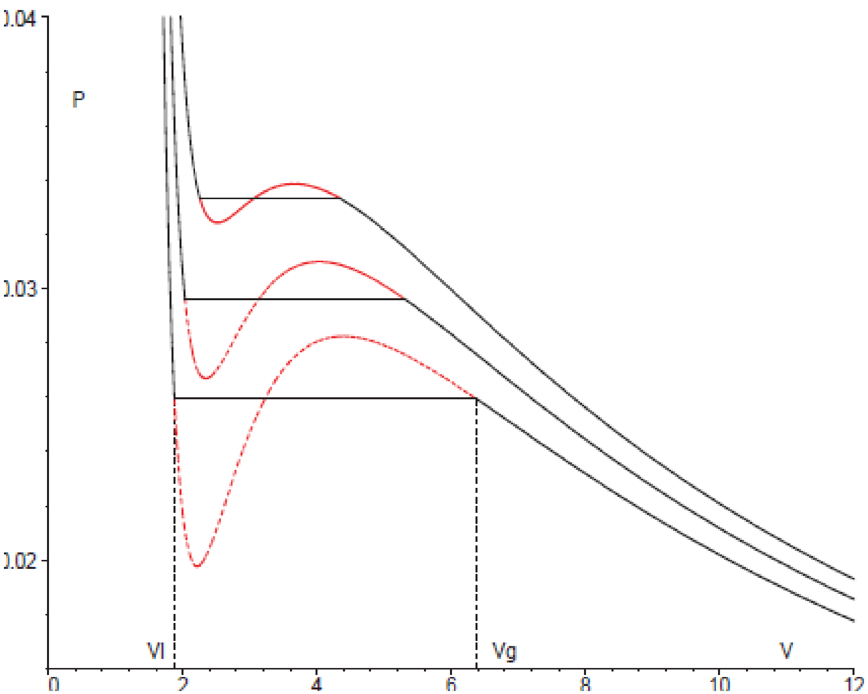


FIG. 2. **Maxwell's equal area law.** The 'oscillating' (dashed) part of the isotherm $T < T_c$ is replaced by an isobar, such that the areas above and below the isobar are equal one another.

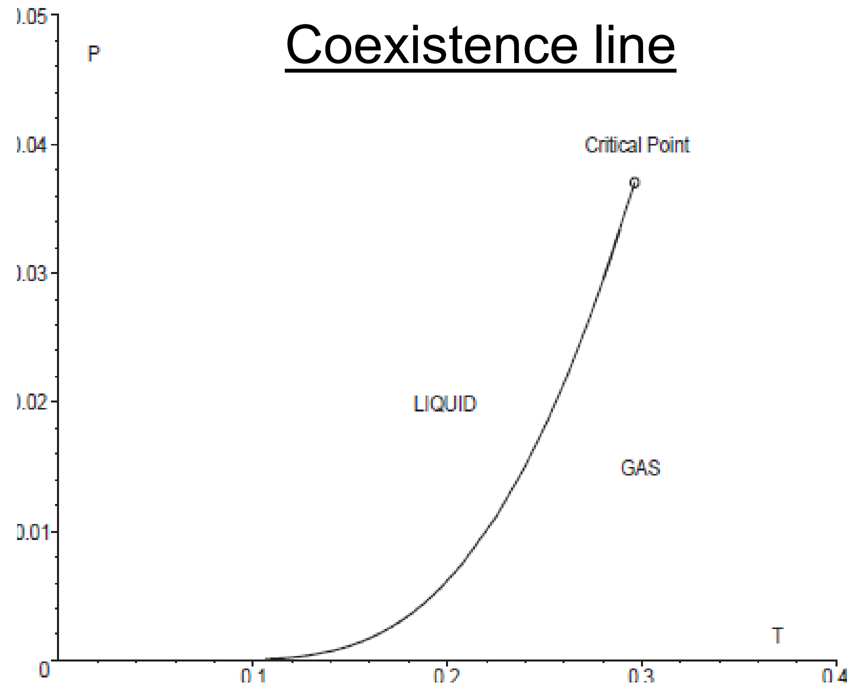


FIG. 5. **Coexistence line of liquid–gas phase.** Fig. displays the coexistence line of liquid and gas phases of the Van der Waals fluid in (P, T)-plane. The critical point is highlighted by a small circle at the end of the coexistence curve.

$$dG = -SdT + vdP$$

Both phases have the same Gibbs free energy

$$\oint vdP = 0$$

Critical exponents

- Describe the behaviour of physical quantities near the critical point
- Universal (depend on dimensionality and/or range of interactions)
- In d ≥ 4 spatial dimensions they can be calculated using MFT (each dof couples to the average of the other dof).

For example, for a fluid with critical Tc, vc, and Pc.

- Exponent α governs the behaviour of the specific heat at constant volume,

$$C_v = T\frac{\partial S}{\partial T}\Big|_v \propto |t|^{-\alpha}\,. \tag{A2}$$

- Exponent β describes the behaviour of the *order parameter* $\eta = v_g - v_l$ (the difference of the volume of the gas v_g phase and the volume of the liquid phase v_l) on the given isotherm

$$\eta = v_g - v_l \propto |t|^{\beta}\,. \tag{A3}$$

- Exponent γ determines the behaviour of the *isothermal compressibility* κ_T,

$$\kappa_T = -\frac{1}{v}\frac{\partial v}{\partial P}\Big|_T \propto |t|^{-\gamma}\,. \tag{A4}$$

- Exponent δ governs the following behaviour on the critical isotherm $T = T_c$:

$$|P - P_c| \propto |v - v_c|^{\delta}\,. \tag{A5}$$

$$\alpha = 0\,, \quad \beta = \frac{1}{2}\,, \quad \gamma = 1\,, \quad \delta = 3$$

The same critical exponents derived for ferromagnets, superfluidity,..
Problem: MFT neglects fluctuations, to go beyond one needs to use RG techniques

Plan for Lecture 2

I. Black holes in AdS

II. Black hole chemistry

III. TD phase transitions of AdS black holes
    I. VdW behavior of charged AdS black holes
    II. Hawking-Page transition
    III. Reentrant phase transitions
    IV. Isolated critical point

IV. Summary

If you want know more:

- DK, R.B. Mann, *P-V criticality of charged AdS black holes*, JHEP 07 (2012) 033; ArXiv:1205:0559.
- DK, R.B. Mann, M. Teo, *Black hole chemistry: thermodynamics with Lambda*, CQG 34 (2017) 063001, Arxiv:1608.0614.

I) Black Holes in Anti de Sitter (AdS)

Lecture 1: asymptotically flat BHs

- First law of black hole thermodynamics:

$$\delta M = T\delta S + \sum_i \Omega_i \delta J_i + \Phi \delta Q$$

- Smarr-Gibbs-Duhem relation:

$$\frac{d-3}{d-2} M = TS + \sum_i \Omega_i J_i + \frac{d-3}{d-2} \Phi Q$$

Basic properties:

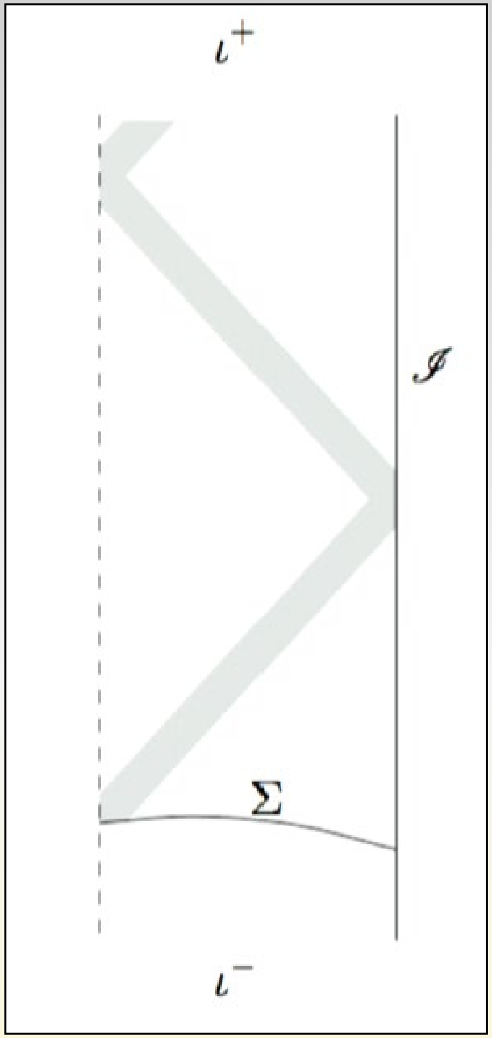
- Thermodynamic ensemble not well defined (C<0)!
- Where is the standard PdV term?
- TD behaviour interesting, yet not exactly analogous to everyday thermodynamics!

Instead: Consider BHs in AdS!

Global AdS: a few basic facts

- **Anti de Sitter (AdS) space** = maximally symmetric solution of EE with negative Lambda:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \qquad \Lambda = -\frac{3}{\ell^2}$$



- metric:

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \qquad f = 1 + \frac{r^2}{\ell^2}$$

- Pull of the negative cosmological constant implies that AdS acts like a **confining box**

- There is a timelike conformal boundary (due to reflective BCs, **nonlinearities do not decay**)

Bizon and Rostworowski, PRL107, 031102 (2011).

Schwarzschild-AdS black hole

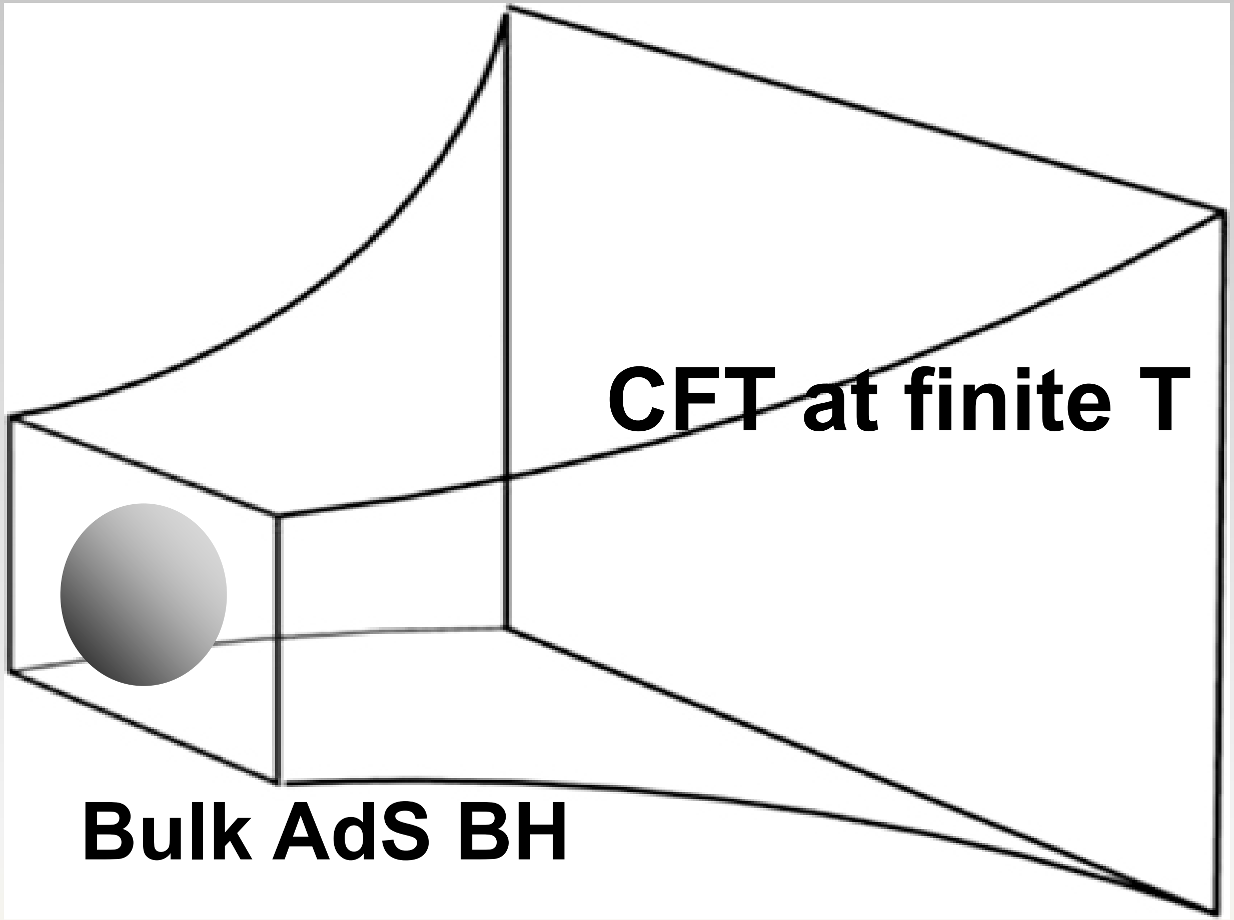
$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_k^2 \qquad f = k - \frac{2m}{r} + \frac{r^2}{\ell^2}$$

