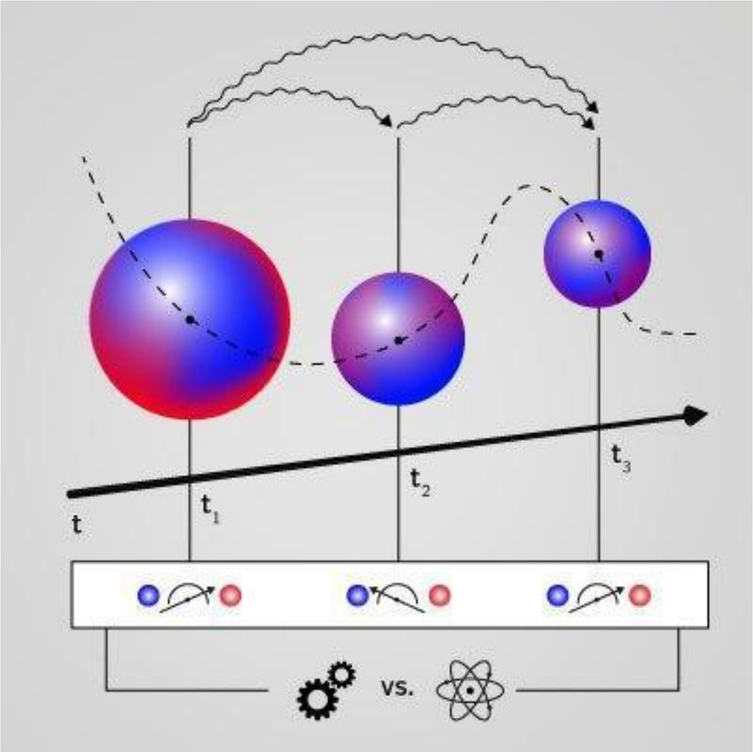
Quantumness and memory effects in multi-time measurements



UNIVERSITÀ DEGLI STUDI DI MILANO

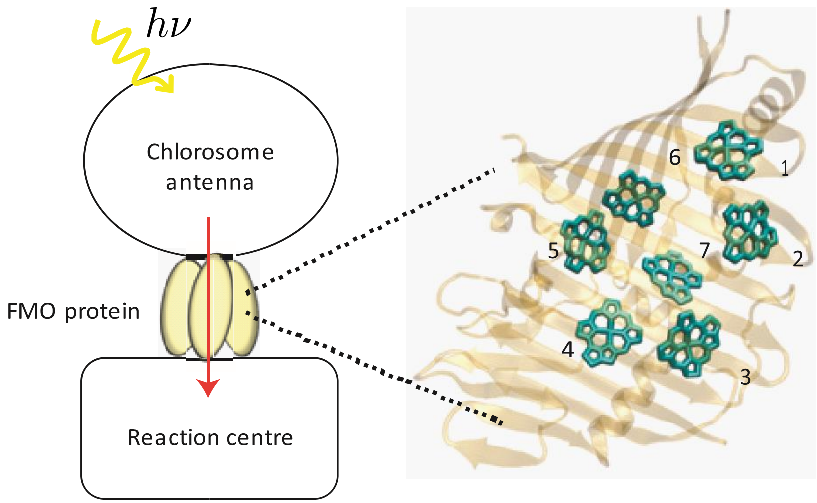
Andrea Smirne

INFN MILANO
Istituto Nazionale di Fisica Nucleare
Sezione di Milano

Naples, 23-27 June 2025

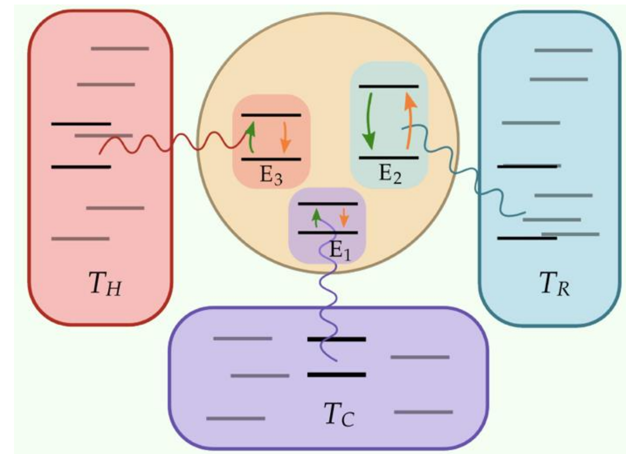
Motivation: which phenomena are intrinsically quantum?

News from Berkley Lab, May 2010
image by Mohan Sarovar and Akihito Ishizaki



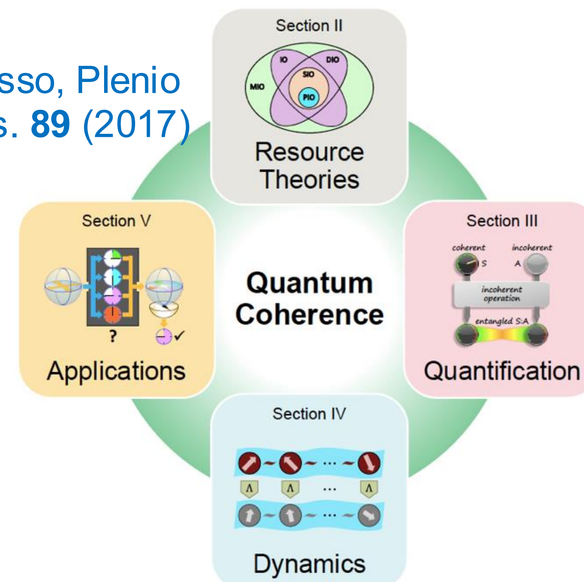
Quantum biology

Goold, Huber, Riera, del Rio & Skrzypczyk, JPA **49** (2016)



Quantum thermodynamics

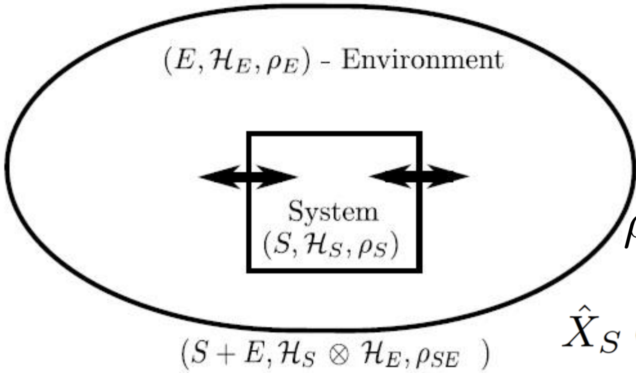
Streltsov, Adesso, Plenio
Rev.Mod.Phys. **89** (2017)



Resource theory of coherence

How can we **irrevocably exclude** the existence of **any classical description**? What is the role played by quantum coherence and by quantum correlations?

