# Lecture 3: Mysterious topological density and its effect on BH TDs

**David Kubizňák**
(Charles University)

$$\mathcal{G} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$$

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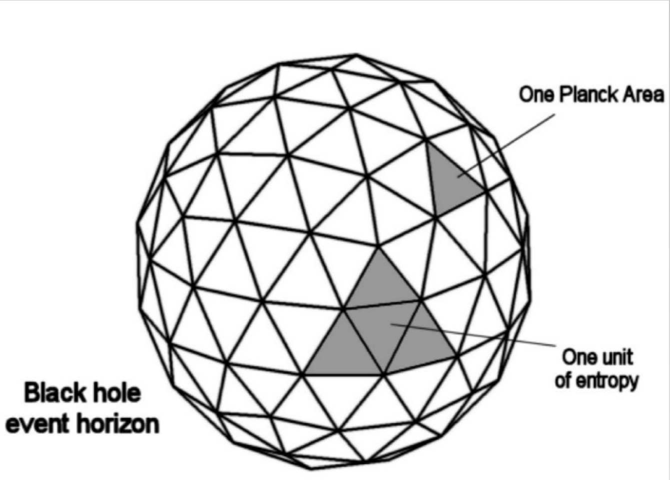
# Black hole entropy

relativity → $c^3$; statistical physics → $k_B$; QM → $\hbar$; gravity → $G_N$

$$S = \frac{A}{4}\frac{c^3 k_B}{\hbar G_N}$$




- Follows **area law** and is huge!

$$S = \frac{k_B}{4}\frac{A}{l_P^2}\,, \qquad l_P = \sqrt{\frac{G\hbar}{c^3}}$$

- It is holographic:

$$S \propto A$$

$$S \leq \frac{A}{4}$$

Bekenstein's (universal) bound

**Accepted wisdom**: The **area law** gets corrections in the presence of higher-curvature corrections to Einstein's gravity.

# Plan for Lecture 3

I. Two more ways to study BH TDs

II. Lovelock gravity: a prototype of higher curvature theory

III. 4D Gauss-Bonnet gravity
  - 4D GB term and its effect on entropy
  - D->4 limit & BH solutions
  - Shift symmetry: area entropy & modified temperature

IV. Back to Lovelock: shall the area law prevail?

V. Summary

## Based on:

- DK, M. Liška, *Shall the area law prevail? Phys. Rev. D108 (2023) L121501;* ArXiv:2307.16201.
- M. Liška, R. Hennigar, DK, *No logarithmic corrections to entropy in shift symmetric Gauss-Bonnet gravity*, JHEP 11 (2023) 195; ArXiv:2309.05629.

$$Z = \int Dg e^{-S_E[g]} \approx e^{-S_E(g_c)}$$

# I) Two more ways to study black hole thermodynamics

- R. Wald, Black hole entropy is the Noether charge, PRD 48, R3427, 1993.
- Braden, Brown, Whiting, York, Charged black hole in a grand canonical ensemble, PRD 42, 3376 (1990).

# Covariant phase space method

- Vary the Lagrangian $$\delta \mathbf{L} = \mathbf{E}\delta\phi + d\mathbf{\Theta}$$
- Variation induced by diffeo ξ: $$\hat{\delta}\phi = \mathcal{L}_\xi \phi$$
  define a **current:** $$\mathbf{j} = \mathbf{\Theta}(\phi, \mathcal{L}_\xi\phi) - \xi \cdot \mathbf{L}$$
- Conserved on-shell:
  $$d\mathbf{j} = -\mathbf{E}\mathcal{L}_\xi \phi \quad \Rightarrow \quad \boxed{\mathbf{j} = d\mathbf{Q}}$$
  $$\delta H = \delta \int_C \mathbf{j} - \int_C d(\xi \cdot \mathbf{\Theta})$$
- For a Killing vector ξ generating the horizon

  **Wald entropy** $$S = \frac{2\pi}{\kappa}\int_H \mathbf{Q}$$ (invariant under adding boundary terms)

# <u>Euclidean path integral a la Brown & York</u>

- Introduce Euclidean time $\tau = it$
- Calculate Euclidean action $S_E$, to get the partition function:

$$Z = \int Dg e^{-S_E[g]} \approx e^{-S_E(g_c)}$$

- **Standardly**: inverse temp β fixed by regularity on horizon
- **<u>Physical twist</u>**: BH off-shell, put a **boundary** and consider β there

$$\partial I / \partial r_+ = 0,$$

**Fixes temperature** (send boundary to infinity)

$$S = \beta \partial_\beta I - I$$

## II) Lovelock gravity: a prototype of higher-curvature gravity theories

$$\mathcal{G} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$$

# Gravitational action

- To write the **Gravitational Action** we want
  - **Scalar Lagrangian** -- diffeomorphism invariance
  - **Second-order (E-L)** equations for the metric

- One possibility is to write the **Einstein-Hilbert action**

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int \sqrt{-g}\mathcal{L}\,, \quad \mathcal{L} = R$$

- Is this just the simplest choice or can we add other scalars?

$$R^2\,,\ R_{\mu\nu}R^{\mu\nu}\,,\ R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}\,,\ \nabla_\mu R\nabla^\mu R\,,\ldots$$

# Lovelock's Theorem

D. Lovelock, *The Einstein Tensor and Its Generalizations"*, *Journal of Mathematical Physics.* **12** *(3): 498–501 (1971).*

**In 4D**, the Einstein-Hilbert action is the only local action (apart from the cosmological constant and topological terms) that leads to the **second order** differential equations for the metric. **In higher D,** we can have **Gauss-Bonnet** (Lovelock) theories.