- It is an **Einstein space** (vacuum with Lambda solution)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \qquad \Lambda = -\frac{3}{\ell^2}$$

- Choices of $k = 0, \pm 1$ correspond to various **horizon topologies** (with $d\Omega_k^2$ the corresponding metric).
- One can add, charge, or rotation – having charge-AdS, or Kerr-AdS black holes in 4 and higher dimensions.
- As we shall see, these have rather interesting properties.

AdS/CFT duality (at finite temperature)



Bulk:	⟺	**Boundary:**
Hawking temp T		CFT at finite T
BH entropy S		CFT entropy S
BH mass M		CFT energy E

Gravitational action in AdS

$$S_E = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}\left[R+\frac{6}{\ell^2}\right]+\frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{-h}\mathcal{K}$$
$$-\frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{-h}\left[\frac{2}{\ell}+\frac{\ell}{2}\mathcal{R}(h)\right],$$

- The 2nd line are the covariant **AdS counterterms** (c.f. \`vague' background subtraction in AF case)

- Variation yields:

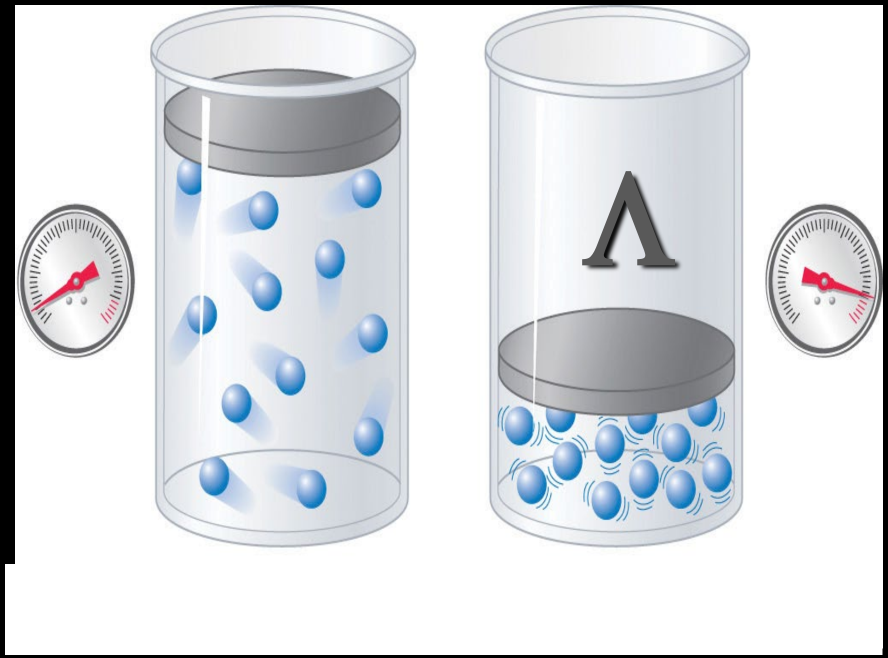
$$\delta S_E = -\frac{1}{2}\int_{\partial\Omega} d^3x\sqrt{-h}\tau_{ab}\delta h^{ab}$$ + bulk EOM

here

$$8\pi\tau_{ab} = \mathcal{K}h_{ab} - \mathcal{K}_{ab} + \ell G_{ab}(h) - \frac{2}{\ell}h_{ab}$$

is up to trivial (infinite) scaling the **holographic stress tensor**

II. Black hole chemistry



Black hole chemistry

Simple idea:

- Consider an asymptotically **AdS black hole spacetime**
- Identify the cosmological constant with a **thermodynamic pressure**

$$P = -\frac{\Lambda}{8\pi G}\,, \quad \Lambda = -\frac{(D-1)(D-2)}{2l^2}$$

- Allow this to be a “**dynamical**” quantity

(Teitelboim and Brown – 1980’s)

Immediate consequences

- **Extended black hole thermodynamics:**

D.Kastor, S.Ray, and J.Traschen, *Enthalpy and the Mechanics of AdS Black Holes*, Class. Quant. Grav. 26 (2009) 195011.

$$\delta M = T\delta S + \Theta\, \delta P + \ldots$$

• Introduces the standard **-PdV term** into black hole thermodynamics

• Black hole mass M no longer identified with energy but rather interpreted as **enthalpy**

$$U = M + \epsilon V = M - PV$$

Immediate consequences

- **Black hole volume:**

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,\dots}$$

Schwarzschild(-AdS):

$$V = \frac{4}{3}\pi r_+^3$$

- More involved for more complicated black holes

- The fact this this provides a good definition of volume is supported by the **Reverse Isoperimetric Inequality** conjecture:

  - M. Cvetic, G.W Gibbons, DK, C.N. Pope, *Black hole enthalpy and an entropy inequality for the thermodynamic volume,* Phys. Rev. D84 (2011) 024037, [arXiv:1012.2888].

  - M. Amo, A.M. Frassino, R.A. Hennigar, *New Inequalities in Extended black hole thermodynamics*, Arxiv:2307.03011.

Immediate consequences

- **Black hole volume:**

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,\ldots}$$

→ Schwarzschild(-AdS):

$$V = \frac{4}{3}\pi r_+^3$$

Reverse isoperimetric inequality

$$\mathcal{R} = \left(\frac{(d-1)\mathcal{V}}{\omega_{d-2}}\right)^{\frac{1}{d-1}} \left(\frac{\omega_{d-2}}{\mathcal{A}}\right)^{\frac{1}{d-2}}$$

$$\omega_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}$$

Conjecture: for any AdS black hole $\mathcal{R} \geq 1$

"For a black hole of given **thermodynamic volume** V, the entropy is maximised for Schwarzschild-AdS"

Consistency between 1st law and Smarr relation (scaling argument)

Euler's theorem:

$$f(\alpha^p x, \alpha^q y) = \alpha^r f(x,y)$$ ⇨ $$r f(x,y) = p\left(\frac{\partial f}{\partial x}\right) x + q\left(\frac{\partial f}{\partial y}\right) y.$$

Mass of black hole: $$M = M(A, P)$$

since $$[P] = L^{-2}\,, \quad [A] = L^{2}\,, \quad [M] = L$$ ⇨

$$M = 2A\left(\frac{\partial M}{\partial A}\right) - 2P\left(\frac{\partial M}{\partial \Lambda}\right)$$ + $$dM = \kappa dA + V dP$$

Smarr relation: $$M = 2(TS - VP)$$

Mass plays the role of **enthalpy** rather than internal energy

Immediate consequences

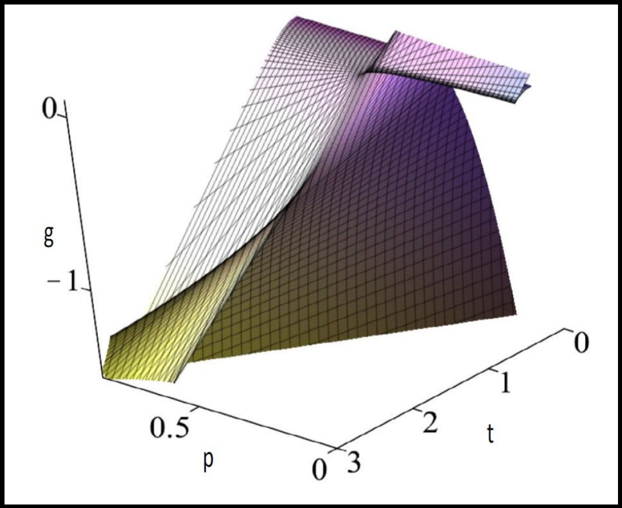
- **Consistent Smarr relation:**

$$\delta M = T\delta S + \sum_i \Omega_i \delta J_i + \Phi \delta Q \quad + V\delta P$$

$$\frac{d-3}{d-2}M = TS + \sum_i \Omega_i J_i + \frac{d-3}{d-2}\Phi Q \quad - \frac{2}{d-2}VP$$

- **Black holes and phase transitions:**

  - AdS black holes can be in **thermal equilibrium**
  - Exhibit interesting **phase transitions**
  - Provide dual description of CFT at finite temperature via **AdS/CFT correspondence**



III. TD phase transitions of AdS black holes

Canonical Example: VdW behavior of charged AdS black holes

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2\,, \quad A = -\frac{Q}{r}dt$$

$$f = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} + \frac{r^2}{l^2}\,,$$

- Basic thermodynamic quantities:

$$M = \frac{r_+(l^2 + r_+^2)}{2l^2 G} + \frac{Q^2}{2r_+}\,, \quad T = \frac{3r_+^4 + l^2 r_+^2 - GQ^2 l^2}{4\pi l^2 r_+^3}$$

$$S = \frac{\pi r_+^2}{G}\,, \quad V = \frac{4\pi r_+^3}{3}\,, \quad \phi = \frac{Q}{r_+}\,, \quad P = -\frac{1}{8\pi}\Lambda = \frac{3}{8\pi}\frac{1}{l^2}$$

$$F = M - TS = \frac{3GQ^2 l^2 + l^2 r_+^2 - r_+^4}{4G r_+ l^2}$$

Example: VdW criticality

- DK, R.B. Mann, *P-V criticality of charged AdS black holes*, JHEP 1207 (2012) 033.