Multi-time probabilities in open quantum systems



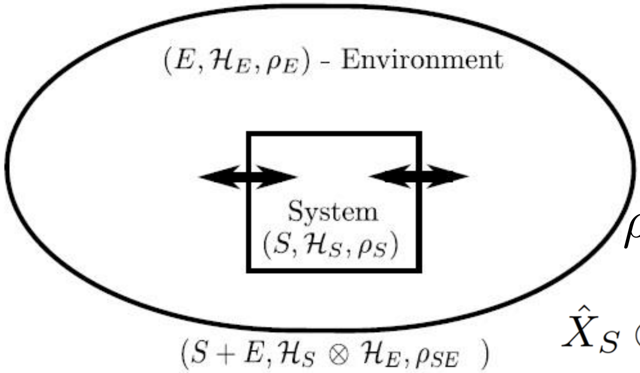
System: degrees of freedom we are interested in

Environment: usually very complex, it is averaged out

$$\rho_S(t) = \mathrm{tr}_E \left\{ U(t)[\rho_S(0) \otimes \rho_E(0)]U^\dagger(t) \right\} = \Lambda_S(t)[\rho_S(t_0)]$$

$\hat{X}_S \otimes \mathbb{1}$ with eigenprojectors $\widehat{\Pi}_x$ $P_1(x,t) = \mathrm{tr}_S \left\{ \widehat{\Pi}_x \Lambda_S(t)[\rho_S(0)] \right\}$

Multi-time probabilities in open quantum systems



System: degrees of freedom we are interested in

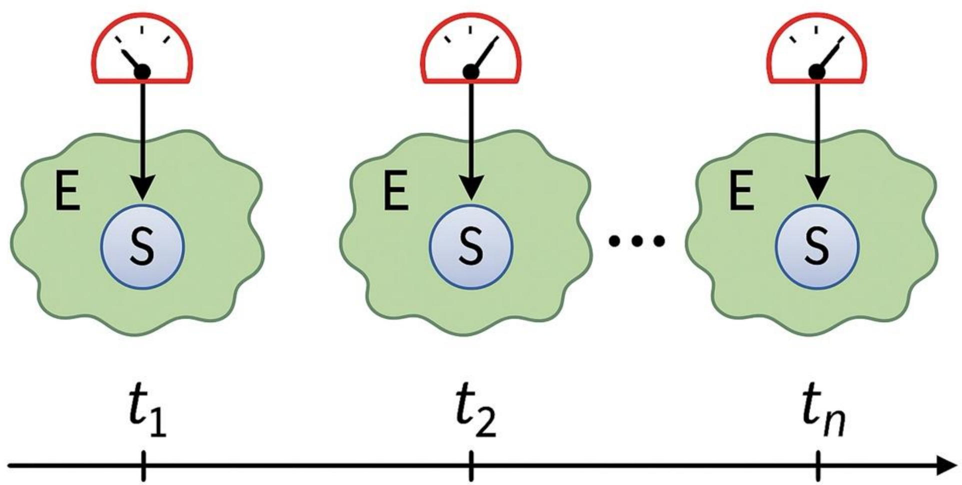
Environment: usually very complex, it is averaged out

$$\rho_S(t) = \mathrm{tr}_E \left\{ U(t)[\rho_S(0) \otimes \rho_E(0)]U^\dagger(t) \right\} = \Lambda_S(t)[\rho_S(t_0)]$$

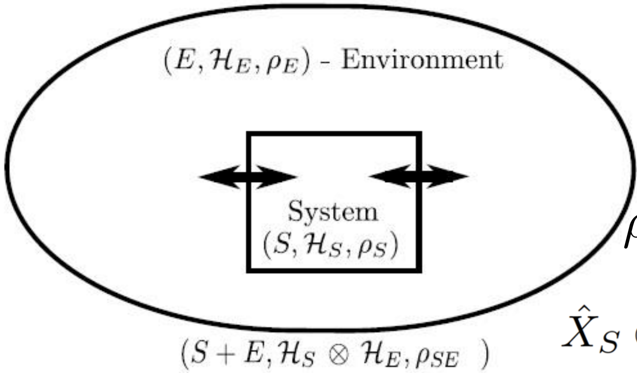
$\hat{X}_S \otimes \mathbb{1}$ with eigenprojectors $\widehat{\Pi}_x$ $P_1(x,t) = \mathrm{tr}_S \left\{ \widehat{\Pi}_x \Lambda_S(t)[\rho_S(0)] \right\}$

What about higher-order probabilities?

$$P_n\left(x_n, t_n; \ldots; x_1, t_1\right)$$



Multi-time probabilities in open quantum systems

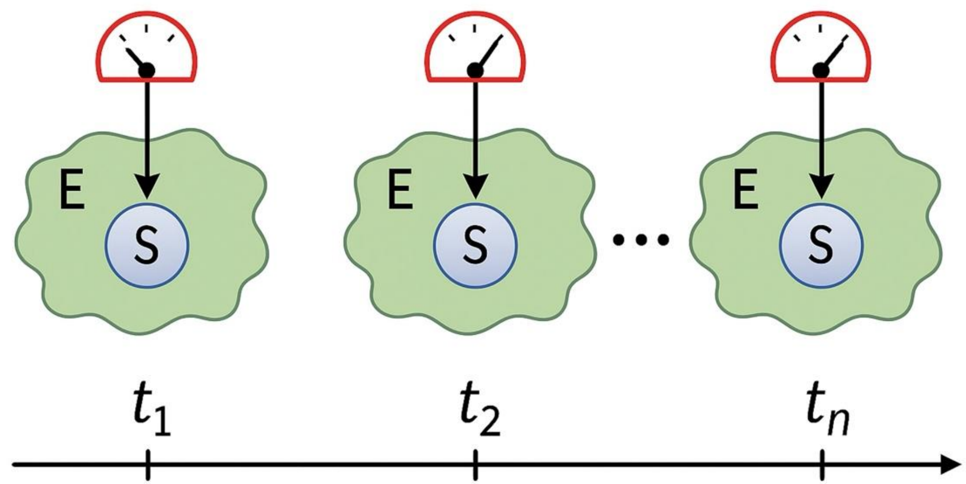


System: degrees of freedom we are interested in

Environment: usually very complex, it is averaged out

$$\rho_S(t) = \mathrm{tr}_E \left\{ U(t)[\rho_S(0) \otimes \rho_E(0)] U^\dagger(t) \right\} = \Lambda_S(t)[\rho_S(t_0)]$$

$\hat{X}_S \otimes \mathbb{1}$ with eigenprojectors $\widehat{\Pi}_x$ $P_1(x,t) = \mathrm{tr}_S \left\{ \widehat{\Pi}_x \Lambda_S(t)[\rho_S(0)] \right\}$



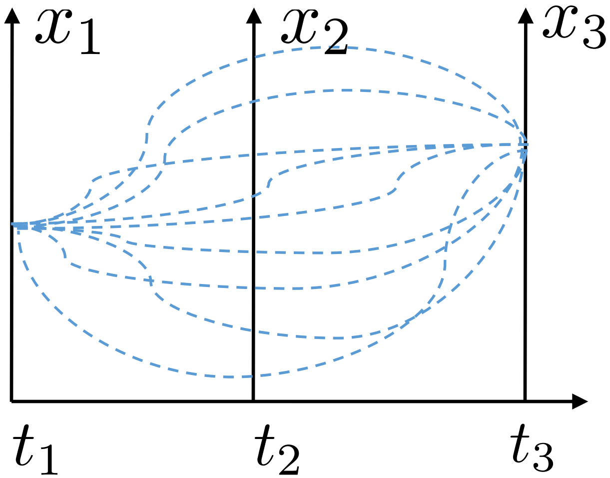
What about higher-order probabilities?

$$P_n\left(x_n, t_n; \ldots; x_1, t_1\right)$$

$$\mathrm{tr}_{SE}\left\{ \widehat{\Pi}_{x_n} U_{SE}(t_n, t_{n-1}) \ldots \widehat{\Pi}_{x_1} U_{SE}(t_1, t_0) \rho_{SE}(t_0) U^\dagger_{SE}(t_1, t_0) \widehat{\Pi}_{x_1} \ldots U^\dagger_{SE}(t_n, t_{n-1}) \widehat{\Pi}_{x_n} \right\}$$

Even though they are referred to open-system quantities, in general <u>one has to deal with the full unitary dynamics</u>

