Modular Theory, Quantization and Non-commutative Space-time

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This talk is inspired by a long-term ongoing collaboration with:

- Roberto Conti (Sapienza Università di Roma)
- Matti Raasakka (Aalto University - Helsinki)

and some related previous and currently developing work with:

- Wicharn Lewkeeratiyutkul (Chulalongkorn University)
- Natee Pitiwan (Satit Chula Bangkok)
- Chatchai Puttirungroj (Sukhothai Thammatirat University)
- Paratat Bejrakarbum (Thammasat University)
- Thitikorn Suwannapeng (TAIST, Kasetsart University)

Abstract

- A link between quantization and (non-commutative) geometry is uncovered via Tomita-Takesaki modular theory for complexified ortho-symplectic spaces.
- We suggest (following arXiv:1007.4094v1) that, in a covariant quantum theory, non-commutative space-time can be a-posteriori recovered from relativistic states over algebras of partial observables.
- Speculative implications for cosmology will be mentioned.

- **Strategy for Modular Algebraic Quantum Gravity**

($\star$ denotes slides with original material)

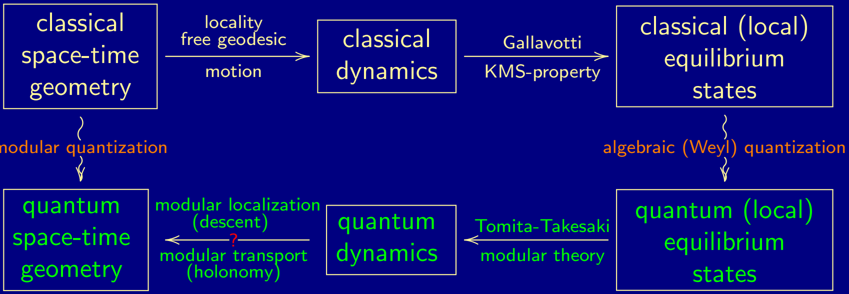
Is there "geometry" at the quantum level?

Modular Algebraic Quantum Theory (arXiv:1007.4094) ⋆

We aim to revert the arrows

geometry ⇒ local equilibrium states

and obtain **geometry from states**: **nc-geometry ⇐ states**



- **Conjecture**: even when starting with a "classical geometry", via modular theory, a new "quantum geometry" is produced.

(Some) Fundamental Physical Principles in MAQT ⋆

- **Modular localization**: quantum subsystems are determined by states via a KMS-equilibrium condition
- **Jacobi-Einstein generalized equivalence principle**: all interactions are subsumed in a background geometry and dynamics is free subject only to the geometric constraints
- **Geometric origin of modular theory**: Tomita-Takesaki modular theory originates in complexification of ortho-symplectic spaces and encodes free dynamics contraint on phase-spaces
- **Modular CPT symmetry**: quantum systems are always in CPT-symmetric thermal double pairs
- **Modular Born geometry**: that is induced by CPT symmetry J, a Born reciprocity Φ and a modular $\mathbb{R}$-polarization Γ

- **Motivations from AQFT**

Algebraic Quantum Field Theory (Araki, Haag, Kastler)[1]

(Dimock, Kay, Wald, Hollands, Brunetti, Fredenhagen, Verch)

- there is a covariant functor from isometric inclusions of open Lorentzian manifolds to unital homomorphisms of C*-algebras:

$$\Big(\mathcal{O}_1 \to \mathcal{O}_2\Big) \mapsto \Big(\mathcal{A}(\mathcal{O}_1) \to \mathcal{A}(\mathcal{O}_2)\Big),$$

- algebras of causally independent regions commute:

$$\mathcal{O}_1 \bowtie \mathcal{O}_2 \subset \mathcal{O} \Rightarrow [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)]_- = \{0_{\mathcal{O}}\},$$