Van der Waals fluid

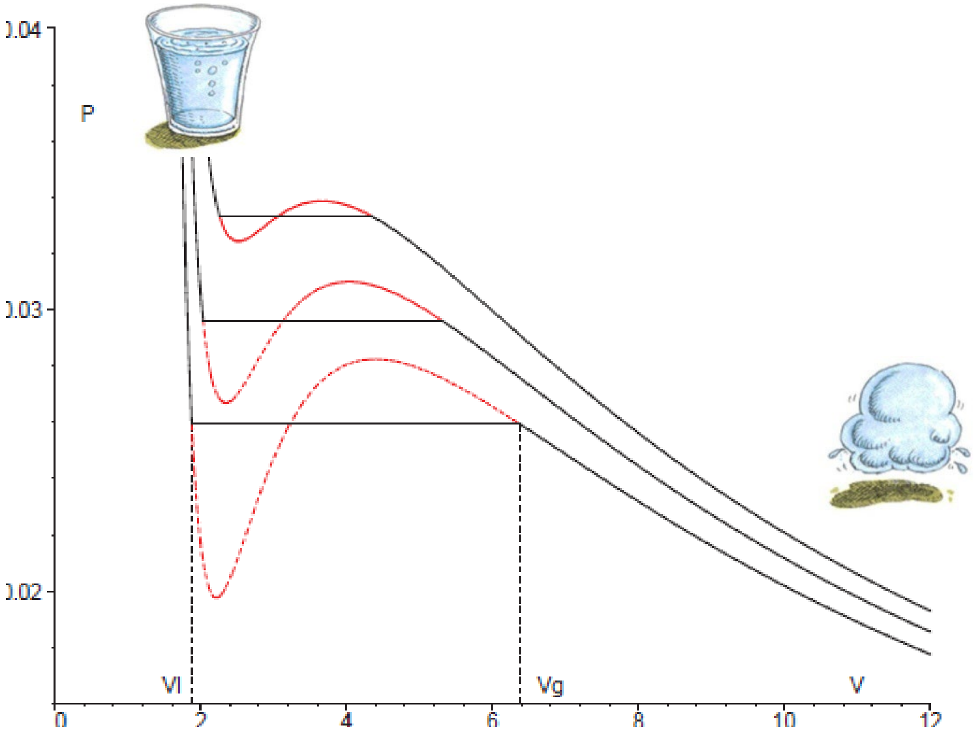


FIG. 2. Maxwell's equal area law. The 'oscillating' (dashed) part of the isotherm $T < T_c$ is replaced by an isobar, such that the areas above and below the isobar are equal one another.

$$\left(P + \frac{a}{v^2}\right)(v - b) = T$$

Parameter a measures the **attraction** between particles (a>0) and b corresponds to "**volume of fluid particles**".

Critical point:

$$\rho_c = \frac{P_c v_c}{T_c} = \frac{3}{8}$$

P-V criticality

- DK, R.B. Mann, *P-V criticality of charged AdS black holes*, JHEP 1207 (2012) 033.

Charged black hole

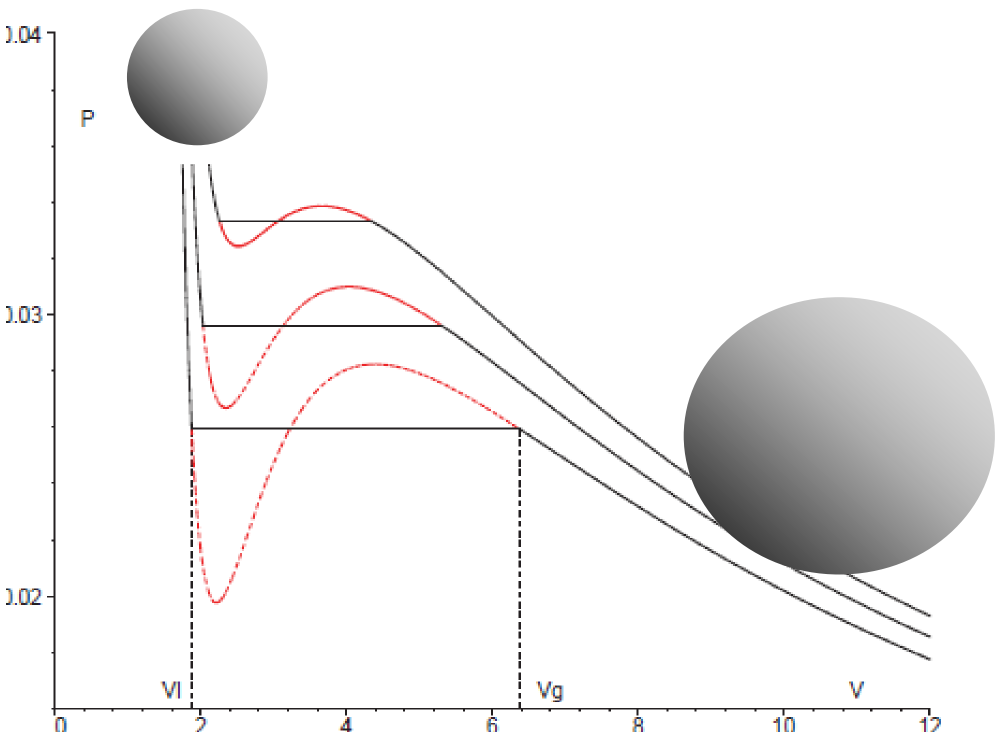


FIG. 2. Maxwell's equal area law. The 'oscillating' (dashed) part of the isotherm $T < T_c$ is replaced by an isobar, such that the areas above and below the isobar are equal one another.

$$\left(P + \frac{a}{v^2}\right)(v - b) = T$$

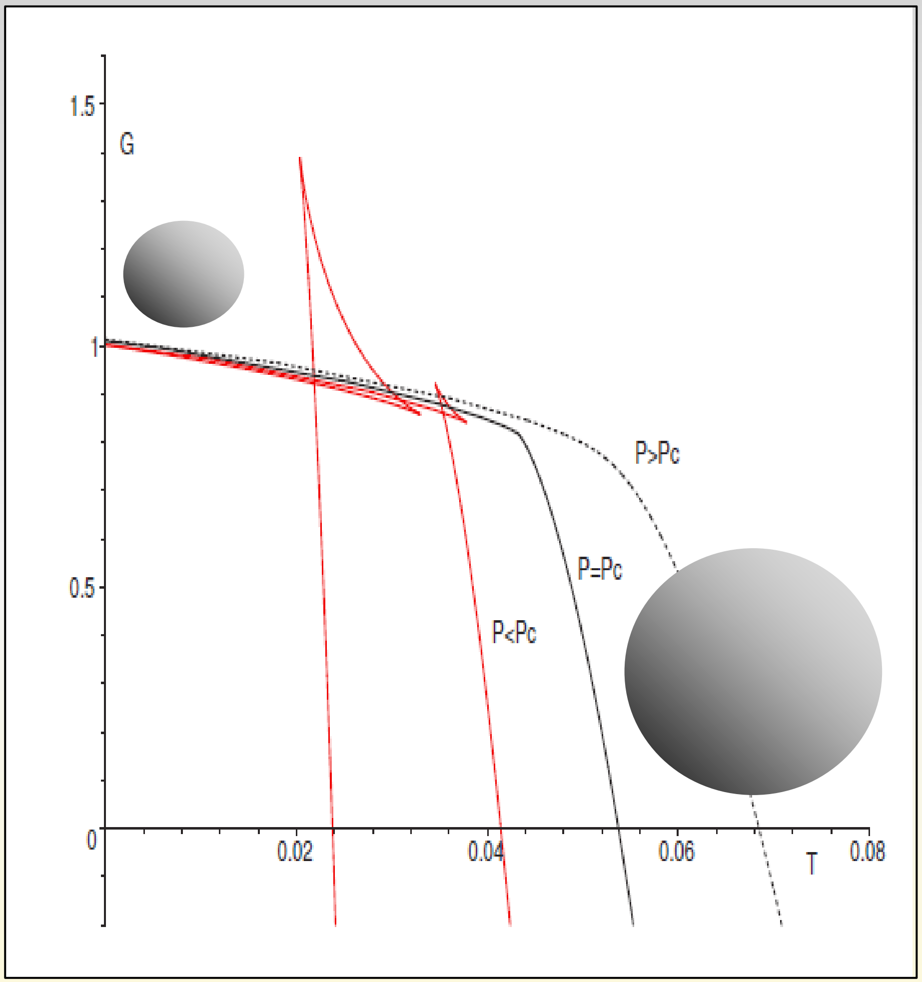
$$P = \frac{T}{v} - \frac{1}{2\pi v^2} + \frac{2Q^2}{\pi v^4}$$

Critical point:

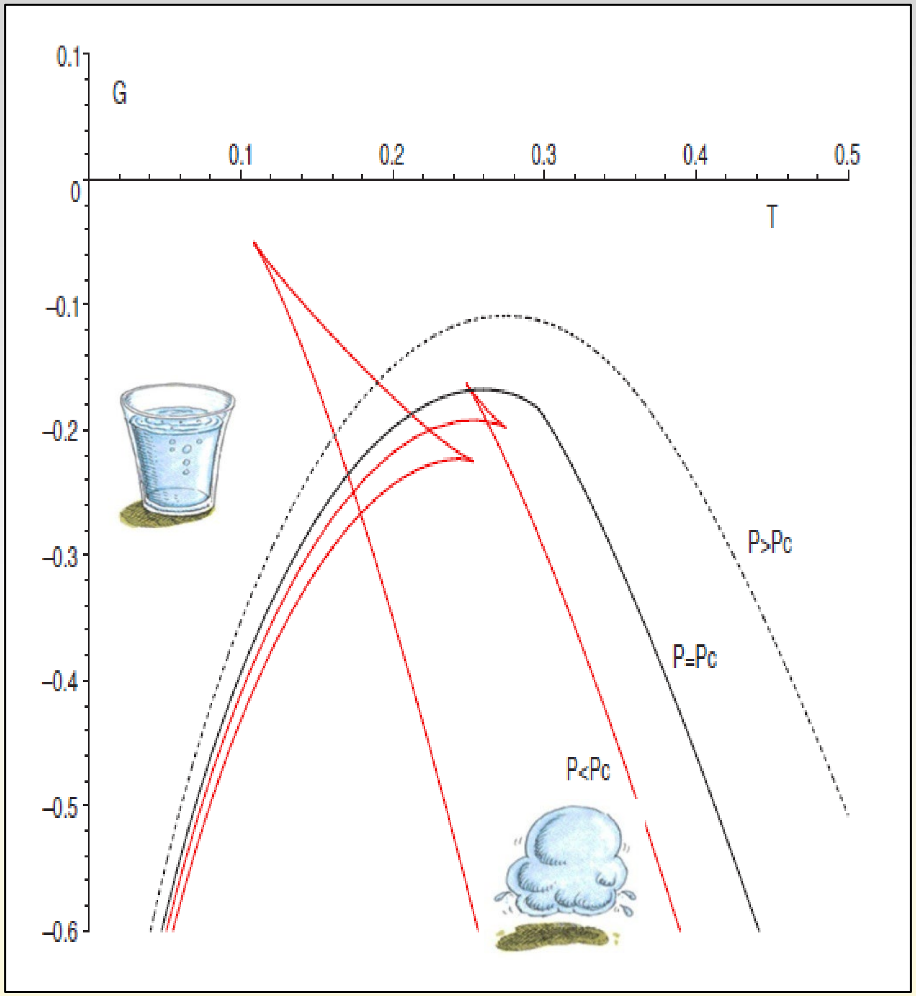
$$\rho_c = \frac{P_c v_c}{T_c} = \frac{3}{8}$$