Classicality of the statistics

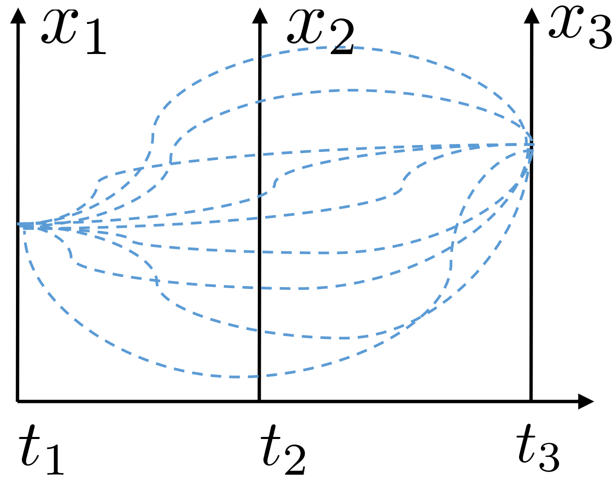


Summing over all the possible intermediate values we obtain the joint probability referred to the initial and final values

$$\sum_{x_2} P_3\left\{x_3, t_3; x_2, t_2; x_1, t_1\right\} = P_2\left\{x_3, t_3; x_1, t_1\right\}$$

Kolmogorov consistency conditions

Classicality of the statistics



Summing over all the possible intermediate values we obtain the joint probability referred to the initial and final values

$$\sum_{x_2} P_3\left\{x_3, t_3; x_2, t_2; x_1, t_1\right\} = P_2\left\{x_3, t_3; x_1, t_1\right\}$$

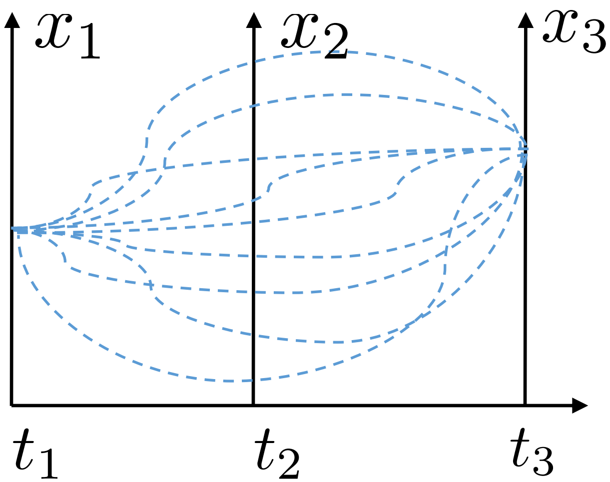
Kolmogorov consistency conditions

$$\boxed{\begin{array}{l} \sum_{x_k} P_n(x_n, t_n; \ldots; x_{k+1}, t_{k+1}; x_k, t_k; x_{k-1}, t_{k-1}; \ldots x_1, t_1) = P_{n-1}(x_n, t_n; \ldots; x_{k+1}, t_{k+1}; \cancel{x_k}, \cancel{t_k}; x_{k-1}, t_{k-1}; \ldots x_1, \\ \qquad \forall k \leq n \in \mathbb{N},\ n > 1; \ \ \forall t_n \geq \ldots \geq t_1 \in \mathbb{R}^+; \ \ \forall x_1, \ldots x_n \end{array}}$$

Kolmogorov extension theorem $\Rightarrow$

There exists a <u>classical stochastic process</u> whose joint probability distributions coincide with P_n for all *n*

Classicality of the statistics



Summing over all the possible intermediate values we obtain the joint probability referred to the initial and final values

$$\sum_{x_2} P_3\{x_3, t_3; x_2, t_2; x_1, t_1\} = P_2\{x_3, t_3; x_1, t_1\}$$

Kolmogorov consistency conditions

$$\sum_{x_k} P_n(x_n, t_n; \ldots; x_{k+1}, t_{k+1}; x_k, t_k; x_{k-1}, t_{k-1}; \ldots x_1, t_1) = P_{n-1}(x_n, t_n; \ldots; x_{k+1}, t_{k+1}; \cancel{x_k, t_k}; x_{k-1}, t_{k-1}; \ldots x_1,$$

$$\forall k \leq n \in \mathbb{N},\ n > 1; \quad \forall t_n \geq \ldots \geq t_1 \in \mathbb{R}^+; \quad \forall x_1, \ldots x_n$$

Kolmogorov extension theorem

There exists a <u>classical stochastic process</u> whose joint probability distributions coincide with P_n for all *n*

- Classicality understood as the possibility to perform **<u>non-invasive measurements</u>**
- Given <u>a collection of multi-time probabilities as input</u>, it allows us to state unambiguously whether there is an alternative classical way to account for them
- In the quantum realm it generally does not hold $\rho \neq \sum_x \widehat{\Pi}_x \rho \widehat{\Pi}_x$: role of coherences!

Quantum Regression Theorem

M. Lax, Phys. Rev. **172**, 350 (1968)
S. Swain, J. Phys. A **14**, 2577 (1981)

- **Only under proper conditions** -- S-E correlations are *irrelevant* -- QRT

$$P_{QRT}\left(x_n, t_n; \ldots; x_1, t_1\right) =$$
$$\mathrm{tr}_S \left\{\widehat{\Pi}_n \Lambda_S(t_n, t_{n-1}) \left[\ldots \Lambda_S(t_2, t_1) \left[\widehat{\Pi}_1 \Lambda_S(t_1, t_0)[\rho_S(t_0)]\widehat{\Pi}_1\right] \ldots\right] \widehat{\Pi}_n\right\}$$

The whole hierarchy of probabilities is fixed by the reduced dynamical maps!!

Propagators of the open-system dynamics

Quantum Regression Theorem

M. Lax, Phys. Rev. **172**, 350 (1968)
S. Swain, J. Phys. A **14**, 2577 (1981)

- **Only under proper conditions** -- S-E correlations are *irrelevant* -- QRT

$$\boxed{\begin{aligned} &P_{QRT}\left(x_n, t_n; \ldots; x_1, t_1\right) = \\ &\mathrm{tr}_S\left\{\widehat{\Pi}_n \Lambda_S(t_n, t_{n-1})\left[\ldots \Lambda_S(t_2, t_1)\left[\widehat{\Pi}_1 \Lambda_S(t_1, t_0)[\rho_S(t_0)]\widehat{\Pi}_1\right]\ldots\right]\widehat{\Pi}_n\right\}\end{aligned}}$$

The whole hierarchy of probabilities is fixed by the reduced dynamical maps!!

Propagators of the open-system dynamics

- **Markovian statistics** $P\left(x_n, t_n | x_{n-1}, t_{n-1}; \ldots; x_1, t_1\right) = P\left(x_n, t_n | x_{n-1}, t_{n-1}\right)$

marginals

$$P_n\left(x_n, t_n; \ldots; x_1, t_1\right)$$

$$\vdots$$

$$P_2\left(x_2, t_2; x_1, t_1\right)$$

$$P_1\left(x_1, t_1\right)$$

<u>In general, this is not possible</u>

Quantum Regression Theorem

M. Lax, Phys. Rev. **172**, 350 (1968)
S. Swain, J. Phys. A **14**, 2577 (1981)

- **Only under proper conditions** -- S-E correlations are *irrelevant* -- QRT

$$P_{QRT}\left(x_n, t_n; \ldots; x_1, t_1\right) =$$
$$\mathrm{tr}_S\left\{\widehat{\Pi}_n \Lambda_S(t_n, t_{n-1})\left[\ldots \Lambda_S(t_2, t_1)\left[\widehat{\Pi}_1 \Lambda_S(t_1, t_0)[\rho_S(t_0)]\widehat{\Pi}_1\right]\ldots\right]\widehat{\Pi}_n\right\}$$

The whole hierarchy of probabilities is fixed by the reduced dynamical maps!!