**Einstein's theory is the unique theory in 4D!**

# Gauss-Bonnet gravity

$$\mathcal{G} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$$

- In **D<4** it **identically vanishes**!

- In 4D, the **Gauss-Bonnet term** is **topological** (total derivative!?!?)**.**

- In **D=5 and higher** dimensions it yields non-trivial EOMs:

$$\begin{aligned} H_{\alpha\beta} \;\; = \;\; & -\frac{1}{2} g_{\alpha\beta}\mathcal{G} + 2RR_{\alpha\beta} - 4R_{\alpha\gamma}R_{\beta}{}^{\gamma} \\ & +4R_{\gamma\alpha\beta\delta}R^{\gamma\delta} + 2R_{\alpha}{}^{\gamma\delta\kappa}R_{\beta\gamma\delta\kappa} = 0\,. \end{aligned}$$

**2$^{nd}$-order PDEs !!!**

# Lovelock gravity

= Unique higher-curvature (with local action) gravity that yields **2nd-order PDEs** for the metric

$$\mathcal{L} = \frac{1}{16\pi G_N} \sum_{k=0}^{K} \alpha_k \mathcal{L}^{(k)} \qquad K = \lfloor \frac{d-1}{2} \rfloor$$

where $\mathcal{L}^{(k)}$ are the 2k-dimensional Euler densities

$$\mathcal{L}^{(k)} = \frac{1}{2^k} \delta^{a_1 b_1 \ldots a_k b_k}_{c_1 d_1 \ldots c_k d_k} R_{a_1 b_1}{}^{c_1 d_1} \ldots R_{a_k b_k}{}^{c_k d_k}$$

- k=0: cosmological term $\Lambda = -\alpha_0/2$
- k=1: Einstein-Hilbert term $R$ **(topological in 2D)**
- k=2: Gauss-Bonnet term $\mathcal{G}$ **(topological in 4D)**
- k=3: 3rd-order Lovelock **(topological in 6D)**

**"Natural generalization of Einstein's theory in higher dimensions"**

# Lovelock gravity

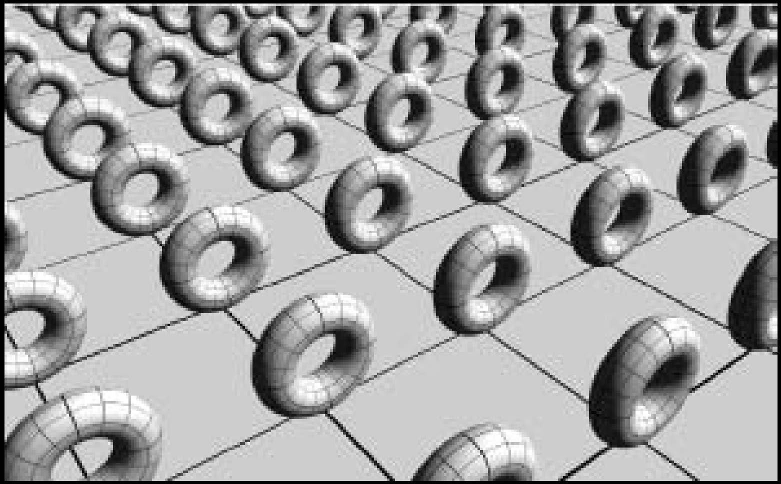
= Unique higher-curvature (with local action) gravity that yields **2nd-order PDEs** for the metric

$$\mathcal{L} = \frac{1}{16\pi G_N} \sum_{k=0}^{K} \alpha_k \mathcal{L}^{(k)} \qquad K = \lfloor \frac{d-1}{2} \rfloor$$

**Wald entropy:**

$$S = \frac{1}{4G_N} \sum_k \hat{\alpha}_k \mathcal{A}^{(k)}, \quad \mathcal{A}^{(k)} = k \int_{\mathcal{H}} \sqrt{\sigma} \mathcal{L}^{(k-1)}$$

# III) 4D Gauss-Bonnet gravity



# Gauss-Bonnet gravity in 4D?

$$S_D = \int d^D x \sqrt{-g}(R - 2\Lambda + \alpha \mathcal{G})$$

$$D \to 4?$$

- In 4D, the Gauss-Bonnet term is **topological** (a total derivative)**.**

$$\mathcal{G} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$$

- Theory remains that of **Einstein**, **but entropy gets modified**!

$$S \to S + \alpha\chi(M)$$

How well is this established? Is it a problem for **BH mergers**?