Free energy: demonstrates standard swallow tail behavior

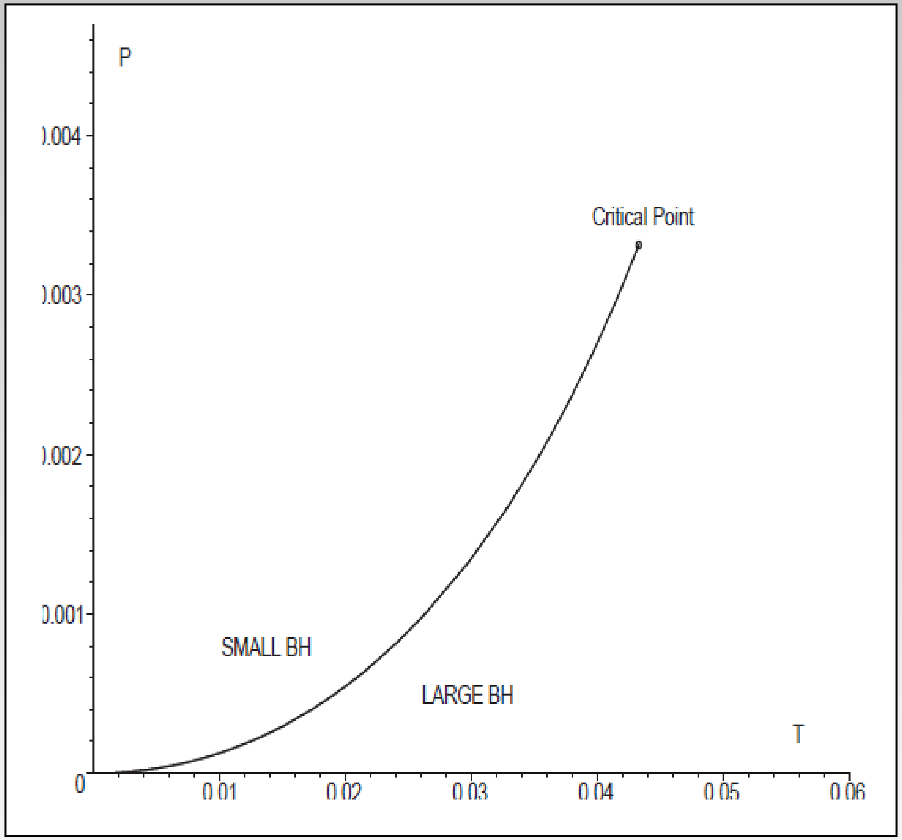
$$F = F(T,P,Q) = \frac{1}{4}\left(r_+ - \frac{8\pi}{3}Pr_+^3 + \frac{3Q^2}{r_+}\right)$$



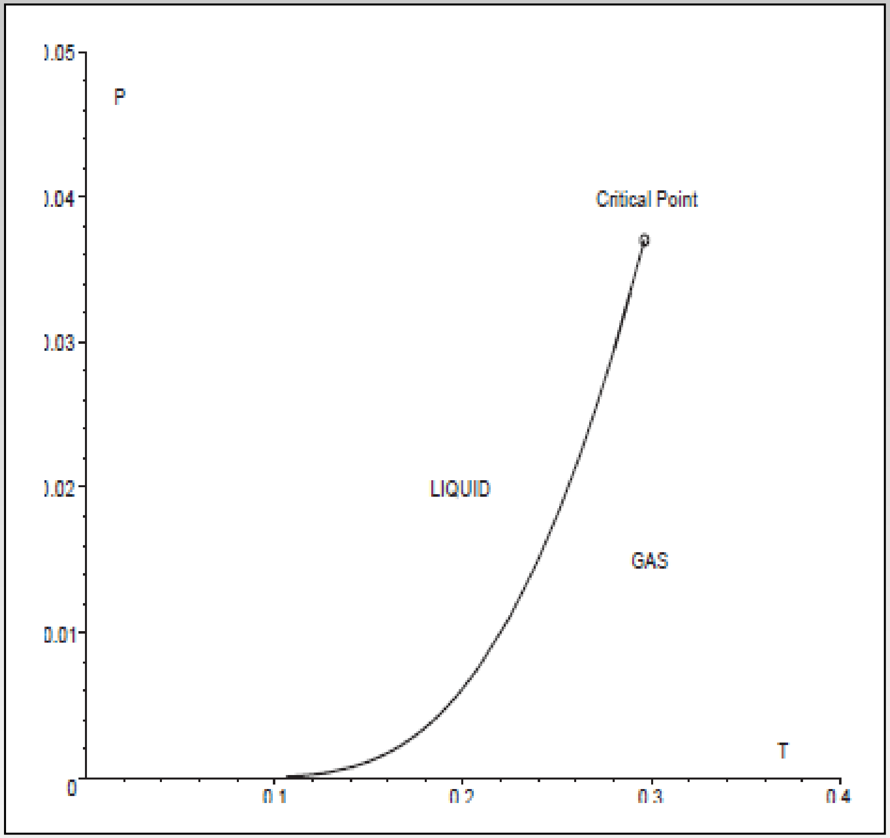
vs.



Phase diagrams: complete analogy



vs.

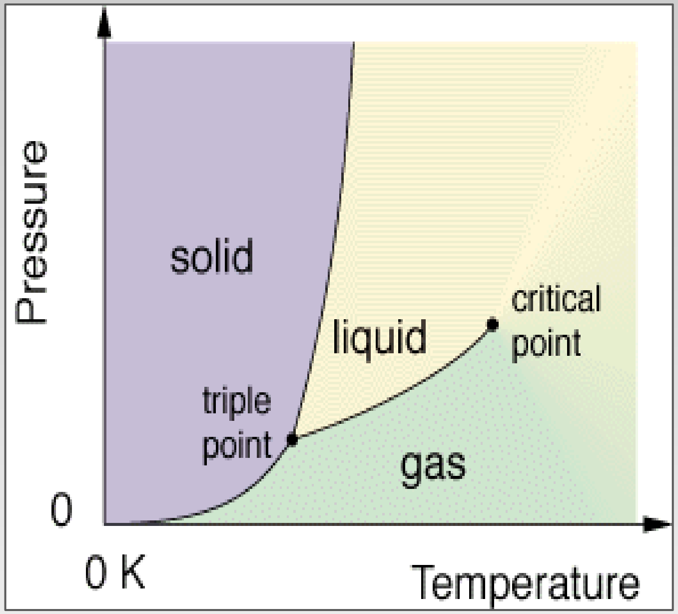


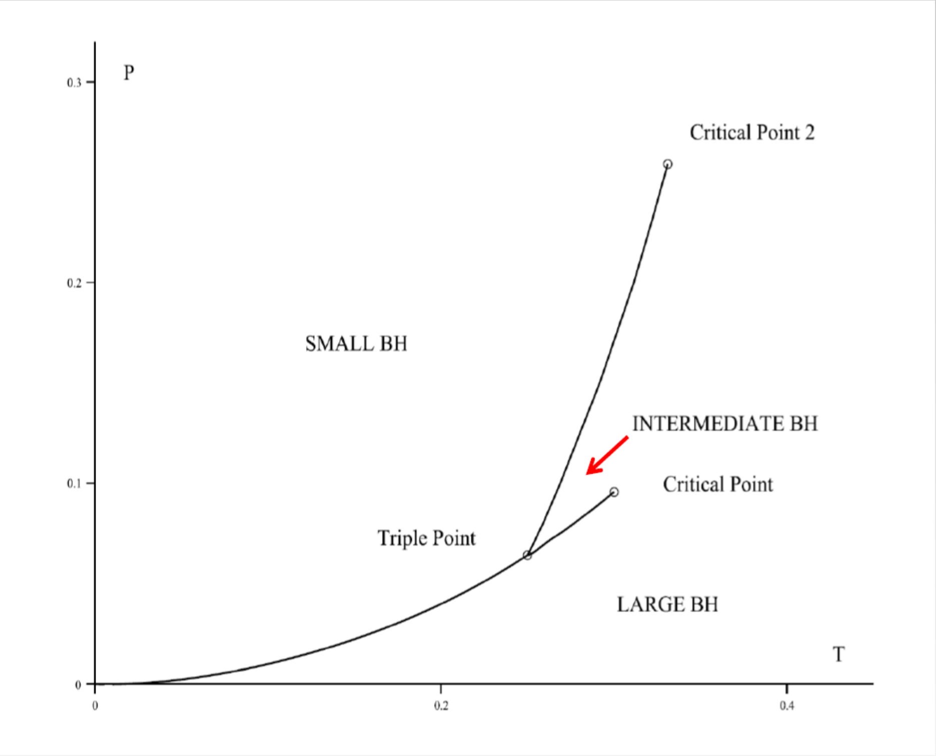
- Coexistence & critical point described by **Clausius-Clapeyron** and **Ehrenfest** equations
- **MFT critical exponents**

$$\alpha = 0\,, \quad \beta = \frac{1}{2}\,, \quad \gamma = 1\,, \quad \delta = 3$$

More generally: black hole chemistry

- **Triple point** and solid/liquid/gas analogue:





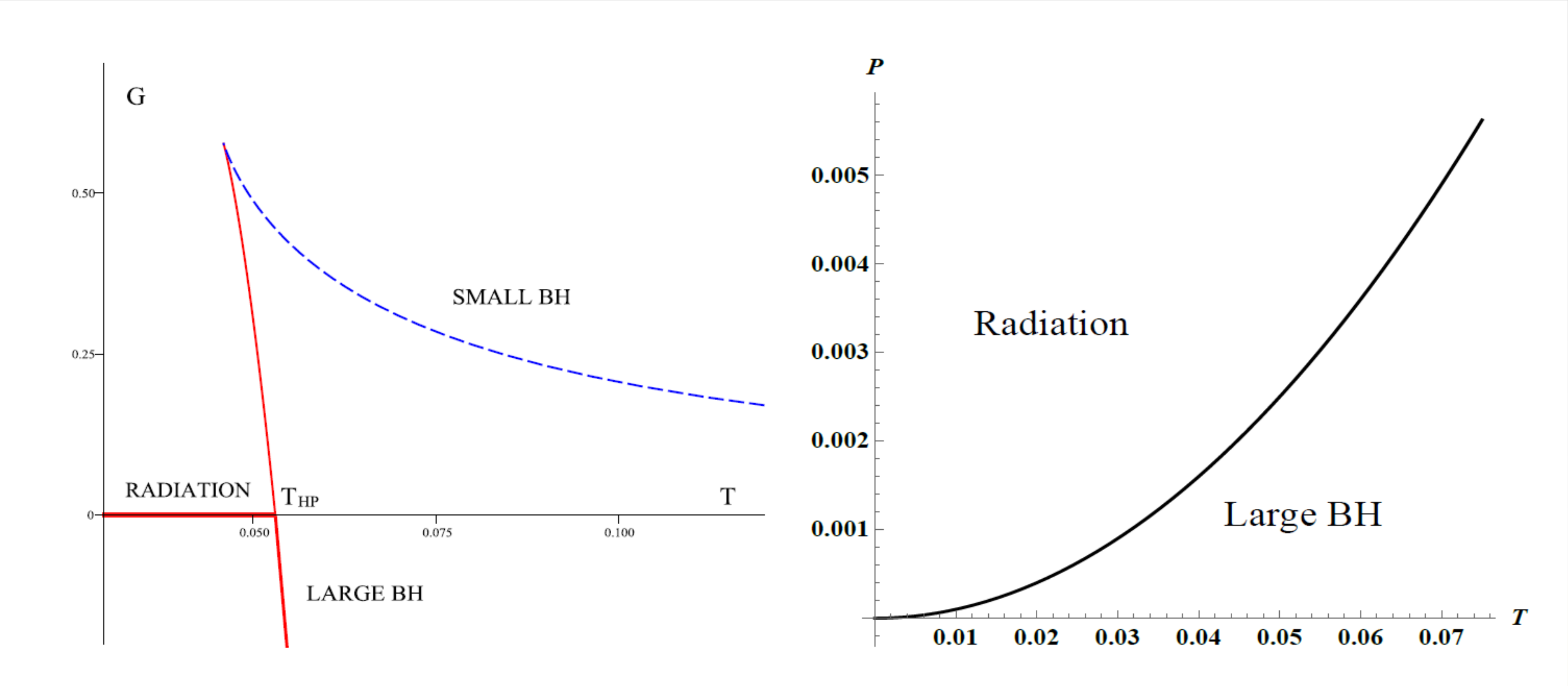
(can have n-tuple points)