Propagators of the open-system dynamics

- **Markovian statistics** $P\left(x_n, t_n | x_{n-1}, t_{n-1}; \ldots; x_1, t_1\right) = P\left(x_n, t_n | x_{n-1}, t_{n-1}\right)$

marginals

$$P_n\left(x_n, t_n; \ldots; x_1, t_1\right)$$
$$\vdots$$
$$P_2\left(x_2, t_2; x_1, t_1\right)$$
$$P_1\left(x_1, t_1\right)$$

$$P_{QRT}\left(x_n, t_n; \ldots; x_1, t_1\right) = \left(\prod_{k=1}^{n-1} P\left(x_{k+1}, t_{k+1} | x_k, t_k\right)\right) P_1\left(x_1, t_1\right)$$

<u>Markovianity</u>

Dynamics of quantum coherences

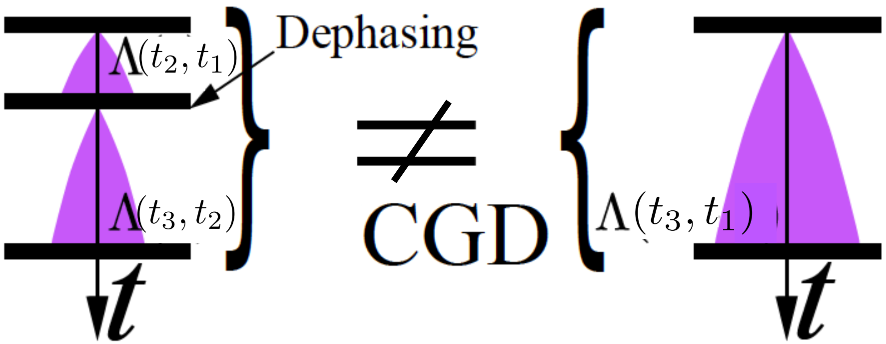
- Definition: *coherence-generating-and-detecting (CGD)* dynamics

The dynamics with propagators $\{\Lambda_S(t,s)\}_{t\geq s\geq 0}$ is CGD iff there are $t_3 \geq t_2 \geq t_1$ s.t.

$$\Delta \circ \Lambda_S(t_3,t_2) \circ \Delta \circ \Lambda_S(t_2,t_1) \circ \Delta \neq \Delta \circ \Lambda_S(t_3,t_1) \circ \Delta$$

Total dephasing $\Delta = \sum_x \mathcal{P}_x$

Dephasing at an intermediate time changes the state transformation with dephasing at the initial and final times



Dynamics of quantum coherences

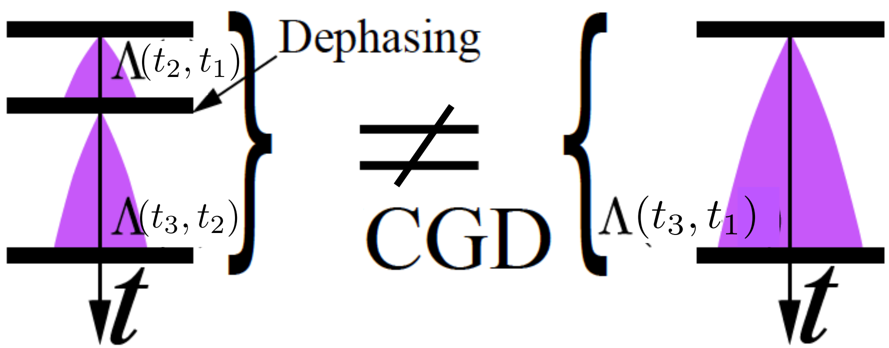
- Definition: *coherence-generating-and-detecting (CGD)* dynamics

The dynamics with propagators $\{\Lambda_S(t,s)\}_{t\geq s\geq 0}$ is CGD iff there are $t_3 \geq t_2 \geq t_1$ s.t.

$$\Delta \circ \Lambda_S(t_3,t_2) \circ \Delta \circ \Lambda_S(t_2,t_1) \circ \Delta \neq \Delta \circ \Lambda_S(t_3,t_1) \circ \Delta$$

Total dephasing $\Delta = \sum_x \mathcal{P}_x$

Dephasing at an intermediate time changes the state transformation with dephasing at the initial and final times



- Coherences are generated by the dynamics and <u>the same coherences</u> are turned into populations
- Direct connection with resource theory of coherence

One-to-one correspondence

Let $\hat{X} = \sum_x x \left|\psi_x\right\rangle\!\left\langle\psi_x\right|$ be a system's non-degenerate observable and $P_n\left(x_n, t_n; \ldots; x_1, t_1\right)$ a Markovian statistics, then the statistics is classical for any initial diagonal state $\rho(0) = \sum_{x_0} p_{x_0} \left|\psi_{x_0}\right\rangle\!\left\langle\psi_{x_0}\right|$ if and only if the dynamics $\{\Lambda_S(t,s)\}_{t \geq s \geq 0}$ is not CGD

AS, D.Egloff, MG Diaz, MB Plenio, SF Huelga Quantum Sci. Technol. **4**, 01LT01 (2019)

One-to-one correspondence

Let $\hat{X} = \sum_x x\,|\psi_x\rangle\langle\psi_x|$ be a system's non-degenerate observable and $P_n\left(x_n, t_n; \ldots; x_1, t_1\right)$ a Markovian statistics, then the statistics is classical for any initial diagonal state $\rho(0) = \sum_{x_0} p_{x_0} |\psi_{x_0}\rangle\langle\psi_{x_0}|$ if and only if the dynamics $\{\Lambda_S(t,s)\}_{t\geq s\geq 0}$ is not CGD

AS, D.Egloff, MG Diaz, MB Plenio, SF Huelga Quantum Sci. Technol. **4**, 01LT01 (2019)

- KEY POINT: we want to connect a property of the dynamics, (N)CGD, with a property of the whole hierarchy of probabilities, (N)CI

$P_n\left(x_n, t_n; \ldots; x_1, t_1\right)$

$\vdots$

$P_2\left(x_2, t_2; x_1, t_1\right)$

$P_1\left(x_1, t_1\right)$

marginals

Classicality concerns the whole hierarchy

Dynamical quantities "live" here

One-to-one correspondence

Let $\hat{X} = \sum_x x\,|\psi_x\rangle\langle\psi_x|$ be a system's non-degenerate observable and $P_n\left(x_n, t_n; \ldots; x_1, t_1\right)$ a <u>Markovian</u> statistics, then the statistics is <u>classical</u> for any initial diagonal state $\rho(0) = \sum_{x_0} p_{x_0}\,|\psi_{x_0}\rangle\langle\psi_{x_0}|$ <u>if and only if</u> the dynamics $\{\Lambda_S(t,s)\}_{t\geq s\geq 0}$ is <u>not CGD</u>

AS, D.Egloff, MG Diaz, MB Plenio, SF Huelga Quantum Sci. Technol. **4**, 01LT01 (2019)

- KEY POINT: we want to connect a property of the dynamics, (N)CGD, with a property of the whole hierarchy of probabilities, (N)CI

$P_n\left(x_n, t_n; \ldots; x_1, t_1\right)$

Classicality concerns the whole hierarchy

$\vdots$

$P_2\left(x_2, t_2; x_1, t_1\right)$

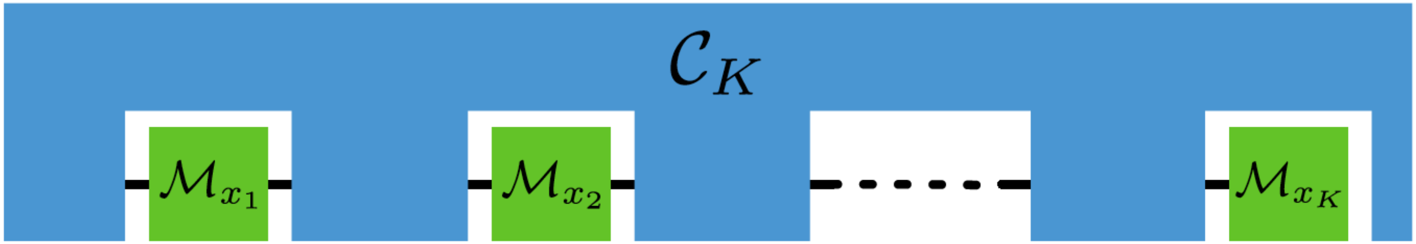
<u>Markovianity is what allows us to do that</u>!