# Glavan & Lin's proposal

PHYSICAL REVIEW LETTERS **124,** 081301 (2020)

## Einstein-Gauss-Bonnet Gravity in Four-Dimensional Spacetime

Dražen Glavan[1,*] and Chunshan Lin[2,†]

[1] *Centre for Cosmology, Particle Physics and Phenomenology (CP3), Université catholique de Louvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium*

[2] *Institute of Theoretical Physics, Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warsaw, Poland*

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In this Letter we present a general covariant modified theory of gravity in $D = 4$ spacetime dimensions which propagates only the massless graviton and bypasses Lovelock's theorem. The theory we present is formulated in $D > 4$ dimensions and its action consists of the Einstein-Hilbert term with a cosmological constant, and the Gauss-Bonnet term multiplied by a factor $1/(D-4)$. The four-dimensional theory is defined as the limit $D \to 4$. In this singular limit the Gauss-Bonnet invariant gives rise to nontrivial contributions to gravitational dynamics, while preserving the number of graviton degrees of freedom and being free from Ostrogradsky instability. We report several appealing new predictions of this theory, including the corrections to the dispersion relation of cosmological tensor and scalar modes, singularity resolution for spherically symmetric solutions, and others.

# Conformal trick: D->4 limit of GB theory

R. Hennigar, DK, R. Mann, C. Pollack, On taking the D->4 limit of Gauss-Bonnet gravity: theory and solutions, KHEP 07 (2020) 027.

- Start with GB $S_D^{\mathrm{GB}} = \alpha \int d^D x \sqrt{-g}\mathcal{G}$ (topological in 4D)
- Evaluate it for the **conformally rescaled** metric $\tilde{g} = e^{\psi} g$

$$S_D^{\mathrm{GB}} = \hat{\alpha} \int d^D x \sqrt{-\tilde{g}}\tilde{\mathcal{G}}$$

- **Expand** around $\epsilon = (D-4)$

$$= \underbrace{\hat{\alpha}\#}_{\text{topological}} + \underbrace{\hat{\alpha}(D-4)\#}_{\text{"new theory"}} + O[(D-4)^2]$$

- **Rescale** the coupling: $\hat{\alpha}(D-4) \to \alpha$ $\quad D \to 4$

# Conformal trick: D->4 limit of GB theory

R. Hennigar, DK, R. Mann, C. Pollack, On taking the D->4 limit of Gauss-Bonnet gravity: theory and solutions, JHEP 07 (2020) 027.

- Formally, one can introduce a **counter-term**

$$S_D^{\mathrm{GB}} = \hat{\alpha} \int d^D x \left( \sqrt{-\tilde{g}} \tilde{\mathcal{G}} - \sqrt{-g} \mathcal{G} \right)$$

- After field redefinition:

$$\psi \to -2\phi \,, \quad g_{ab} \to -\frac{1}{2} g_{ab} \,, \quad \alpha \to \alpha/2$$

$$S = \int d^4 x \sqrt{-g} \Big[ R - 2\Lambda + \alpha \Big( \phi \mathcal{G} + 4 G^{ab} \partial_a \phi \partial_b \phi - 4 (\partial \phi)^2 \Box \phi + 2 ((\nabla \phi)^2)^2 \Big) \Big] \,,$$

# The new theory

$$S=\int d^4x\sqrt{-g}\Big[R-2\Lambda+\alpha\Big(\phi\mathcal{G}+4G^{ab}\partial_a\phi\partial_b\phi$$
$$-4(\partial\phi)^2\Box\phi+2((\nabla\phi)^2)^2\Big)\Big]\,,$$

- Is a **scalar-tensor** theory of **Horndeski-type**

  G. W. Horndeski, *Second-order scalar-tensor field equations in a four-dimensional space*, International Journal of Theoretical Physics 10 (1974) 363-384.

- It can also be derived by **Kaluza-Klein compactification** in the limit of vanishing extra dimensions:

$$ds^2_D=ds^2_4+e^{2\phi}d\Sigma^2_{D-4}$$

H. Lu and Y. Pang, *Horndeski Gravity as D->4 Limit of Gauss-Bonnet*, 2003.11552. (see also T. Kobayashi, 2003.12771)

# Solutions: GB Black Hole

H. Lu and Y. Pang, *Horndeski Gravity as D->4 Limit of Gauss-Bonnet*, 2003.11552.

- Constructed the SSS solutions of the **Horndeski-GB theory**

$$ds^2 = -f dt^2 + \frac{dr^2}{fh} + r^2 d\Omega^2$$

- **Special class** of solutions has $h = 1$

$$f_\pm = 1 + \frac{r^2}{2\alpha}\left(1 \pm \sqrt{1 + \frac{4}{3}\alpha\Lambda + \frac{8\alpha M}{r^3}}\right)$$

$$\phi(r) = \ln\left(\frac{r}{L}\right) \pm \int_{r_+}^{r} \frac{1}{\rho\sqrt{f(\rho)}} \mathrm{d}\rho.$$

- Metric **coincides** with the naïve D->4 limit.