- **Many other**:
  - Hawking-Page PT
  - Isolated critical point
  - Reentrant PT
  - superfluid PT

- DK, Mann, Teo, *Black hole chemistry: thermodynamics with Lambda*, CQG 34 (2017) 063001, Arxiv:1608.0614.

a) Hawking-Page transition

S.W. Hawking & D.N. Page, *Thermodynamics of black holes in anti-de-Sitter space*, Commun. Math. Phys. 87, 577 (1983).



1st-order radiation/large black hole phase transition

(dual to **confinement/deconfinement** PT of QGP)

b) Reentrant phase transition

A system undergoes an RPT if a **monotonic** variation of any thermodynamic quantity results in two (or more) phase transitions such that the **final state is macroscopically similar** to the initial state.

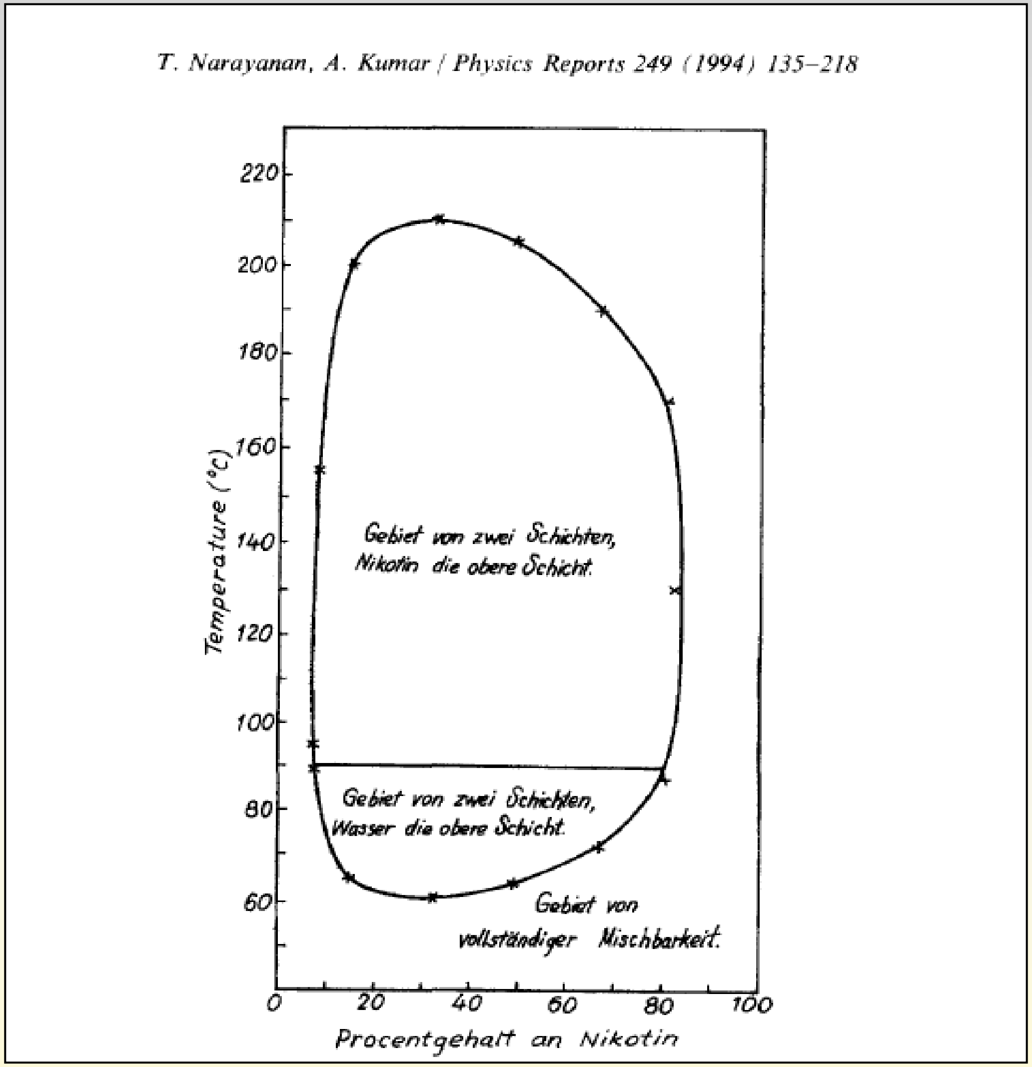
First observed by Hudson (1904) in a nicotine/water mixture

Z. Phys. Chem. 47 (1904) 113.

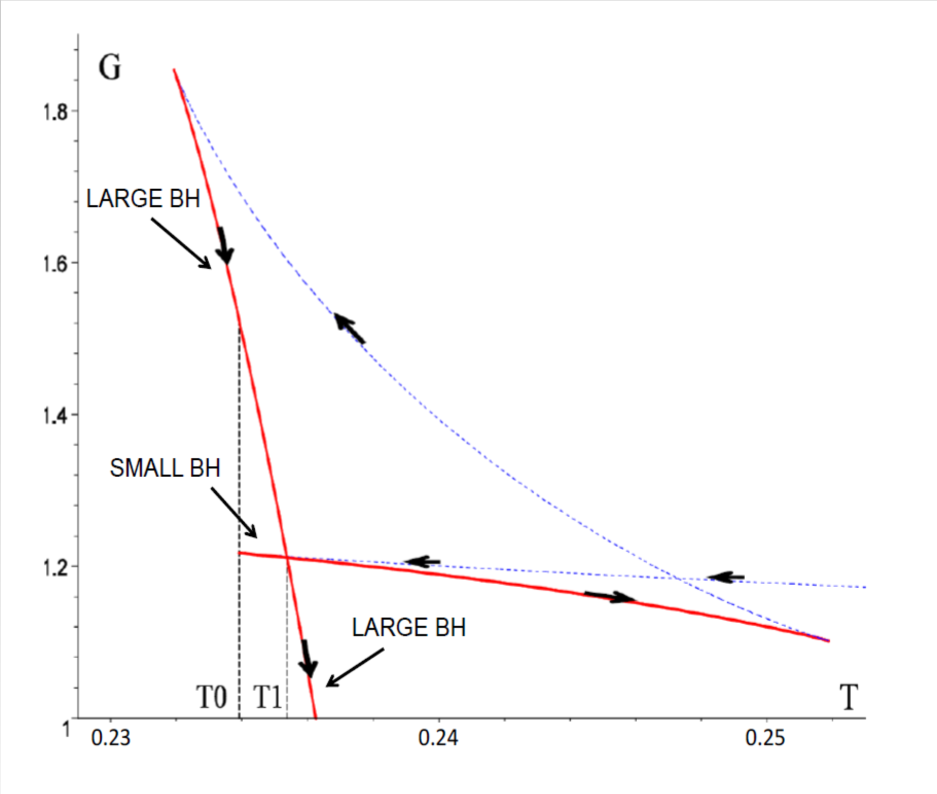
Since then:

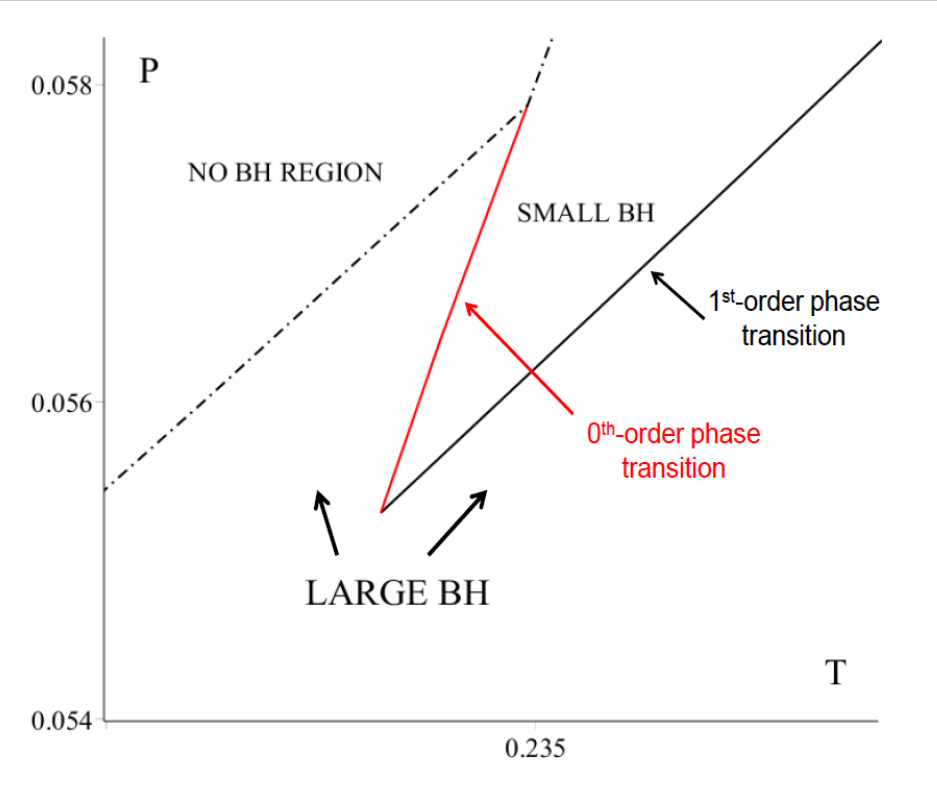
multicomponent fluid systems, gels, ferroelectrics, liquid crystals, and binary gases

T. Narayanan and A. Kumar, Reentrant phase transitions in multicomponent liquid mixtures, Physics Reports 249 (1994) 135–218.