$P_1\left(x_1, t_1\right)$

Dynamical quantities "live" here

marginals

Quantum combs

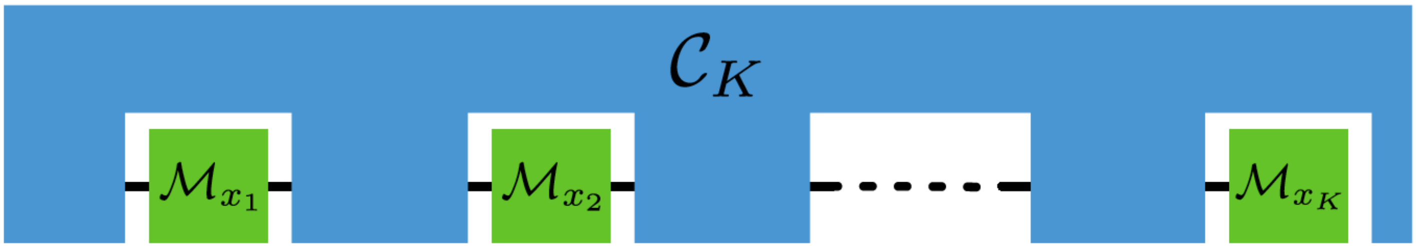


$$\mathbb{P}_K(x_K, t_k; \ldots; x_1, t_1) = \mathcal{C}_K[\mathcal{M}_{x_K}, \ldots, \mathcal{M}_{x_1}]$$

Chiribella, D'Ariano, Perinotti, PRA **80**, 022339 (2009)

- Multi-linear functional on the set of quantum operations

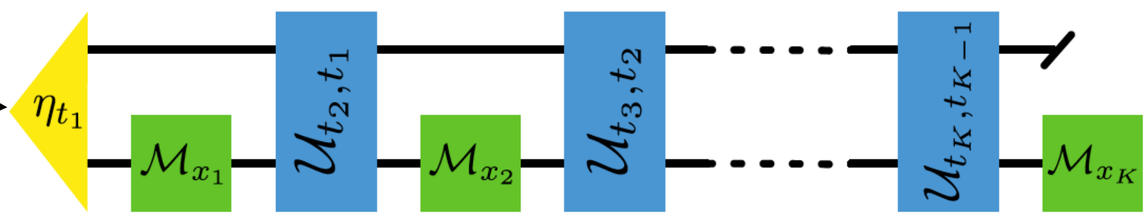
Quantum combs



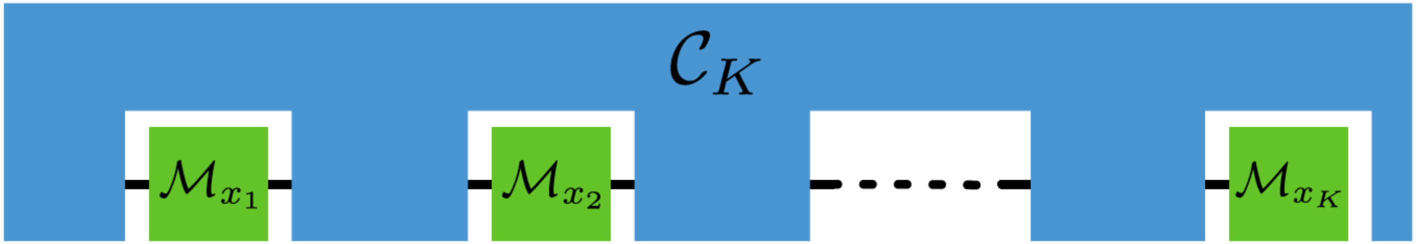
$$\mathbb{P}_K(x_K, t_k; \ldots; x_1, t_1) = \mathcal{C}_K[\mathcal{M}_{x_K}, \ldots, \mathcal{M}_{x_1}]$$

Chiribella, D'Ariano, Perinotti, PRA **80**, 022339 (2009)

- Multi-linear functional on the set of quantum operations
- It is always possible to represent it via a unitary dilation



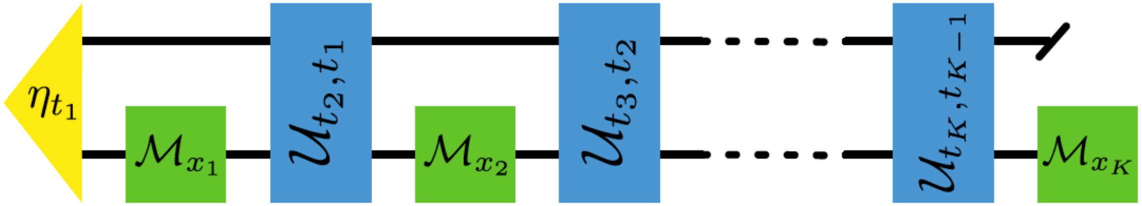
Quantum combs



$$\mathbb{P}_K(x_K, t_k; \ldots; x_1, t_1) = \mathcal{C}_K[\mathcal{M}_{x_K}, \ldots, \mathcal{M}_{x_1}]$$

Chiribella, D'Ariano, Perinotti, PRA **80**, 022339 (2009)

- Multi-linear functional on the set of quantum operations
- It is always possible to represent it via a unitary dilation

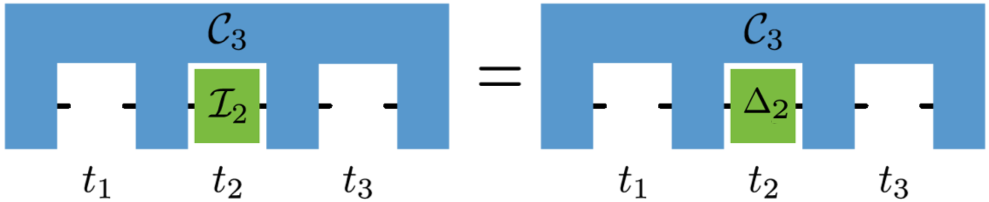


- It can be tomographically reconstructed via measurements on the open system *only*
- Strictly related to non-Markovian quantum stochastic processes by Lindblad

Commun. Math. Phys. 65, 281 (1979)

Discord and multi-time nonclassicality

- Full characterization of combs yielding a classical statistics: <u>Kolmogorov iff</u>

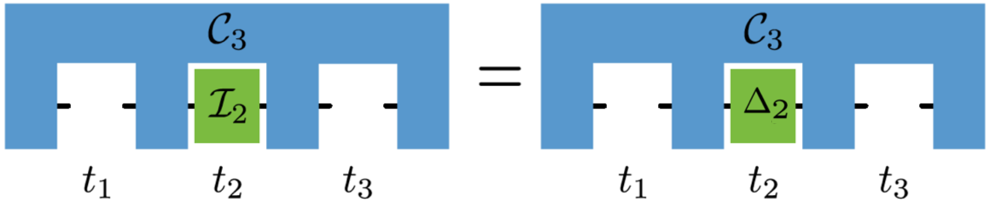


$$\mathcal{C}_K\left[\bigotimes_{t_j\in\mathcal{T}'}\mathcal{I}_j,\ \bigotimes_{t_k\in\mathcal{T}\setminus\mathcal{T}'}\mathcal{P}_{x_k}\right]=\mathcal{C}_K\left[\bigotimes_{t_j\in\mathcal{T}'}\Delta_j,\ \bigotimes_{t_k\in\mathcal{T}\setminus\mathcal{T}'}\mathcal{P}_{x_k}\right]$$

S.Milz, D.Egloff, P.Taranto, T.Theurer, M.B.Plenio, AS, S.F.Huelga, Phys. Rev. **X** 10, 041049 (2020)

Discord and multi-time nonclassicality

- Full characterization of combs yielding a classical statistics: Kolmogorov iff



$$\mathcal{C}_K\left[\bigotimes_{t_j\in\mathcal{T}'}\mathcal{I}_j, \bigotimes_{t_k\in\mathcal{T}\setminus\mathcal{T}'}\mathcal{P}_{x_k}\right] = \mathcal{C}_K\left[\bigotimes_{t_j\in\mathcal{T}'}\Delta_j, \bigotimes_{t_k\in\mathcal{T}\setminus\mathcal{T}'}\mathcal{P}_{x_k}\right]$$