- algebras of regions sharing causal closures are isomorphic:

$$\mathcal{A}(\mathcal{O}) \simeq \mathcal{A}(\mathcal{O}''),$$

[1] Haag R (1996) **Local Quantum Physics** Springer

(Scalar) QFT on Curved Space-time

(Wald,[3] Bär, Ginoux, Pfäffle)

globally hyperbolic Lorentzian submanifold $(\mathcal{O}, g)$ ⟼ symplectic $\mathbb{R}$-space $(\Gamma_{\mathcal{O}}, \omega)$ of solutions of KG-equation $\Box_g \phi = m\phi$ ⟼ complexified Weyl algebra $\mathbb{W}\mathrm{y}(\Gamma_{\mathcal{O}}, \omega)$

- the assignment $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) := \mathbb{W}\mathrm{y}(\Gamma_{\mathcal{O}}, \omega)$ is an AQFT [2]
- the complexification $\Gamma_{\mathcal{O}}^{\mathbb{C}}$ of $(\Gamma_{\mathcal{O}}, \omega)$ is a Kreĭn space
- a choice of Kähler polarization (equivalently a quasi-free state on $\mathbb{W}\mathrm{y}(\Gamma_{\mathcal{O}}, \omega)$) selects a one-particle Hilbert space $\mathcal{H}^{+}$.

[2] Actually $\mathcal{A}(\mathcal{O})$ is the C*-algebra of "exponentiated Weyl CCR".

[3] Wald R (1994) **Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics** Chicago University Press.

Rereh-Schlieder Theorem

- whenever the open region $\mathcal{O} \neq \varnothing$ has a non-trivial causal complement $\mathcal{O}' \neq \varnothing$, the Bosonic Fock space $\mathcal{H} := \mathcal{F}_-(\mathcal{H}^+)$ of $\mathcal{H}^+$ is a standard Hilbert space for the von Neumann algebra $\mathcal{A}(\mathcal{O})''$ generated by $\mathcal{A}(\mathcal{O})$, with vacuum $\xi_o \in \mathcal{H}$ that is cyclic for both $\mathcal{A}(\mathcal{O})$ and its commutant $\mathcal{A}(\mathcal{O})'$.

The vacuum vector $\xi_o \in \mathcal{H}$ is cyclic and separating for every local algebra $\mathcal{A}(\mathcal{O})$ of a region $\mathcal{O} \subset \mathbb{M}$ such that $\mathcal{O}' \neq \varnothing \neq \mathcal{O}$.

Tomita-Takesaki modular theory (next section) will provide each $\mathcal{A}(\mathcal{O})''$ with a one-parameter modular group of automorphisms ... a mysterious intrinsic dynamics whose physical/geometrical significance, since the '70s, is still escaping understanding!

Heretic conjecture: the modular dynamics of quantum fields is a "Coriolis' effect" of the non-commutative geometry of space-time.

- **Tomita-Takesaki Modular Theory**

  (the missing ingredient)

Modular Theory - von Neumann Algebras (Tomita-Takesaki)

For a von Neumann algebra $\mathcal{M} = \mathcal{M}'' \subset \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, for every vector state $\phi(T) := \langle \xi \mid T(\xi) \rangle$, with $\xi \in \mathcal{H}$ that is:

$$\textit{cyclic}: \quad \overline{(\mathcal{M}\xi)} = \mathcal{H}, \qquad \textit{separating}: \quad A\xi = 0 \Rightarrow A = 0,$$

the $\mathbb{R}$-subspace $\{T(\xi) \mid T = T^*, \; T \in \mathcal{M}\} \subset \mathcal{H}$ is standard and

- its modular one-parameter unitary group

$$t \mapsto \Delta_\xi^{it} = \exp(itK_\xi) \in \mathcal{B}(\mathcal{H}),$$

- its modular conjugation antilinear isometry $\mathcal{H} \xrightarrow{J_\xi} \mathcal{H}$, satisfy:

$$\boxed{\sigma_t^{\phi}(\mathcal{M}) := \Delta_\xi^{it} \mathcal{M} \Delta_\xi^{-it} = \mathcal{M}, \quad \forall t \in \mathbb{R}, \qquad J_\xi \mathcal{M} J_\xi = \mathcal{M}',}$$

where the **commutant** $\mathcal{M}'$ of $\mathcal{M}$ is defined by:

$$\mathcal{M}' := \{A' \in \mathcal{B}(\mathcal{H}) \mid [A', A]_- = 0, \forall A \in \mathcal{M}\}.$$

Notice that $\mathcal{M}'$ is a von Neumann algebra and $\mathcal{M}'' = \mathcal{M}$.