AdS analogue: large/small/large black hole phase transition in Kerr-AdS BH in 6 dimensions (Altamirano et al 2013)





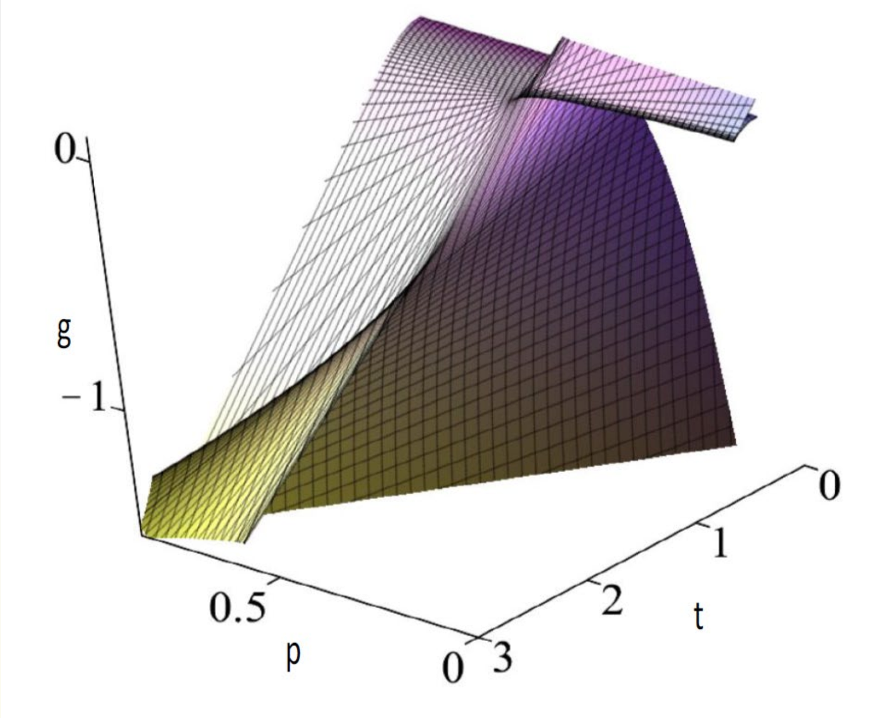
Low T	Medium T	High T
mixed	water/nicotine	mixed
large BH	small BH	large BH

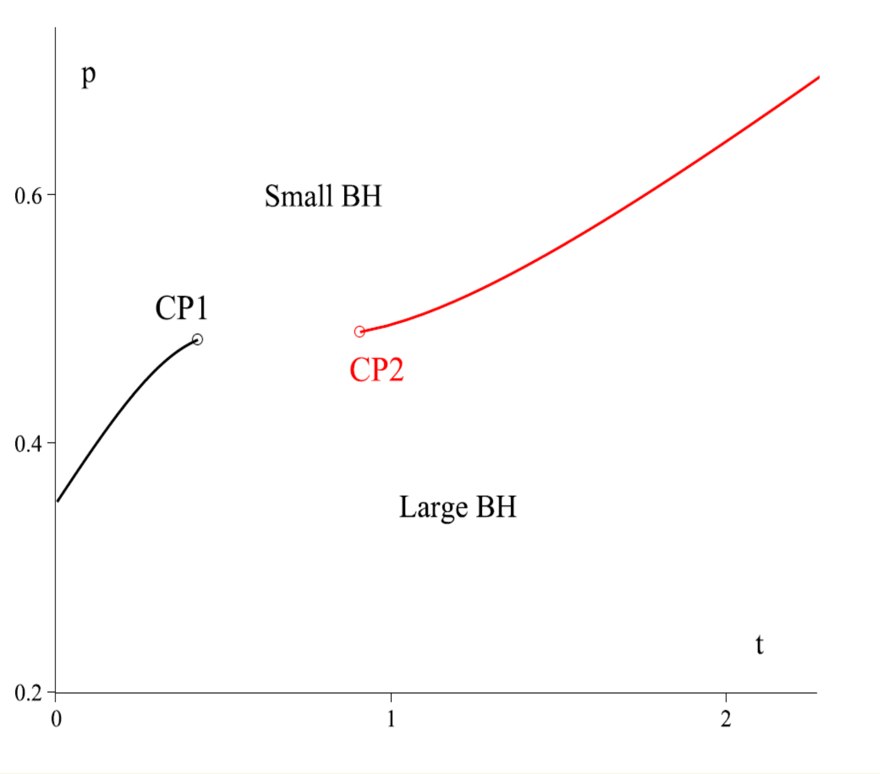
c) Isolated critical point

- Lovelock higher curvature gravity

$$\mathcal{L} = \frac{1}{16\pi G_N} \sum_{k=0}^{K} \hat{\alpha}_{(k)} \mathcal{L}^{(k)}$$

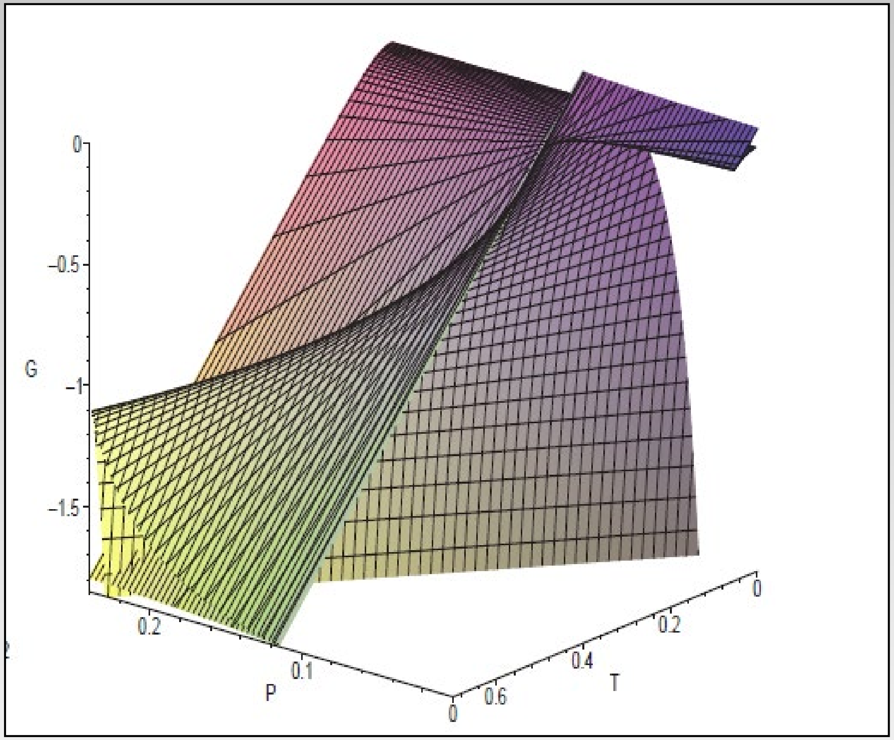
- odd-order K

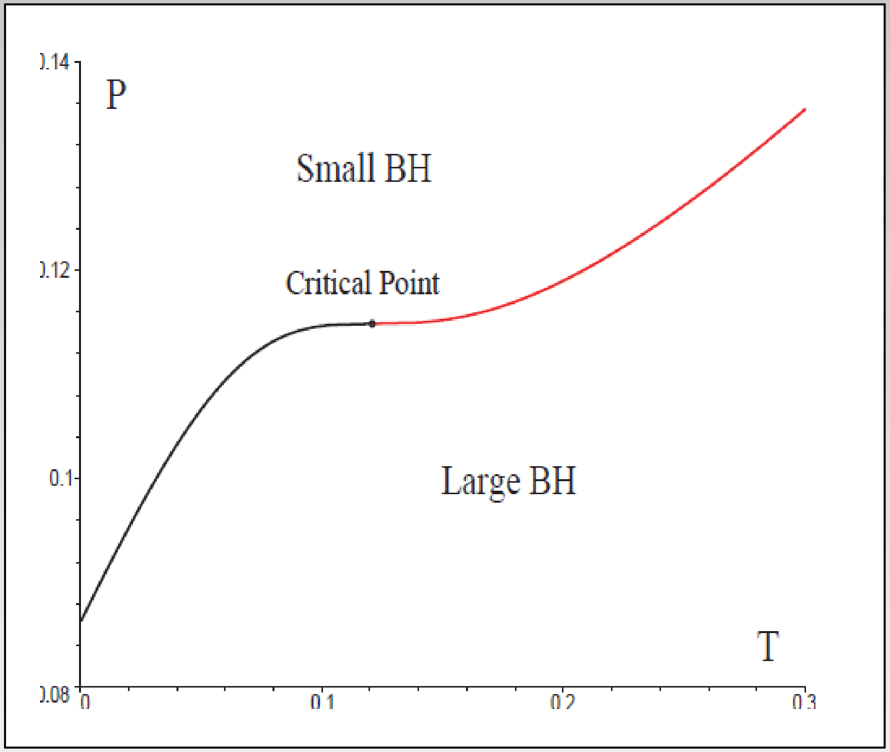




B. Dolan, A. Kostouki, DK, R. Mann, CQG31 (2014) 242001.

• For a special tuned Lovelock coupling we get





<u>**Critical exponents:**</u>

$$\alpha = 0\,, \quad \beta = 1\,, \quad \gamma = K - 1\,, \quad \delta = K\,.$$

• cf. mean field theory critical exponents

$$\alpha = 0\,, \quad \beta = \frac{1}{2}\,, \quad \gamma = 1\,, \quad \delta = 3$$

Topological interpretation: "vortex/anti-vortex creation"