S.Milz, D.Egloff, P.Taranto, T.Theurer, M.B.Plenio, AS, S.F.Huelga, Phys. Rev. **X** 10, 041049 (2020)

- Zero-discord state in the measurement basis

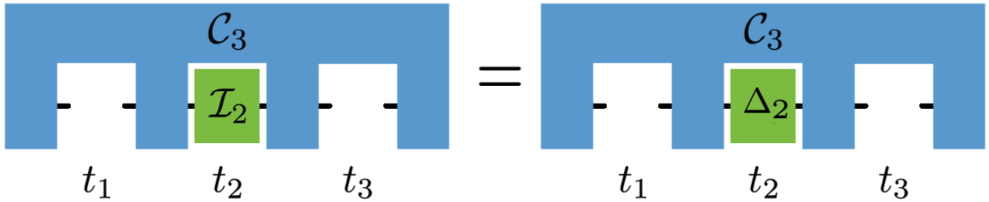
$$\rho_{SE}(t_j) = \sum_x p_x(t_j)\,|x\rangle\langle x| \otimes \xi_x(t_j)$$

Dephasing in the measurement basis is non-invasive

$$\Delta \otimes \mathbb{1}_E\,[\rho_{SE}(t)] = \rho_{SE}(t)$$

Discord and multi-time nonclassicality

- Full characterization of combs yielding a classical statistics: Kolmogorov iff



$$\mathcal{C}_K\left[\bigotimes_{t_j\in\mathcal{T}'}\mathcal{I}_j, \bigotimes_{t_k\in\mathcal{T}\setminus\mathcal{T}'}\mathcal{P}_{x_k}\right] = \mathcal{C}_K\left[\bigotimes_{t_j\in\mathcal{T}'}\Delta_j, \bigotimes_{t_k\in\mathcal{T}\setminus\mathcal{T}'}\mathcal{P}_{x_k}\right]$$

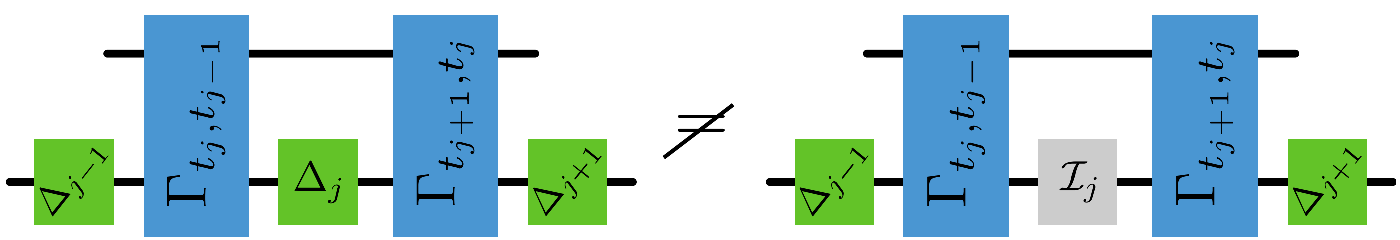
S.Milz, D.Egloff, P.Taranto, T.Theurer, M.B.Plenio, AS, S.F.Huelga, Phys. Rev. **X** 10, 041049 (2020)

- Zero-discord state in the measurement basis

$$\rho_{SE}(t_j) = \sum_x p_x(t_j)\,|x\rangle\langle x| \otimes \xi_x(t_j)$$

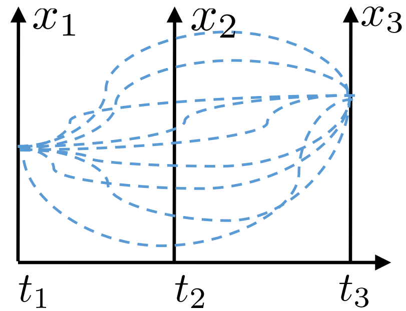
Dephasing in the measurement basis is non-invasive $\qquad \Delta \otimes \mathbb{1}_E\,[\rho_{SE}(t)] = \rho_{SE}(t)$

- Non-classicality $\longleftrightarrow$ the **global S-E dynamics generates** quantum discord that can be **later detected** via measurements on the open system



Summary

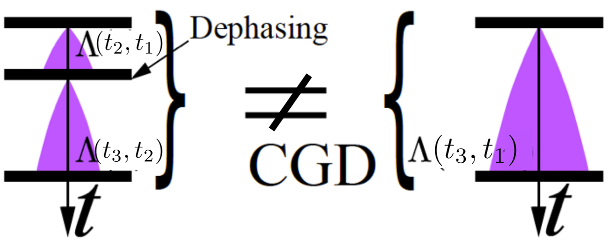
- Markovian multi-time statistics: one-to-one correspondence



Non-classicality of the multi-time statistics, in the sense of Kolmogorov

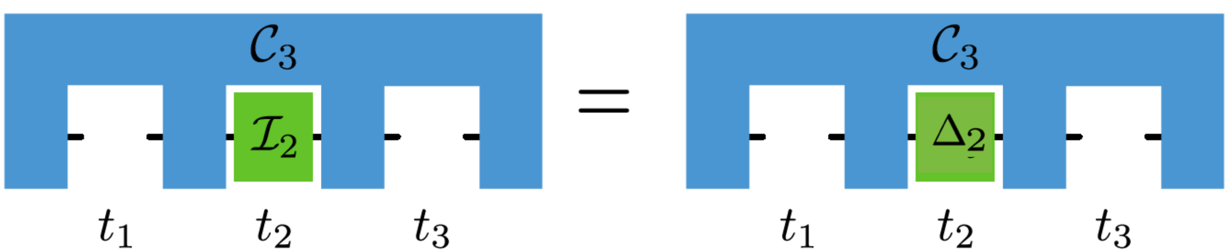
↕

Capability of the dynamics to generate and detect quantum coherence



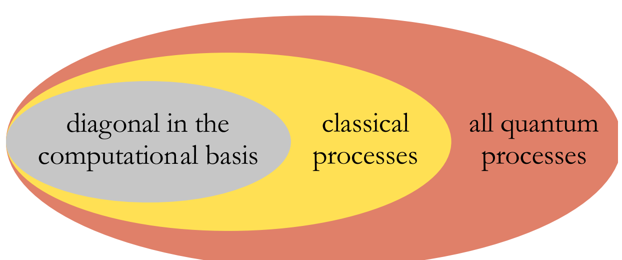
$$\Delta \circ \Lambda_S(t_3, t_2) \circ \Delta \circ \Lambda_S(t_2, t_1) \circ \Delta \neq \Delta \circ \Lambda_S(t_3, t_1) \circ \Delta$$

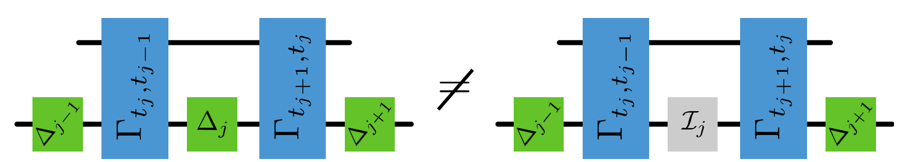
- Classical non-Markovian statistics and quantum combs



Role of discord

Structural characterization





Acknowledgments

EEQU: Efficienza Energetica con risorse QUantistiche

QuReCo: Quantum Reservoir Computing

IOP Publishing Quantum Sci. Technol. 4 (2019) 01LT01 https://doi.org/10.1088/2058-9565/aaebd5

Quantum Science and Technology

RECEIVED
31 July 2018

ACCEPTED FOR PUBLICATION
26 October 2018

PUBLISHED
21 November 2018

LETTER

Coherence and non-classicality of quantum Markov processes

A Smirne[1], D Egloff[1], M G Díaz[2], M B Plenio[1] and S F Huelga[1]

[1] Institute of Theoretical Physics, Universität Ulm, Albert-Einstein-Allee 11, D-89069 Ulm, Germany
[2] Departament de Física, Grup d'Informació Quàntica, Universitat Autònoma de Barcelona, ES-08193 Bellaterra (Barcelona), Spain

E-mail: andrea.smirne@uni-ulm.de

Keywords: quantum coherence, non-classicality, quantum Markovianity, multi-time statistics

PHYSICAL REVIEW X **10,** 041049 (2020)

When Is a Non-Markovian Quantum Process Classical?