Modular Theory - First Quantized Level

(originated in M.Rieffel - see for example R.Longo lecture notes)

Any complex Hilbert space $\mathcal{H}_{\mathbb{C}}$ with a **standard real subspace** $V_{\mathbb{R}}$:

$$\mathcal{H}_{\mathbb{C}} = \overline{V_{\mathbb{R}} \oplus i \cdot V_{\mathbb{R}}}, \qquad V_{\mathbb{R}} \cap i \cdot V_{\mathbb{R}} = \{0_{\mathcal{H}}\},$$

canonically determines the following data:

- a densely defined Tomita conjugation
  $S_V : v \oplus (i \cdot w) \mapsto v \oplus (-i \cdot w), \quad x, y \in V_{\mathbb{R}}$
  with polar Hilbert decomposition $\overline{S}_V = J_V \circ \Delta_V^{\frac{1}{2}}$,
- a unitary one-parameter group $t \mapsto \Delta_V^{it} = \exp(itK_V)$ on $\mathcal{H}_{\mathbb{C}}$, leaving $V_{\mathbb{R}}$ stable: $\Delta_V^{it}(V_{\mathbb{R}}) = V_{\mathbb{R}}$, for all $t \in \mathbb{R}$,
- an anti-unitary operator J_V mapping $V_{\mathbb{R}}$ on its symplectic complement $V' := \{h \in \mathcal{H}_{\mathbb{C}} \mid \mathrm{Im}\langle h \mid v\rangle_{\mathcal{H}} = 0, v \in V_{\mathbb{R}}\}$.

complexifications of orthosymplectic spaces	$\leftrightarrow$	standard $\mathbb{R}$-subspaces of $\mathbb{C}$-Hilbert spaces

Modular Theory - Second Quantized Level

Second Quantization Functor

symplectic $\mathbb{R}$-spaces	$\mapsto$	Weyl $\mathbb{R}$-algebras of CCR

Complexification Functor

modular theory of a standard $\mathbb{R}$-space	$\mapsto$	modular theory induced by the Fock vacuum on the von Neumann algebra generated by second quantized Weyl algebra

$$K_V \mapsto \bigoplus_{n=0}^{+\infty} \sum_{k=1}^{n} \left(\overbrace{I_V \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} I_V}^{k-1} \otimes_{\mathbb{C}} K_V \otimes_{\mathbb{C}} \overbrace{I_V \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} I_V}^{n-k} \right),$$

$$J_V \mapsto \bigoplus_{n=0}^{+\infty} \left(\underbrace{J_V \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} J_V}_{n} \right).$$

Kubo-Martin-Schwinger KMS-Condition

- A **C*[W*]-dynamical system** $(\mathcal{A}, \alpha)$ is a one paramenter group of $*$-authomorphisms $\mathbb{R} \xrightarrow{\alpha} \mathrm{Aut}(\mathcal{A})$ of a unital C*[W*]-algebra $\mathcal{A}$ that is strongly[weakly] continuous.
- A **state** $\mathcal{A} \xrightarrow{\phi} \mathbb{C}$ is a $\mathbb{C}$-linear map that is normalized: $\phi(1_{\mathcal{A}}) = 1_{\mathbb{C}}$ and positive: $\phi(x^* x) \geq 0_{\mathbb{C}}$, for all $x \in \mathcal{A}$.
- A **KMS-state, at inverse temperature** β, is a state over a C*[W*]-dynamical system $(\mathcal{A}, \alpha)$ such that there exists a "C*[W*]-dense" subalgebra $\mathcal{A}_\phi \subset \mathcal{A}$ such that:
  - $\mathcal{A}_\phi$ is α-invariant: $\alpha_t(\mathcal{A}_\phi) \subset \mathcal{A}_\phi$, for all $t \in \mathbb{R}$,
  - $z \mapsto \alpha_z(x)$ is $\mathbb{C}$-analytic, for all $x \in \mathcal{A}_\phi$,
  - $\omega(x \cdot \alpha_{-i\beta}(y)) = \omega(y \cdot x)$, for all $x, y \in \mathcal{A}_\phi$.