- **Uniqueness** – Fernandes et al, Arxiv:2107.00046

# Solutions: GB Black Hole

- The same spacetime considered as "**quantum gravity** corrected metric"
  - Y. Tomozawa, Quantum corrections to gravity, 1107.1424.
  - G. Cognola, R. Myrzakulov, L. Sebastiani and S. Zerbini, Einstein gravity with Gauss-Bonnet entropic corrections, Phys. Rev. D 88 024006, [1304.1878].

- **Have a theory** – so can use Wald's formalism to calculate **entropy**, which picks up *logarithmic corrections*:

$$S_0 = \frac{\text{Area}}{4} + \frac{\alpha}{4}\int_{\mathcal{H}} \phi\Big(\epsilon^{ab}R - 4\epsilon^{ca}R_c{}^b + \epsilon_{cd}R^{bacd}\Big)\epsilon_{ba}\,\mathrm{d}^2\mathcal{A}\,.$$

$$S = \frac{1}{G_4}\left(\pi r_+^2 + 4\alpha\pi\log\frac{r_+}{L}\right)$$

# Shift symmetry

$$I \propto \int d^4x \sqrt{-\mathfrak{g}} \left( R + \alpha \phi \mathcal{G} + \dots \right)$$

$$\mathcal{G} = \nabla_\mu \mathcal{G}^\mu$$

- Is a **shift-symmetric** theory $\phi \to \phi + C$.
- However, such a **change shifts entropy by a constant** (this may now become negative?)
- Consider instead a manifestly **shift-symmetric action**

$$I_{\text{inv}} = I_{\text{EGB}} - \frac{1}{16\pi} \int \nabla_\mu \left( \alpha \phi \mathcal{G}^\mu \right) \sqrt{-\mathfrak{g}} \mathrm{d}^4 x.$$

(DBC for metric & NBC for scalar)

- For **4D GB theory**, we have on-shell covariant expression (up to superpotential – that does not contribute to Noether charges)

$$\mathcal{G}^{\mu} = 8G^{\mu\nu}\nabla_{\nu}\phi - 8\nabla^{\nu}\nabla_{\nu}\phi\nabla^{\mu}\phi + 8\nabla^{\nu}\phi\nabla_{\nu}\nabla^{\mu}\phi + 8\nabla_{\nu}\phi\nabla^{\nu}\phi\nabla^{\mu}\phi.$$

$\Rightarrow$ $$S = \frac{\text{Area}}{4}$$ (Wald entropy for the shift-invariant action)

see also: M. Minamitsuji, K. Maeda, Arxiv:2308.01082.

- **Validity of 1st law**: then requires **modified temperature**

$$T_{\text{mod}} = T_0\left(1 + \frac{2\alpha}{r_+^2}\right), \quad T_0 = \frac{\kappa}{2\pi} = \frac{|f'(r_+)|}{4\pi}$$

…follows from the Brown-York Euclidean action calculation!

# Summary of thermodynamics

$$f = 1 + \frac{r^2}{2\alpha}\left(1 - \sqrt{1 + \frac{8\alpha M}{r^3} - \frac{4\alpha Q^2}{r^4}}\right)$$

$$\delta M = T\delta S$$

$$M = m = \frac{r_+^2 + \alpha}{2r_+}$$

## "standard TDs"

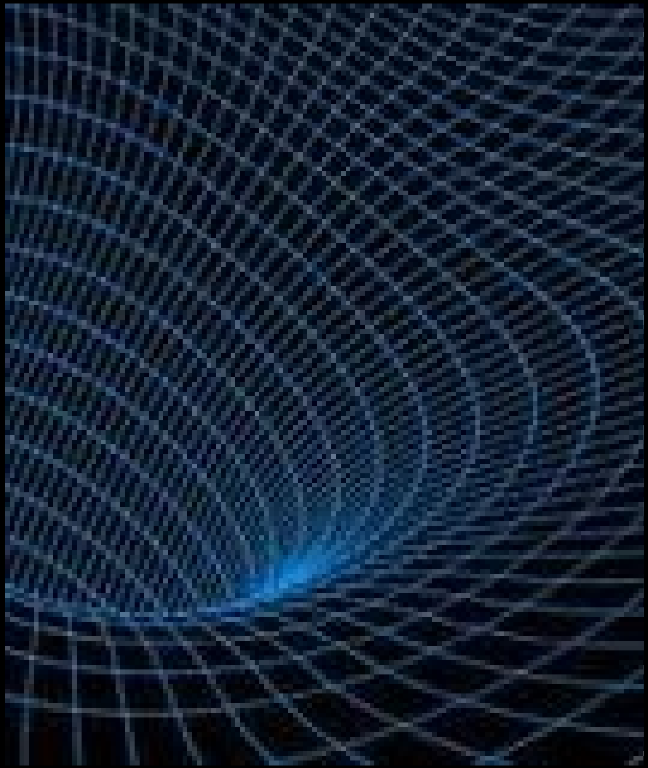
$$T_0 = \frac{|f'(r_+)|}{4\pi} = \frac{r_+^2 - \alpha}{4\pi r_+(r_+^2 + 2\alpha)}$$