Simon Milz,[1,2,*] Dario Egloff,[3,4,†] Philip Taranto,[2] Thomas Theurer,[3] Martin B. Plenio,[3] Andrea Smirne,[3,5,6] and Susana F. Huelga[3,‡]

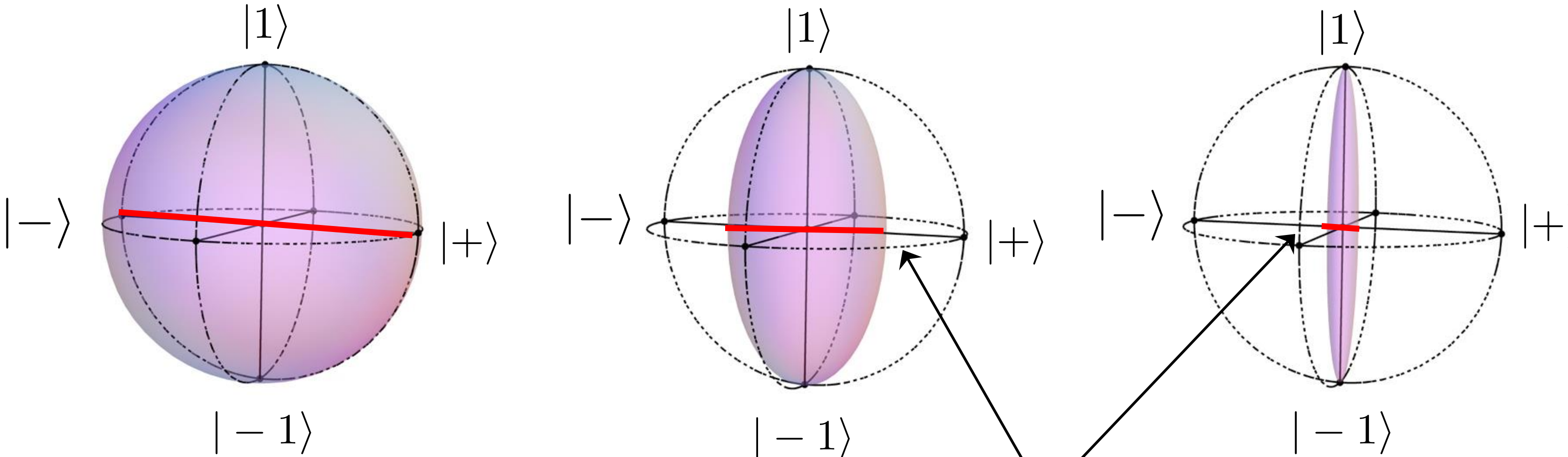
[1]*School of Physics and Astronomy, Monash University, Clayton, Victoria 3800, Australia*
[2]*Institute for Quantum Optics and Quantum Information, Austrian Academy of Sciences, Boltzmanngasse 3, 1090 Vienna, Austria*
[3]*Institute of Theoretical Physics and IQST, Universität Ulm, Albert-Einstein-Allee 11, D-89069 Ulm, Germany*
[4]*Institute of Theoretical Physics, Technical University Dresden, D-01062 Dresden, Germany*
[5]*Dipartimento di Fisica "Aldo Pontremoli," Università degli Studi di Milano, via Celoria 16, I-20133 Milan, Italy*
[6]*Istituto Nazionale di Fisica Nucleare, Sezione di Milano, via Celoria 16, I-20133 Milan, Italy*

Non-classicality without quantum coherence ❗

- Beyond Markov, the hypothesis of our Theorem does not apply, in fact…
- The statistics is non-classical: 2-time Kolmogorov does not hold

$$\rho(0) = |+\rangle\langle+| \qquad \boxed{\sum_y P_2^{\hat{\sigma}_x}(+,t;y,s) \neq P_1^{\hat{\sigma}_x}(+,t)}$$

- The dynamics is NCGD (with respect to $\hat{\sigma}_x$)



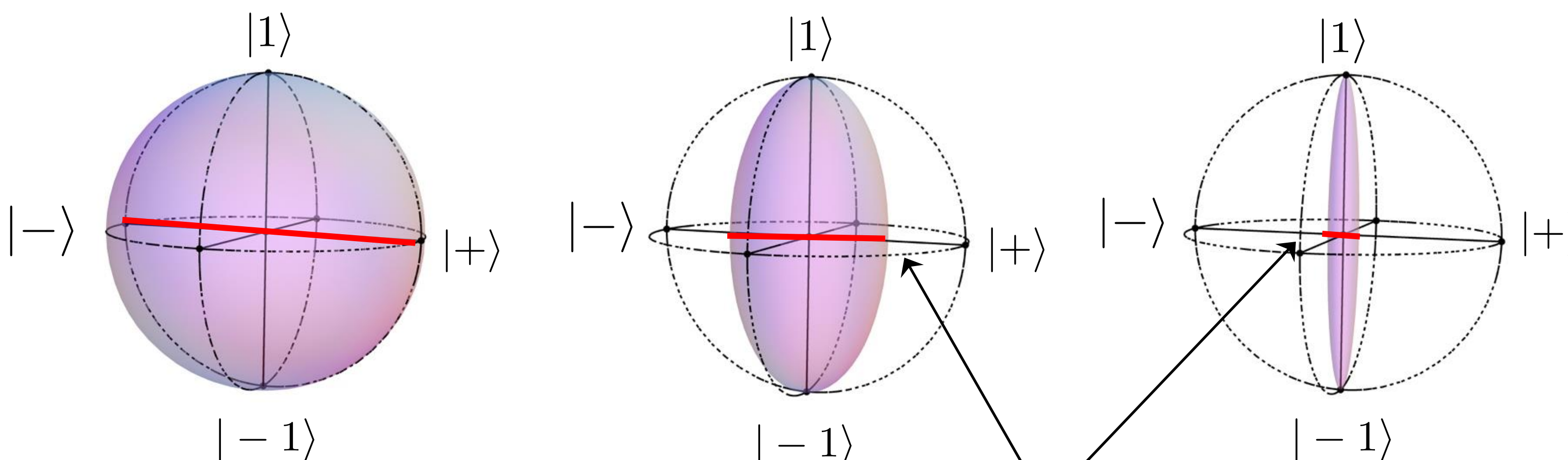
Quantum coherence is not even generated!!

Non-classicality without quantum coherence ❗

- Beyond Markov, the hypothesis of our <u>Theorem does not apply</u>, in fact…
- The statistics is <u>non-classical</u>: 2-time Kolmogorov does not hold

$$\rho(0) = |+\rangle\langle+| \qquad \boxed{\sum_y P_2^{\hat{\sigma}_x}(+,t;y,s) \neq P_1^{\hat{\sigma}_x}(+,t)}$$

- The dynamics is NCGD (with respect to $\hat{\sigma}_x$)

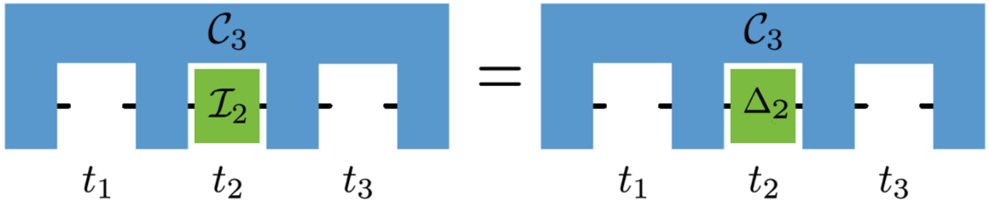


Quantum coherence is not even generated!!