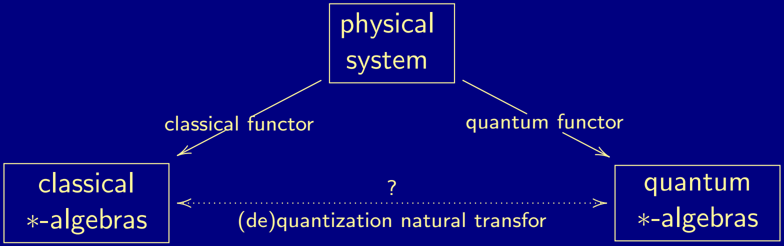
The modular authomorphism group σ^ϕ of ϕ is the unique such that ϕ satisfies the KMS-condition at temperature $\beta = 1$.

- **Modular Quantization**

Classical / Quantum Theories 1.

Let (X, g_X) be a **classical spacetime** and let $M := T^*X$ the associated **classical extended phase-space** (M, g_M, ω).

- **classical** observables on X $\leftrightarrow$ elements of $\mathcal{C}(M)$ a **commutative** $*$-algebra
- **quantum** observables on X $\leftrightarrow$ elements of $\mathcal{Q}(M)$ a **non-commutative** $*$-algebra

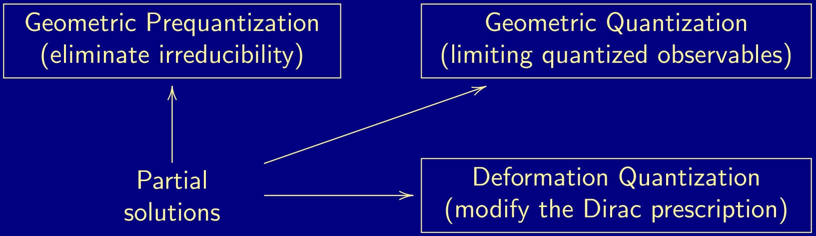


quantum $\sim$ non-commutativization $+$ complexification $+$?

Classical / Quantum Theories 2.

- $\mathcal{C}(M)$ and $\mathcal{Q}(M)$ have common superselected subalgebras!
- there is a dequantization "convergence" $\mathcal{Q}(M) \xrightarrow{\hbar \to 0} \mathcal{C}(M)$
- a Dirac quantization transformation $\mathcal{C}(M) \xrightarrow{\mathfrak{q}} \mathcal{Q}(M)$ such that:
  - $i\{f, g\} \mapsto [\mathfrak{q}(f), \mathfrak{q}(g)]_-$ (Dirac prescription)
  - $1_{\mathcal{C}(M)} \mapsto 1_{\mathcal{Q}(M)}$
  - $\mathfrak{q}$ is $\mathbb{C}$-linear and $\mathfrak{q}(\overline{f}) = \mathfrak{q}(f)^*$
  - $\mathfrak{q}(\mathcal{C}(M))$ acts irreducibly on a complex Hilbert space

does not exist (Groenewold - van Howe)