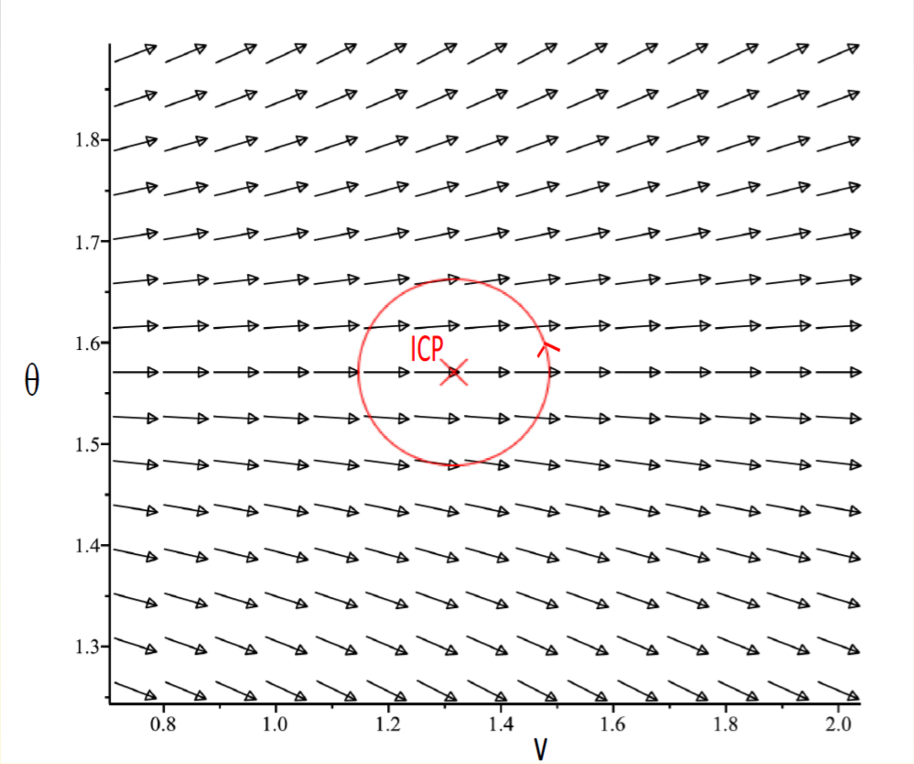
- Topological charge

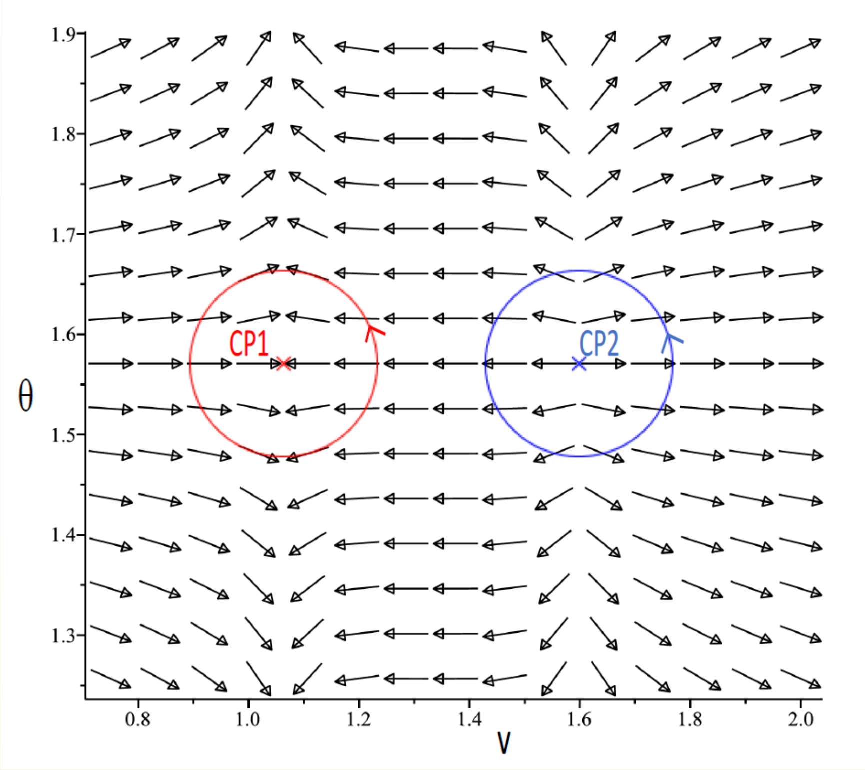
$$Q = \frac{1}{2\pi}\int_0^{2\pi} \epsilon_{ab} n^a \partial_\vartheta n^b d\vartheta$$

$$Q(\mathrm{CP1}) = -1\,, \quad Q(\mathrm{CP2}) = +1$$

$$Q(\mathrm{ICP}) = 0$$

- Suggestive (but a bit misleading) pictures:





M.B. Ahmed, DK, R.B. Mann, arXiv:2207.02147.

Summary

1) **AdS black holes** are an interesting generalization of their asymptotically flat cousins.

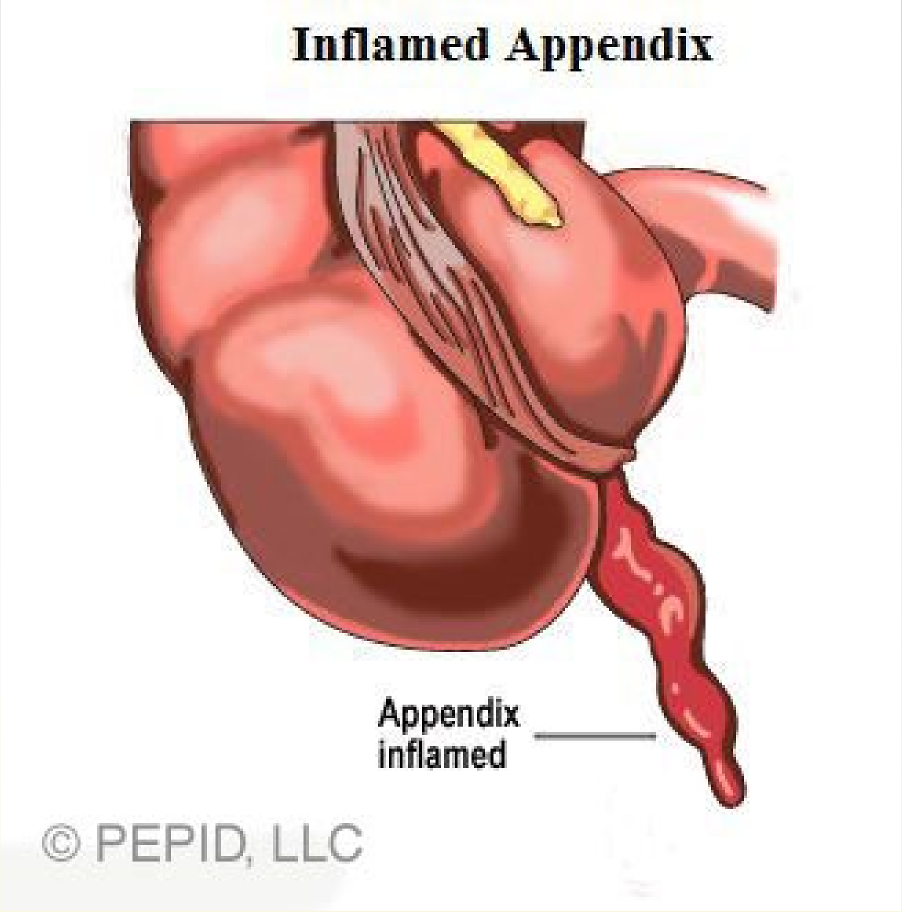
2) **Black hole chemistry** (TDs with variable Lambda) provides an interesting framework for AdS black hole thermodynamics:

   - **Extended first law** consistent with Smarr
   - Black hole mass is **enthalpy**
   - Definition of **black hole volume**
   - Uncovers various **phase transitions** & similarities with TDs of ordinary systems:

   e.g. **Van der Waals criticality** of charged BHs

3) All these **phase transitions** have a **dual description** via the AdS/CFT duality. Can they be **observed** on the CFT side?

Appendices



a) Global AdS and its (in)stability

Global AdS: (vacuum with Λ<0)

$$ds^2 = -\left(1+\frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{1+\frac{r^2}{l^2}} + r^2 d\Omega^2$$

Conjectured to be **nonlinearly unstable** (Dafermos, Anderson -2006)

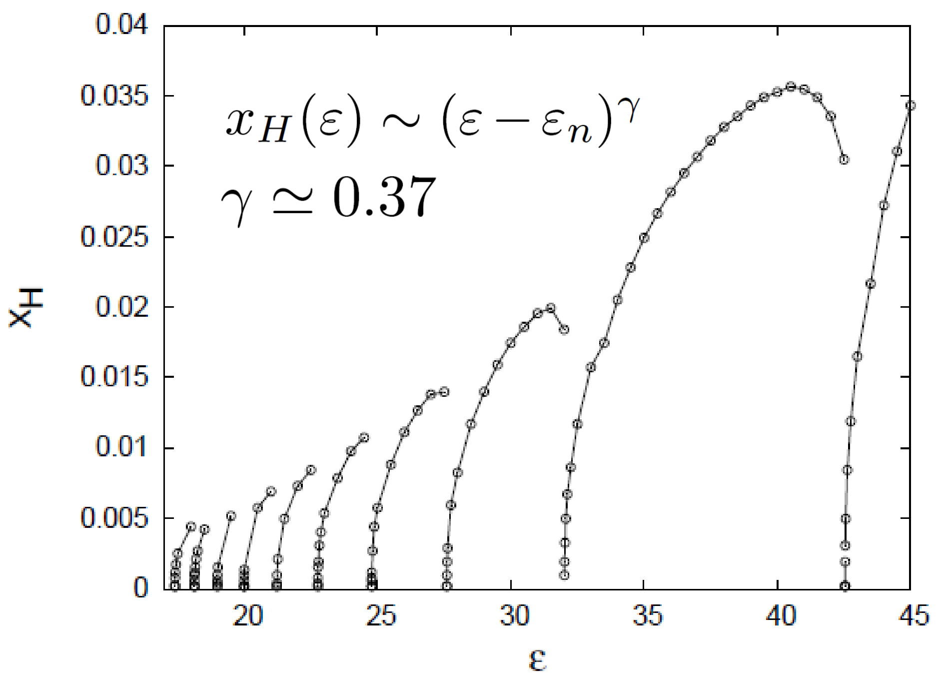
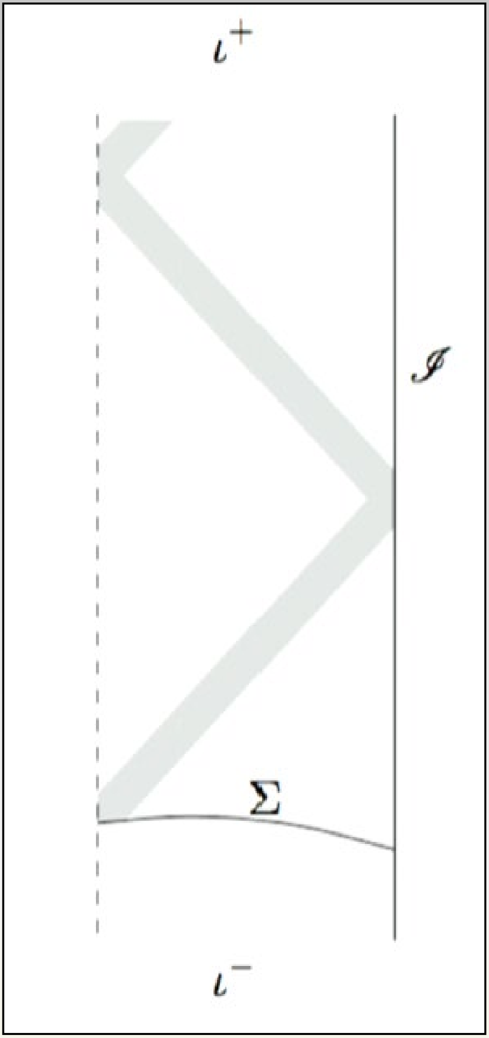


FIG. 1: Horizon radius vs amplitude for initial data (9). The number of reflections off the AdS boundary before collapse varies from zero to nine (from right to left).



Bizon and Rostworowski, *On weakly turbulent instability of anti-de Sitter space*, Phys. Rev. Lett. 107, 031102 (2011).