The open-system dynamics no longer accounts for multi-time statistics: a more general perspective is needed!

Non-Markovian classical combs

- Full characterization of combs yielding a classical statistics: <u>Kolmogorov iff</u>

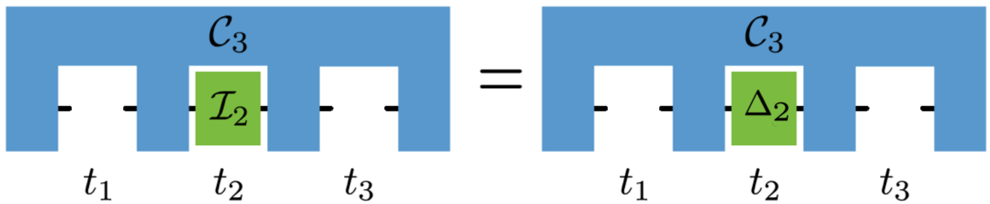


$$\mathcal{C}_K\left[\bigotimes_{t_j\in\mathcal{T}'}\mathcal{I}_j,\ \bigotimes_{t_k\in\mathcal{T}\setminus\mathcal{T}'}\mathcal{P}_{x_k}\right]=\mathcal{C}_K\left[\bigotimes_{t_j\in\mathcal{T}'}\Delta_j,\ \bigotimes_{t_k\in\mathcal{T}\setminus\mathcal{T}'}\mathcal{P}_{x_k}\right]$$

S.Milz, D.Egloff, P.Taranto, T.Theurer, M.B.Plenio, AS, S.F.Huelga, Phys. Rev. **X** 10, 041049 (2020)

Non-Markovian classical combs

- Full characterization of combs yielding a classical statistics: Kolmogorov iff



$$\mathcal{C}_K\left[\bigotimes_{t_j\in\mathcal{T}'} \mathcal{I}_j, \bigotimes_{t_k\in\mathcal{T}\setminus\mathcal{T}'} \mathcal{P}_{x_k}\right] = \mathcal{C}_K\left[\bigotimes_{t_j\in\mathcal{T}'} \Delta_j, \bigotimes_{t_k\in\mathcal{T}\setminus\mathcal{T}'} \mathcal{P}_{x_k}\right]$$

S.Milz, D.Egloff, P.Taranto, T.Theurer, M.B.Plenio, AS, S.F.Huelga, Phys. Rev. **X** 10, 041049 (2020)

- Choi-Jamiołkowski isomorphism for quantum combs

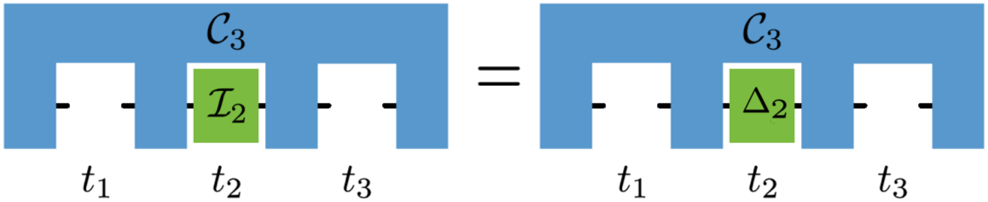
$$\mathcal{C}_K[\mathcal{M}_{x_K}, \dots, \mathcal{M}_{x_1}] = \mathrm{tr}\left[\left(M_{x_K}^{\mathrm{T}} \otimes \cdots \otimes M_{x_1}^{T}\right) C_K\right]$$

Choi state of the CP map $\mathcal{M}_{x_K}$

”Multi-time” Choi state

Non-Markovian classical combs

- Full characterization of combs yielding a classical statistics: <u>Kolmogorov iff</u>



$$\mathcal{C}_K\left[\bigotimes_{t_j\in\mathcal{T}'} \mathcal{I}_j, \bigotimes_{t_k\in\mathcal{T}\setminus\mathcal{T}'} \mathcal{P}_{x_k}\right] = \mathcal{C}_K\left[\bigotimes_{t_j\in\mathcal{T}'} \Delta_j, \bigotimes_{t_k\in\mathcal{T}\setminus\mathcal{T}'} \mathcal{P}_{x_k}\right]$$

S.Milz, D.Egloff, P.Taranto, T.Theurer, M.B.Plenio, AS, S.F.Huelga, Phys. Rev. **X** 10, 041049 (2020)

- Choi-Jamiołkowski isomorphism for quantum combs

$$\mathcal{C}_K[\mathcal{M}_{x_K},\ldots,\mathcal{M}_{x_1}] = \mathrm{tr}\left[(M_{x_K}^{\mathrm{T}} \otimes \cdots \otimes M_{x_1}^{T})\, C_K\right]$$

Choi state of the CP map $\mathcal{M}_{x_K}$

"Multi-time" Choi state

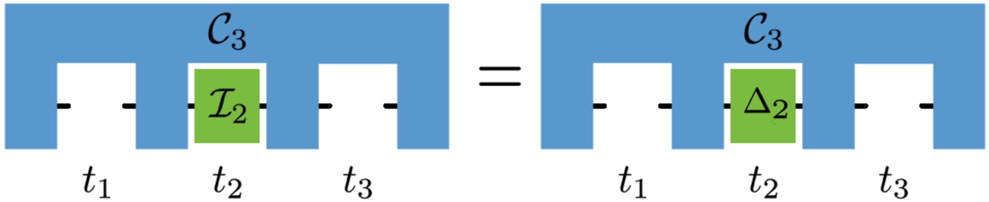
- Structural property of classical combs

Any comb: $C_K = \widetilde{C}_K^{\mathrm{Cl.}} + \chi$

$$\widetilde{C}_K^{\mathrm{Cl.}} = \sum_{x_K,\ldots,x_1} \mathbb{P}_K(x_K,\ldots,x_1) P_{x_K} \otimes \cdots \otimes P_{x_1}$$

Non-Markovian classical combs

- Full characterization of combs yielding a classical statistics: <u>Kolmogorov iff</u>



$$\mathcal{C}_K\left[\bigotimes_{t_j\in\mathcal{T}'}\mathcal{I}_j,\ \bigotimes_{t_k\in\mathcal{T}\setminus\mathcal{T}'}\mathcal{P}_{x_k}\right]=\mathcal{C}_K\left[\bigotimes_{t_j\in\mathcal{T}'}\Delta_j,\ \bigotimes_{t_k\in\mathcal{T}\setminus\mathcal{T}'}\mathcal{P}_{x_k}\right]$$

S.Milz, D.Egloff, P.Taranto, T.Theurer, M.B.Plenio, AS, S.F.Huelga, Phys. Rev. **X** 10, 041049 (2020)

- Choi-Jamiołkowski isomorphism for quantum combs

$$\mathcal{C}_K[\mathcal{M}_{x_K},\ldots,\mathcal{M}_{x_1}]=\mathrm{tr}\left[(M_{x_K}^{\mathrm{T}}\otimes\cdots\otimes M_{x_1}^{T})\,C_K\right]$$

Choi state of the CP map $\mathcal{M}_{x_K}$

"Multi-time" Choi state

- Structural property of classical combs

Any comb: $C_K=\widetilde{C}_K^{\mathrm{Cl.}}+\chi$

$$\widetilde{C}_K^{\mathrm{Cl.}}=\sum_{x_K,\ldots,x_1}\mathbb{P}_K(x_K,\ldots,x_1)P_{x_K}\otimes\cdots\otimes P_{x_1}$$

<u>Classical combs</u>: $\mathrm{tr}\left[\left(\bigotimes_{t_j\in\mathcal{T}'}(\mathbb{1}_j-\Delta_j)\bigotimes_{t_k\in\mathcal{T}\setminus\mathcal{T}'}P_{x_k}\right)\chi\right]=0$