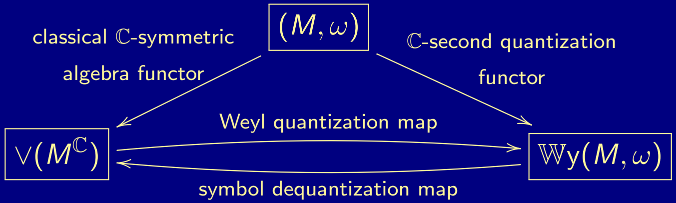


Free Theories - Algebraic Second Quantization

- A physical system is **globally free** (from interaction) if its phase space (M, ω) is a linear symplectic space
- globally free systems have a $\mathbb{C}$-**second quantization functor**:

$$(M, \omega) \mapsto \mathbb{W}\mathrm{y}(M, \omega) \sim \frac{\bigotimes M^{\mathbb{C}}}{< [x, y]_- - 2i\omega(x, y) >}$$

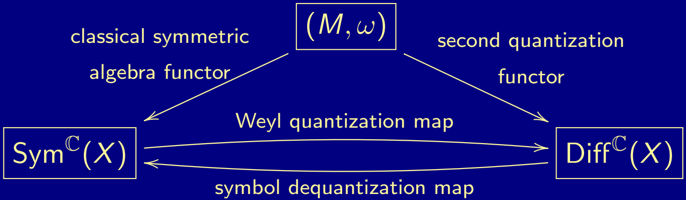
where the $\mathbb{C}$-**Weyl-Heisenberg algebra** is the universal complex associative algebra under the linear inclusion of M



Non-linear Second Quantization on Cotangent Spaces

(Weyl, Wigner, Segal, Kohn, Nirenberg, Vinogradov)

We can generalize to cotangent symplectic manifolds $M := T^*X$



- $\mathrm{Diff}^{\mathbb{C}}(X)$ is the algebra of differential operators on X that is the universal complex enveloping associative algebra of the Lie-Rinehart algebra $C^\infty(X) \oplus \mathrm{Der}(X)$ with $[V, f]_- := V(f)$.
- $\mathrm{Sym}^{\mathbb{C}}(X)$ is the algebra of symbols of differential operators: the graded algebra associated to the filtered algebra $\mathrm{Diff}^{\mathbb{C}}(X)$.

Real Polarization, Duality, Born Symmetry ⋆

- When $M = T^*X$ we have a global $\mathbb{R}$-**polarization** of M and, given a metric (X, g_X), the Ehresmann Levi-Civita connection on T^*X induces a canonical decomposition
$T_mM \simeq T_{\pi(m)}X \oplus T^*_{\pi(m)}X; \quad R : v \oplus w^* \mapsto v \oplus (-w^*)$.
- The space TM decomposes as direct sum of Riesz-dual Lagrangian spaces $TM \simeq TX \oplus T^*X$ with $\mathbb{R}$-polarization R.
- The symplectic form ω induces (modulo sign) a canonical **Born symmetry** $TM \xrightarrow{B} TM$ (given by $(v, w^*) \mapsto (-w, v^*)$), where $v \mapsto v^*$ denotes the **Riesz duality isomorphism**).
- $\omega(B(x), B(y)) = \omega(x, y) \Rightarrow B \in \mathrm{Sp}(TM, \omega)$ furthermore:
$$B^2 = -\,\mathrm{Id}, \quad B \circ R = -R \circ B,$$
almost split-quaternionic structure, with generators B, R.

$$\begin{array}{ccc}
\mathbb{W}\mathrm{y}(M,\omega) & \xleftrightarrow{\text{canonical isomorphism}} & \mathrm{Diff}^{\mathbb{C}}(X) \\
\Big\downarrow {\scriptstyle \text{Bogolubov Born symmetry}} & & \Big\downarrow {\scriptstyle \text{Fourier transform}} \\
\mathbb{W}\mathrm{y}(M,\omega) & \xleftrightarrow{\text{canonical isomorphism}} & \mathrm{Diff}^{\mathbb{C}}(X')
\end{array}$$

Gauge Modular Flow of a Classical Riemannian Geometry ⋆

(forthcoming work with R.Conti)