$$S_0 = \pi r_+^2 + 4\pi\alpha \log(r_+/L)\,.$$

## "shift symmetric TDs"

$$T_{\rm mod} = T_0\left(1 + \frac{2\alpha}{r_+^2}\right) = \frac{r_+^2 - \alpha}{4\pi r_+^3}$$

$$S = \pi r_+^2\,.$$



# IV) Back to Lovelock: shall the Bekenstein’s area law prevail?

# Area law for Lovelock black holes

- If the **area law holds** for the 4D GB theory, shouldn't it also hold before the limit in higher dimensions?
- After-all every Lovelock gravity can be reduced to **shift-symmetric** scalar-tensor theory in the **critical dimension**.
- **Spec: spherical black holes - observation**

$$S_0 = S + \frac{\Sigma_{d-2}^{(\kappa)}(d-2)}{4}\sum_{k=2}^{K}\frac{k\kappa^{k-1}\alpha_k r_+^{d-2k}}{d-2k}$$

$$S = \frac{\Sigma_{d-2}^{(\kappa)}\alpha_1 r_+^{d-2}}{4}$$

$$T_0 = \frac{|f'(r_+)|}{4\pi} = \frac{T_{\rm mod}}{\Delta}$$

$$\Delta = \sum_{k=1}^{K} k\alpha_k\left(\kappa/r_+^2\right)^{k-1}$$

$$T_0 \to T_{\rm mod} = T_0\Delta\,, \quad S_0 \to S$$

# Area law for Lovelock black holes

- **In 4D GB**: $\mathcal{G} = \nabla_a \mathcal{G}^a$

  (covariant expression for G^a not known!!!)

- If the spacetime possesses a KV ξ, then (A. Yale & T. Padmanabhan 2011)

$$\mathcal{G}^a = -2 \frac{\partial \mathcal{G}}{\partial R_{abcd}} \xi_b \nabla_c \xi_d$$

- **In higher dimensions**:

$$I_2 = I_1 + \frac{\alpha}{8\pi} \int \nabla_a \left( \frac{\partial \mathcal{G}}{\partial R_{abcd}} \xi_b \nabla_c \xi_d \right) \sqrt{-\mathfrak{g}} \mathrm{d}^5 x$$

⇒ $S = \frac{\text{Area}}{4}$ $\quad T_{\text{mod}} = \frac{1}{2\pi r_+}$ (Wald & Brown-York)

# Summary of Lecture 3

1) While the **GB term** becomes **topological in 4D**, its presence is believed to **shift** the **BH entropy** by a constant proportional to the genus of the manifold. (Problem with BH formation, in which the entropy can subsequently decrease?)

2) One can take a singular limit D->4, rescaling the GB coupling. The resultant theory is a scalar-tensor theory known as **4D GB gravity:**

$$S = \int d^4x \sqrt{-g}\Big[R - 2\Lambda + \alpha\Big(\phi\mathcal{G} + 4G^{ab}\partial_a\phi\partial_b\phi - 4(\partial\phi)^2\Box\phi + 2((\nabla\phi)^2)^2\Big)\Big]\,,$$

3) The corresponding BHs have been claimed to feature **logarithmic corrections** to entropy. This seems to be a consequence of **breaking the shift-symmetry** at the action level. When shift-symmetric action is written down, we recover **Bekenstein's area law:**

$$S = \frac{\text{Area}}{4} \qquad T_{\text{mod}} = T_0\left(1 + \frac{2\alpha}{r_+^2}\right)$$

# Summary of Lecture 3

4. **Similarly:** One can cook up a **certain boundary term** for **higher-dimensional Lovelock** gravity so that the **area law prevails** and the **BH temperature is modified**.

5. What is the **origin of modified temperature**?

   a) **Modified speed of gravitons** on the horizon? See Hajian, Liberati, Sheikh-Jabbari, M. Vahidinia, PLB 812, 136002 (2021).
   b) "**Screening effect**" of strong gravitational field?
   c) Effect of certain boundary terms on the "**interpretation of TD ensemble**"?
   d) Assumed **fixed topology** in standard calculations?