Intermezzo 1: Calculation of mass

- Komar integration does not work

$$M = -\frac{1}{8\pi}\int_{S_\infty} *dk\,, \quad k^a = (\partial_t)^a$$

- Nor does the background substracted Komar

Conformal method

(Ashtekar & Das, Clas. Qauntum Gravity 17 (2000) L17)

$$Q(\xi) = \frac{\ell}{8\pi}\lim_{\bar{\Omega}\to 0}\oint \frac{\ell^2}{\bar{\Omega}} N^\alpha N^\beta \bar{C}^\nu{}_{\alpha\mu\beta}\xi_\nu d\bar{S}^\mu$$

where $\bar{g}_{\mu\nu} = \bar{\Omega}^2 g_{\mu\nu}$ is the conformal completion of g (divergencies removed)

$N_\mu = \partial_\mu \bar{\Omega}$ is the normal to the boundary

Holographic stress tensor

$$8\pi \mathcal{T}_{ab} = \ell \mathcal{G}_{ab}(h) - \frac{2}{\ell} h_{ab} - \mathcal{K}_{ab} + h_{ab}\mathcal{K}$$

- **AdS boundary** parametrized by **Fefferman-Graham** coordinates:

$$ds^2 = \frac{\ell^2}{\rho^2} d\rho^2 + \frac{\rho^2}{\ell^2}\left(\gamma^{(0)}_{ab} + \frac{1}{\rho^2}\gamma^{(2)}_{ab} + \ldots\right) dx^a dx^b$$

- Expectation value of the CFT3 energy momentum:

$$\langle \mathcal{T}_a^b \rangle = \lim_{\rho \longrightarrow \infty} \frac{\rho}{\ell} \mathcal{T}_a^b$$

- Mass is an integral over the energy density

$$M = \int \rho_E \sqrt{-\gamma^{(0)}}$$

Thermodynamic machinery

- **Study**: charged and rotating AdS black holes in a canonical (fixed Q or J) ensemble. **Not an analogy!** (compare same physical quantities)

- The corresponding thermodynamic potential is **Gibbs free energy**

$$G = M - TS = G(P, T, J_1, \ldots, J_N, Q)\,.$$

equilibrium state corresponds to the **global minimum** of G.

- **Local thermodynamic stability**: positivity of the specific heat

$$C_P \equiv C_{P,J_1,\ldots,J_N,Q} = T\left(\frac{\partial S}{\partial T}\right)_{P,J_1,\ldots,J_N,Q}$$

- **Phase diagrams**: P-T diagrams

- **Critical points:** calculate critical exponents,….

Scalar field condensate: holographic superconductors

No-hair theorems: "All black holes" uniquely characterized by four asymptotic charges (mass, electric and magnetic charges, and angular momentum). All other information "disappears".

Wheeler in 70's **"Black holes have no hair"**

possible:

not possible:

AdS: holographic superconductors: need a black hole that has hair at low temperatures but no hair at low temperatures (Gubser 2008-charged scalar field in charged AdS BH)

H^3 Theory (Hartnoll, Herzog, Horowitz, "Building a Holographic Superconductor," Phys. Rev. Lett. 101, 031601, 2008)

Thermodynamics of dS black holes

2 problems:
- 2 horizons at different temperatures
- No timelike KF outside the BH and hence there is no asymptotic mass

B.P. Dolan, D. Kastor, DK, R.B. Mann, J. Traschen, Thermodynamic Volumes and Isoperimetric Inequalities for de Sitter Black Holes, arXiv:13001.5926 (2013).

Identify:

$$P = -\frac{1}{8\pi}\Lambda < 0$$

Hamiltonian analysis gives 3 first laws and Smarr relations

$$\delta M = T_{bh}\delta S_{bh} + V_{bh}\delta P$$
$$\delta M = -T_c\delta S_c + V_c\delta P$$
$$0 = T_{bh}\delta S_{bh} + T_c\delta S_c - V\delta P$$

TD volume $V = V_c - V_{bh}$ conjectured to obey ISO inequality.

Example: RPT and classical instabilities

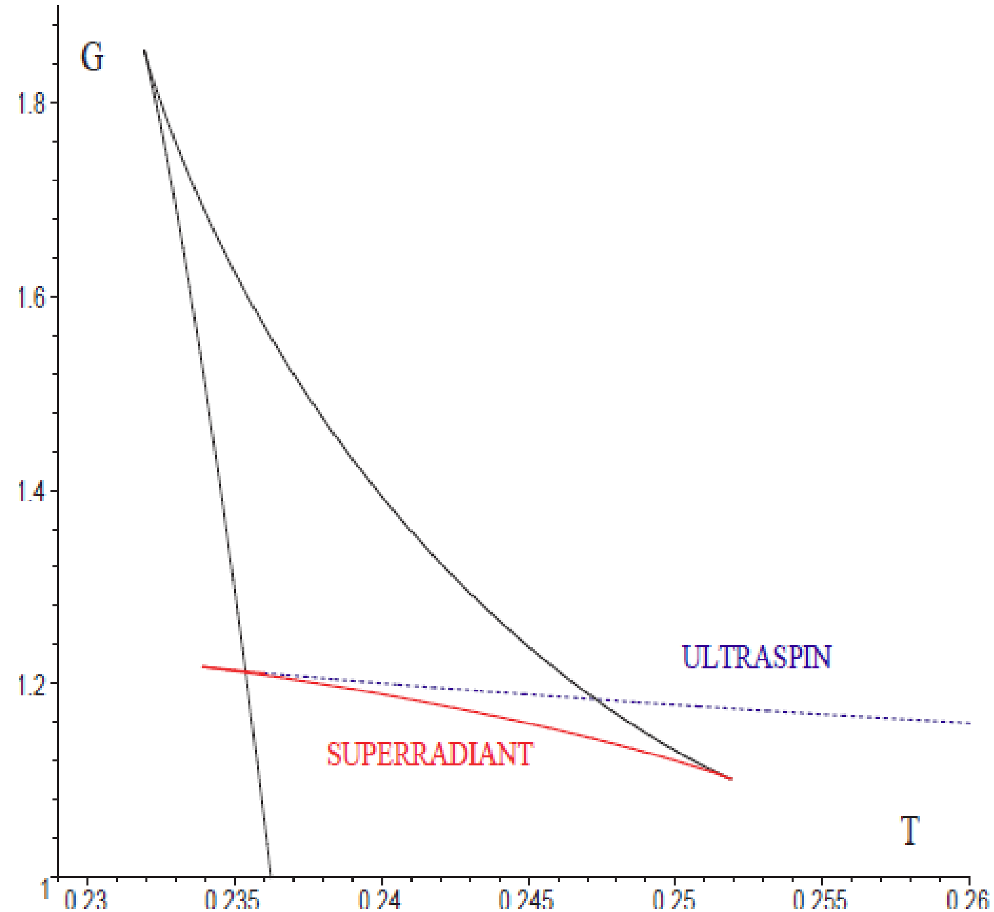
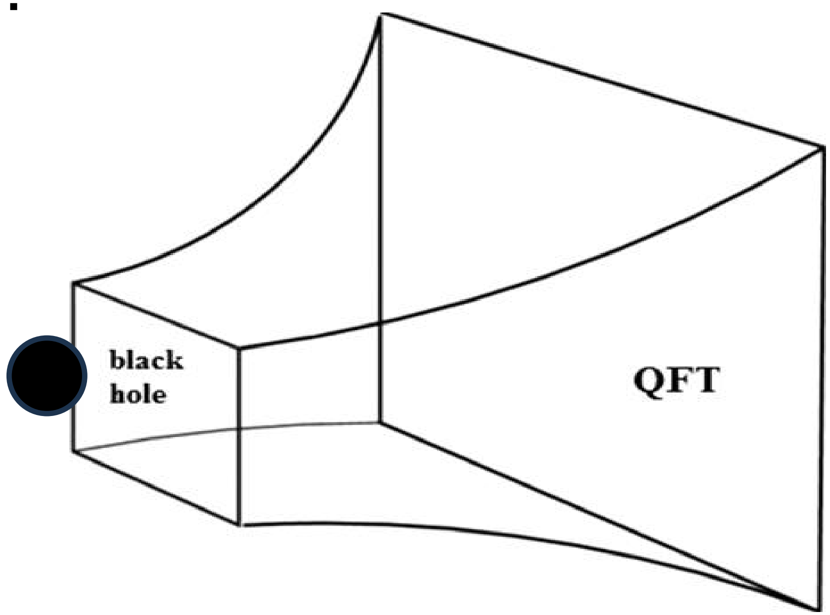


FIG. 1. **RPT and classical instabilities.** The blue curve indicates the potential presence of ultraspinning instabilities, derived from thermodynamic considerations (the existence of additional zero modes). The red curve displays the branch of black holes subject to superradiant instabilities.

Moral

- We have a very **rich structure** of phase transitions
- **Thermodynamic pressure P** (cosmological constant) plays a role of "**control parameter**"
- What is the **interpretation of VdP on the dual CFT side**?



AdS/CFT interpretation

- Einstein gravity

$$C \propto \frac{L^{d-2}}{G_N}$$

- So it seems that

$$V\delta P \leftrightarrow \mu\delta C$$

- C. Johnson, Holographic Heat engines, CQG 31 (2014) 205002, Arxiv:1404:5982.
- B. Dolan, Bose condensation and branes, JHEP 10 (2014) 179, ArXiv:1406.7267.
- D.Kastor, S.Ray, J.Traschen, Chemical potential in the frst law of holographic entanglement entropy, Arxiv:1409.3521.

AdS/CFT interpretation

- However, this is not the full story, as the **CFT volume** in principle also changes

- A. Karch and B. Robinson, *Holographic black hole chemistry*, JHEP 12 (2015 073, Arxiv:1510.0247.
- M. Visser*, Holographic thermodynamics requires a chemical potential for color*, Axiv:2101.04145.

$$ds^2 = -\frac{r^2}{L^2}dt^2 + \frac{L^2}{r^2}dr^2 + r^2 d\Omega^2_{d-2}$$

⇓ conformal completion

$$ds^2 = \omega^2\left(-dt^2 + L^2 d\Omega^2_{k,d-2}\right)$$

AdS/CFT interpretation

- **Standardly,** one choses

$$\omega = 1 \Rightarrow \mathcal{V} = \Omega_{d-2} L^{d-2}$$

and the volume also changes with variations of L.

- **Problematic**: On CFT side, **varying Lambda** corresponds to both – varying the **central charge C** and varying the **CFT volume**.

$$\delta\Lambda \Leftrightarrow \delta C \;\&\; \delta\mathcal{V}$$

- **To make these independent:** vary also the gravitational constant:

$$dG_N \quad \delta\Lambda \Leftrightarrow \delta C \;\&\; \delta\mathcal{V}$$

AdS/CFT interpretation

- **Alternatively:** "preserve the conformal symmetry of the dual CFT" by keeping general ω – and promote it to a new **thermodynamic parameter**

$$\mathcal{V} \propto (\omega L)^{d-2}$$

$$\delta\Lambda \Leftrightarrow \delta C \;\&\; \delta\mathcal{V}$$

… now independent

- Can hold **G fixed**! (dual to extended bulk thermodynamics)

Holographic thermodynamic laws

$$\delta\tilde{E} = \tilde{T}\delta S + \tilde{\Omega}\delta J + \tilde{\Phi}\delta\tilde{Q} + \mu\delta C - p\delta\mathcal{V}$$

- Here, E is the **internal energy** not **enthalpy**!
- Allows to study μ-C criticality

- **Accompanied by**

$$\mu = \frac{1}{C}(\tilde{E} - \tilde{T}S - \tilde{\Omega}J - \tilde{\Phi}\tilde{Q})$$

$$p = \frac{\tilde{E}}{(d-2)\mathcal{V}}\,.$$

- The first equality – **holographic Smarr (Euler relation)** – comes from “central charge extensivity” (note that it does not have D-dependent factors!)

- The second equality - **equation of state** - comes from the dimensional analysis