Riemannian manifold (X, g_X)	$\Longrightarrow$	modular gauge groups (for $0 < \gamma < 1$) $\sigma^{g,\gamma}$

- A Riemannian classical geometry (X, g_X), for $0 < \gamma < 1$, has canonically associated (first-quantized) gauge modular flows $\sigma^{g,\gamma}$ on the **complexified** $\mathcal{H}_X := (TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C}$ **of its Hitchin-Gualtieri generalized tangent bundle** $TM|_X$:
  - $TM|_X \simeq TX \oplus T^*X$ is an orthosymplectic bundle with simplectic structure $\omega(v_1 \oplus w_1^*, v_2 \oplus w_2^*) := w_1^*(v_2) - w_2^*(v_1)$, inner product $g := g_X \oplus g_{X'}$ with $g_{X'}(w_1^*, w_2^*) := g_X(w_1, w_2)$,
  - $TM|_X$ is a standard $\mathbb{R}$-subbundle of the Hilbert bundle $\mathcal{H}_X$ with inner product $h^g + i\gamma h^\omega$ where h^g and h^ω are the sesquilinear forms induced respectively by g and ω.
- **What is the physical/geometrical meaning of σ^g?**

Tulczyjew Double Bundles (see for example: arXiv:1505.0748)

- Classical dynamics is implicitly given by a constraint on Tulczyjew double bundle $TM := T(T^*X) \simeq TX \oplus T^*X$.
- $TM \to M = T^*X \to X$ restricting the Tulczyjew bundle to X, gives the Hitchin-Gualtieri generalized bundle.
- The flow $\sigma^{g,\gamma}$ is actually on the complexified Tulczyjew bundle $T(X \times X') \simeq T(X) \oplus T(X')$ of $X \times X'$, a **double copy** of X.

$$
\begin{array}{ccc}
T(T^*X) \simeq T(X) \oplus T^*(X) & \xleftrightarrow[\ ]{\text{Tulczyjew } *\text{-isomorphism}} & T^*(X') \oplus T(X') \simeq T(T^*X') \\
\downarrow & & \downarrow \\
(T(X), g_X) & \xleftrightarrow[\ ]{\text{Riesz } *\text{-isomorphism}} & (T(X'), g_{X'}) \\
\downarrow & & \downarrow \\
(X, g_X) & & (X', g_{X'})
\end{array}
$$

Modular Born Geometries and Hypercomplex Actions ⋆

More generally, given an orthosymplectic space [4] (V, ω, g)

- its complexification has a canonical modular flux preserving V:

$$\Delta_V^{it} = \exp(iK_V t), \quad K_V = \Gamma_V \circ \Theta_V, \quad \Theta_V := |K_V|,$$

- $\Phi_V := i \cdot \Gamma_V = B \oplus B$ is a **modular Born symmetry** that coincides with the phase of the polar decomposition of the complexification of the polarizers P, Q of (V, ω, g):

$$\omega(v, w) = g(Pv, w), \quad g(v, w) = \omega(v, Qw),$$
$$P = B \circ |P|, \quad Q = B \circ |Q|, \quad |Q| = |P|^{-1},$$

- the operators $J_V, \Phi_V, T := i\cdot$, generate, on $\mathbb{C} \otimes_{\mathbb{R}} V$, the action of the hypercomplex $\mathbb{R}$-algebra $\mathbb{M}_2(\mathbb{C})$ of the q-bit, a Φ_V-complexification of the split-quaternions of J_V, T.
- On the double $V_{\mathbb{R}} \oplus V'_{\mathbb{R}}$ there is a generalized Born geometry.

[4] We assume the continuity of ω with respect to the norm induced by g.