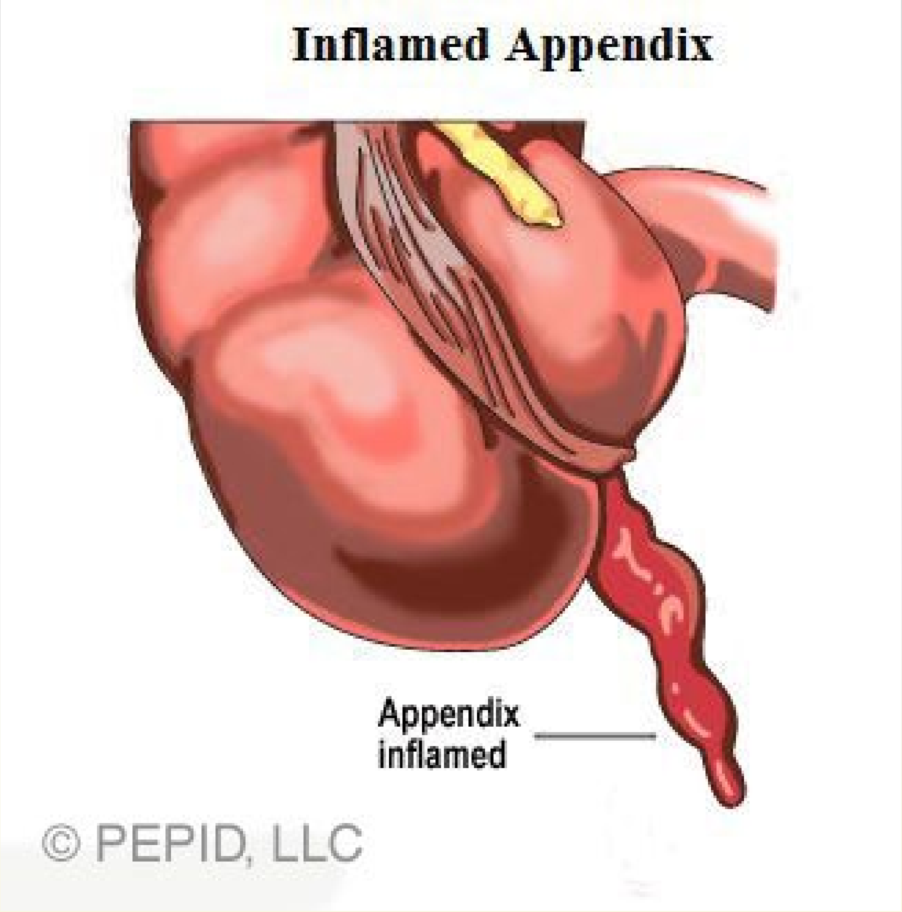
**Maybe: The area law shall prevail!**

# Mini course summary: Existential advice

And if anyone knows anything about anything—it's Owl who knows something about something, said Bear to himself—or my name's not Winnie-the-Pooh—he said—which it is, he added—so there you are.

Winnie-the-Pooh, A. A. Milne (1926)

# Appendices



# Brown-York approach

Braden, Brown, Whiting, York, *Charged black hole in a grand canonical ensemble*, PRD 42, 3376 (1990).

$$ds^2 = b^2 d\tau^2 + \alpha^2 dy^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

$$A_\mu dx^\mu = A_\tau(y) d\tau$$

where $b$, $\alpha$, and $r$ are functions only of the radial coordinate $y \in [0,1]$. The "Euclidean time" $\tau$ is chosen to have period $2\pi$; the angles $\theta$ and $\varphi$ are the usual coordinates of the unit sphere. By "black-hole topology" we mean that the four-geometries $(M,g)$ are regular with product topology $R^2 \times S^2$, boundary $S^1 \times S^2$ at $y=1$, and Euler number $\chi = 2$.

- Boundary data: $\{\beta, \phi, r_B\}$

$$\beta = T^{-1} = \int_0^{2\pi} b(1) d\tau = 2\pi b(1) \qquad A_\tau(1) = \frac{\beta\phi}{2\pi i}$$

# Brown-York approach

Braden, Brown, Whiting, York, *Charged black hole in a grand canonical ensemble*, PRD 42, 3376 (1990).

$$ds^2 = b^2 d\tau^2 + \alpha^2 dy^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

$$A_\mu dx^\mu = A_\tau(y) d\tau$$

- Regular (horizon) center:

$$b(0) = 0 \qquad (\alpha^{-1} b')|_{y=0} = 1 \qquad A_\tau(0) = 0$$

In order to have $\chi = 2$, the condition

$$[\alpha^{-2}(r')^2]_{y=0} = 0$$

# Brown-York approach

$$I_g = -\frac{1}{16\pi}\int_M d^4x\,\sqrt{g}\,R + \frac{1}{8\pi}\int_{\partial M} d^3x\,\sqrt{\gamma}(K - K^0)$$

$$I_A = \frac{1}{16\pi}\int d^4x\,\sqrt{g}\,F_{\mu\nu}F^{\mu\nu}\,,$$

$$I_g = -\frac{1}{2}\int d\tau\,dy\left[\frac{2}{\alpha}rr'b' + \frac{b}{\alpha}(r')^2 + \alpha b - 2(br)'\right] - \frac{1}{2}\int_{y=0} d\tau\left[\frac{1}{\alpha}(br^2)' - 2br\right].$$

$$I_A = \frac{1}{2}\int d\tau\,dy\left[\frac{r^2}{\alpha b}A'_\tau\right]A'_\tau$$

- Impose Gauss and Hamiltonian constraints:

$$\left[\frac{r^2}{\alpha b}A'_\tau\right]' = 0 \quad\Rightarrow\quad \frac{r^2}{b\alpha}A'_\tau = -ie$$

$$G^\tau_\tau - 8\pi T^\tau_\tau = 0 \quad\Rightarrow\quad \left(\frac{r'}{\alpha}\right)^2 = 1 - \frac{C}{r} + \frac{e^2}{r^2}$$

$$\chi = 2 \quad\Rightarrow\quad C = r_+ + \frac{e^2}{r_+}$$

# Brown-York approach

- Reduced action $I_* = I_*(\beta, \phi, r_B; e, r_+)$

Stationary points w.r.t. e and r+ fixes potential and temperature:

$$\frac{\partial I_*}{\partial e} = 0 \quad \Rightarrow \quad \phi = \frac{e}{r_+}\left[1 - \frac{r_+}{r_B}\right]^{1/2}\left[1 - \frac{e^2}{r_+ r_b}\right]^{-1/2},$$

$$\frac{\partial I_*}{\partial r_+} = 0 \quad \Rightarrow \quad \beta = 4\pi r_+\left[1 - \frac{e^2}{r_+^2}\right]^{-1}\left[1 - \frac{r_+}{r_B}\right]^{1/2}\left[1 - \frac{e^2}{r_+ r_B}\right]^{1/2}$$