Quantum $\mathbb{R}$-Polarizations and Non-commutative Spaces ⋆

(extremely speculative conjecture)

- Whenever we start from a classical geometry (X, g_X), we always have a **classical $\mathbb{R}$-polarization** R that assures that the "thermal double" $X \oplus X'$ upon quantization provides Weyl algebras that are isomorphic to $\mathrm{Diff}(X) \otimes \mathrm{Diff}(X')$.
- As we have seen, generalized modular Born geometry exists also in cases where a classical $\mathbb{R}$-polarization is absent.
- We conjecture that **"quantum" $\mathbb{R}$-polarizations** Γ_V will induce non-commutative geometries of "configuration spaces" given (see Paschke Kopf) by thermal doubles $\mathcal{C} \otimes_{\mathbb{R}} \mathcal{C}'$, with $\mathcal{C}' := J_V \mathcal{C} J_V$ that will provide as isomorphism of the thermal Weyl double $\mathbb{W}\mathrm{y}(V) \otimes \mathbb{W}\mathrm{y}(V')$ with an algebra of "non-commutative differential operators"[5] $\overrightarrow{\mathrm{Diff}}(\mathcal{C}) \otimes \overleftarrow{\mathrm{Diff}}(\mathcal{C}')$.

[5]Some steps in such direction will be taken in PB, R.Conti, C.Puttirungroj "Contravariant Non-commutative Geometry" (based on CP PhD thesis).

Classical/Quantum Dynamics: Liouvillians (conjectural)

(Inspired by: Mauro; Gallavotti;[6] Streater;[7] Emch;[8] Ammari Ratsimanetrimanana)

- Classical (free) dynamics is given by the Liouvillian equation on Koopman von Neumann Hilbert space $L^2(X \oplus X^*, \omega^{\wedge n})$ that carries commuting representations of the Weyl algebra (Streater, Emch).
- Quantum (equilibrium) dynamics is given by the modular Liouvillian K_ω on the GNS-Hilbert space of the KMS-state.
- $\boxed{\text{quantum KMS condition}} \xrightarrow{\hbar \to 0} \boxed{\text{Gallavotti KMS condition}}$
- Weyl quantization reproduces Mauro's prescription (at algebraic level) giving $\mathbb{W}\mathrm{y}(X \oplus X^*)'' \otimes_{\mathbb{C}} \mathbb{W}\mathrm{y}(X \oplus X^*)'$.

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[6] Gallavotti G (1976) **Classical KMS Condition and Tomita-Takesaki theory** *International Conference on Dynamical Systems in Mathematical Physics (Rennes, 1975)* 89-94 *Astérisque* 40 Soc. Math. France

[7] Strater R (1966) **Canonical Quantization** *Communications in Mathematical Physics* 2:354-374

[8] Emch, Gérard G (1981) **Prequantization and KMS Structures** *International Journal of Theoretical Physics* 20(12):891-904

Non-commutative Space-time from Modular Gauge Flow ?

(extremely speculative conjectures)

- minimal coupling in gauge theory $p_k \mapsto \hat{p}_k := p_k + c \cdot A_k(x)$ induces non-commutativity of momenta:[9]

$$[\hat{p}_k, \hat{p}_j] = ic \cdot F_{kj} \quad \text{with field strength} \quad F_{kj} := \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}$$

- under Born symmetry the (Abelian) gauge theory also induces non-commutativity of coordinates (e.g. Guendelman Singleton):

$$[\hat{x}_k, \hat{x}_j] = ic' \cdot F'_{kj} \quad \text{with field strength} \quad F'_{kj} := \frac{\partial A'_k}{\partial p_j} - \frac{\partial A'_j}{\partial p_k}$$

conjugate modular fluxes on a spaceoid over the groupoid $X \times X'$

geometric operators on X/X' $\xleftrightarrow{\text{CPT+Born}}$ gauge fluxes on X'/X

[9] Here for Abelian gauge groups.