- Thermodynamic potential:

$$\ln Z_{GC} \approx -\tilde{I}(\beta, \phi, r_B),$$

# Brown-York approach

- **Thermodynamic potential:**

$$\ln Z_{GC} \approx -\tilde{I}(\beta, \phi, r_B),$$

- **Remaining TD quantities:**

$$\langle Q \rangle = \beta^{-1} \left[ \frac{\partial(\ln Z_{GC})}{\partial \phi} \right]_{\beta, r_B} \approx -\beta^{-1} \left[ \frac{\partial \tilde{I}}{\partial \phi} \right]_{\beta, r_B}$$

$$S \approx \beta \left[ \frac{\partial \tilde{I}}{\partial \beta} \right]_{\phi, r_B} - \tilde{I} = \pi r_+^2$$

$$\langle E \rangle \approx \left[ \frac{\partial \tilde{I}}{\partial \beta} \right]_{\beta\phi, r_B}$$

$$d\langle E \rangle = T\, dS - \lambda\, dA + \phi\, d\langle Q \rangle$$

As $r_b$ sent to infinity, the middle term vanishes, and we recover the standard TDs. **All quantities were derived from the action!**

# 2) Lovelock Thermodynamics

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_{d-2}^2 \,, \qquad F = \frac{Q}{r^{d-2}} dt \wedge dr \,,$$

$$\sum_{k=0}^{K} \alpha_k \left(\frac{\kappa - f}{r^2}\right)^k = \frac{16\pi M}{(d-2)\Sigma_{d-2}^{(\kappa)} r^{d-1}} - \frac{8\pi Q^2}{(d-2)(d-3)} \frac{1}{r^{2d-4}} \,,$$

$$T_{\text{mod}} = \frac{1}{4\pi r_+}\left[\sum_{k=0}^{K} \kappa\alpha_k (d-2k-1)\left(\frac{\kappa}{r_+^2}\right)^{k-1} - \frac{8\pi Q^2}{(d-2) r_+^{2(d-3)}}\right] \,,$$

$$S = \frac{\Sigma_{d-2}^{(\kappa)} \alpha_1 r_+^{d-2}}{4} \,.$$

$$\Delta = \sum_{k=1}^{K} k\alpha_k \left(\kappa / r_+^2\right)^{k-1}$$

$$M = \frac{\Sigma_{d-2}^{(\kappa)}(d-2)}{16\pi} \sum_{k=0}^{K} \alpha_k \kappa^k r_+^{d-1-2k} + \frac{\Sigma_{d-2}^{(\kappa)}}{2(d-3)} \frac{Q^2}{r_+^{d-3}} \,,$$

$$T_0 = \frac{|f'(r_+)|}{4\pi} = \frac{T_{\text{mod}}}{\Delta} \,, \qquad \Phi = \frac{\Sigma_{d-2}^{(\kappa)} Q}{(d-3) r_+^{d-3}} \,,$$

$$S_0 = S + \frac{\Sigma_{d-2}^{(\kappa)}(d-2)}{4} \sum_{k=2}^{K} \frac{k\kappa^{k-1}\alpha_k r_+^{d-2k}}{d-2k} \,, \qquad 0.17)$$

# 3) 4D GB gravity: equations of motion

$$
\begin{aligned}
\mathcal{E}_\phi = & -\mathcal{G} + 8G^{ab}\nabla_b\nabla_a\phi + 8R^{ab}\nabla_a\phi\nabla_b\phi - 8(\Box\phi)^2 + 8(\nabla\phi)^2\Box\phi + 16\nabla^a\phi\nabla^b\phi\nabla_b\nabla_a\phi \\
& + 8\nabla_b\nabla_a\phi\nabla^b\nabla^a\phi - 24\lambda^2 e^{-4\phi} - 4\lambda R e^{-2\phi} + 24\lambda e^{-2\phi}\left[(\nabla\phi)^2 - \Box\phi\right] \\
= & \, 0\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}_{ab} = & \Lambda g_{ab} + G_{ab} + \alpha\Big[\phi H_{ab} - 2R\left[(\nabla_a\phi)(\nabla_b\phi) + \nabla_b\nabla_a\phi\right] + 8R^c_{(a}\nabla_{b)}\nabla_c\phi + 8R^c_{(a}(\nabla_{b)}\phi)(\nabla_c\phi) \\
& - 2G_{ab}\left[(\nabla\phi)^2 + 2\Box\phi\right] - 4\left[(\nabla_a\phi)(\nabla_b\phi) + \nabla_b\nabla_a\phi\right]\Box\phi - \left[g_{ab}(\nabla\phi)^2 - 4(\nabla_a\phi)(\nabla_b\phi)\right](\nabla\phi)^2 \\
& + 8(\nabla_{(a}\phi)(\nabla_{b)}\nabla_c\phi)\nabla^c\phi - 4g_{ab}R^{cd}\left[\nabla_c\nabla_d\phi + (\nabla_c\phi)(\nabla_d\phi)\right] + 2g_{ab}(\Box\phi)^2 - 2g_{ab}(\nabla_c\nabla_d\phi)(\nabla^c\nabla^d\phi) \\
& - 4g_{ab}(\nabla^c\phi)(\nabla^d\phi)(\nabla_c\nabla_d\phi) + 4(\nabla_c\nabla_b\phi)(\nabla^c\nabla_a\phi) + 4R_{acbd}\left[(\nabla^c\phi)(\nabla^d\phi) + \nabla^d\nabla^c\phi\right] \\
& + 3\lambda^2 e^{-4\phi}g_{ab} - 2\lambda e^{-2\phi}\left(G_{ab} + 2(\nabla_a\phi)(\nabla_b\phi) + 2\nabla_b\nabla_a\phi - 2g_{ab}\Box\phi + g_{ab}(\nabla\phi)^2\right)\Big] \\
= & \, 0\,,
\end{aligned}
$$

Interesting consequence:

$$
0 = g^{ab}\mathcal{E}_{ab} + \frac{\alpha}{2}\mathcal{E}_\phi = 4\Lambda - R - \frac{\alpha}{2}\mathcal{G}
$$

# Some properties of the theory

- **Is the theory well posed?** May be – probably not?
  - A.D. Kovacs, H.S. Reall, *Well-posed formulation of Lovelock and Horndeski theories*, PRD 101, 124003 (2020).

- **Asymptotic structure:** no propagating scalar dof
  - H. Lu, P. Mao, *Asymptotic structure of Einstein-Gauss-Bonnet theory in lower dimensions*, ArXiv:2004.14400.

- **Observational constraints**
  - T. Clifton, P. Carrilho, P.G.S. Fernandes, D.J. Muryne, *Observational constraints on the regularized 4D Einstein-Gauss-Bonnet theory of gravity*, ArXiv:2006.15017.
  - J-X. Feng, B-M Gu, F-W. Shu, *Theoretical and observational constraints on regularized 4D Einstein-Gauss-Bonner gravity*, ArXiv:2006.16751.

# 4) Kaluza-Klein approach

H. Lu and Y. Pang, *Horndeski Gravity as D->4 Limit of Gauss-Bonnet*, 2003.11552. (see also T. Kobayashi, 2003.12771)

- Start with EHGB

$$S_D = \int d^D x \sqrt{-g}\big(R - 2\Lambda + \hat{\alpha}\mathcal{G}\big)$$

- **Compactify** on

$$ds_D^2 = ds_p^2 + e^{2\phi} d\Sigma_{D-p,\,\lambda}^2$$

$$R_{abcd} = \lambda(g_{ac}g_{bd} - g_{ad}g_{bc})$$

- Resultant **effective p-dimensional** action is

$$\begin{aligned} S_p = \frac{1}{16\pi G_p}\int d^p x\sqrt{-g}e^{(D-p)\phi}\Big\{ & R - 2\Lambda_0 + (D-p)(D-p-1)\big((\partial\phi)^2 + \lambda e^{-2\phi}\big) \\ & + \alpha\Big(\mathrm{GB} - 2(D-p)(D-p-1)\left[2G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \lambda R e^{-2\phi}\right] \\ & - (D-p)(D-p-1)(D-p-2)\left[2(\partial\phi)^2\Box\phi + (D-p-1)((\partial\phi)^2)^2\right] \\ & + (D-p)(D-p-1)(D-p-2)(D-p-3)\left[2\lambda(\partial\phi)^2 e^{-2\phi} + \lambda^2 e^{-4\phi}\right]\Big)\Big\}, \end{aligned}$$

# Kaluza-Klein approach

- In $p \leq 4$ one can substract topological (zero) term

$$-\frac{\alpha}{16\pi G_p}\int d^p x\sqrt{-g}\,\mathrm{GB}$$

- Rescale the coupling alpha and take the limit:

$$\alpha \to \frac{\alpha}{D-p} \qquad D \to p$$

(Limit of **0-dim.** internal space)

## Gauss-Bonnet in $p \leq 4$

$$S = \int d^p x\sqrt{-g}\Big[R - 2\Lambda + \alpha\Big(\phi\mathcal{G} + 4G^{ab}\partial_a\phi\partial_b\phi - 4(\partial\phi)^2\Box\phi + 2((\nabla\phi)^2)^2\Big)\Big],$$

$$S_\lambda = \int d^p x\sqrt{-g}\Big(-2\lambda R e^{-2\phi} - 12\lambda(\partial\phi)^2 e^{-2\phi} - 6\lambda^2 e^{-4\phi}\Big)$$