- **Physical Predictions ?**

GQuEST Interferometry Experiments

Some theoretical developments, broadly inspired by (variations of) AdS/CFT (celestial) holography, claim that:

- the quantum nature of the geometry induced by modular theory might actually be soon testable by the GQuEST collaboration proposed interferometry experiment based on: [10]

K.Zurek, E.Verlinde et al. arXiv:1902.08207 arXiv:1911.02018 arXiv:2208.01059 arXiv:2205.01799 arXiv:2012.05870 arXiv:2305.11224

The purpose of such experiments is to observe the quantum superposition/fluctuation nature of geometry induced by the vacuum state of (conformal) free quantum scalar field theory on Minkowski spacetime via the modular theory of double cones algebras (seen as conformal equivalent to black holes).

- positive results of GQuEST should give support to our proposed modular algebraic quantum gravity.

[10] I became aware (at QG 2023 Nijmegen K.Zurek's talk) of this very intriguing ongoing experimental attempts.

Tomita CPT, Contravariant NCG, Mirror Cosmology ⋆

A quite intriguing aspect of modular quantum gravity is that:

- **CPT symmetry** J_ϕ is a built-in feature of local **reducible** standard GNS-representations of a covariant vacuum state ϕ: they are representations of **thermal doubles** $\mathcal{A} \otimes J_\phi \mathcal{A} J_\phi$
- such chiral symmetry seems also a direct byproduct of non-commutativity of space-time: non-commutative vector fields (derivations) and connections split in right/left variants (A.Boroviec arXiv:q-alg/9710006 van den Bergh arXiv:math/0410528) [11]
- L.Boyle and N.Turok have proposed an alternative semi-classical CPT-symmetric cosmological model: arXiv:1803.08928 arXiv:1803.08930 arXiv:2109.06204
  Modular Algebraic Quantization can provide a quantum-level justification for Boyle-Turok CPT mirror cosmologies.

[11] See also PB, R.Conti, C.Puttirungroj "Contravariant Non-commutative Geometry" (work in progress partially based on C.P. PhD thesis).

This talk is broadly inspired by these previous papers:

- M.Raasakka (2019) **Spacetime Granularity from Finite-dimensionality of Local Observable Algebras** arXiv:1908.09293
- PB, R.Conti, N.Pitiwan (2019) **Discrete Non-commutative Gel'fand-Naĭmark Duality** East West Journal of Mathematics 21(2)
- M.Raasakka (2017) **Local Lorentz Covariance in Finite-dimensional Local Quantum Physics** arXiv:1705.06711
- M.Raasakka (2016) **Spacetime-Free Approach to Quantum Theory and Effective Spacetime Structure** arXiv:1605.03942
- PB (2014) **Categorical Operator Algebraic Foundations of Relational Quantum Theory** arXiv:1412.7256
- PB, R.Conti, W.Lewkeeratiyutkul (2011) **A Horizontal Categorification of Gel'fand Duality** Advances in Mathematics 226(1):584-607
- PB, R.Conti, W.Lewkeeratiyutkul (2010) **Modular Theory, Non-commutative Geometry and Quantum Gravity** arXiv:1007.4094v1
- PB, R.Conti, W.Lewkeeratiutkul (2008) **Non-commutative Geometry, Categories and Quantum Physics** arXiv:0801.2826

... and by several ongoing/forthcoming works:

- **Modular Quantization and Non-commutative Space-time**
  PB, R.Conti
- **Complexification, Quantization and Modular Theory**
  PB, R.Conti, C.Puttirungroj
- **Algebraic Formalism for Covariant Quantum Theory**
  PB, T.Suwannapeng
- **The Modular Gauge of a Riemannian Manifold**
  PB, R.Conti
- **Contravariant Non-commutative Geometry**
  PB, R.Conti, C.Puttirungroj
- **Non-commutative Gel'fand-Naĭmark Duality**
  PB, R.Conti, N.Pitiwan
- **Modular Algebraic Quantum Gravity**
  PB, R.Conti, M.Raasakka

Thank You for Your Kind Attention!

Very special thanks to R.Conti: if some progress has been achieved, it is due mostly to His incredible patience and tolerance ;-)

This file has been realized using the “beamer” LaTeX package of the TeX-live distribution and TeXstudio editor on Ubuntu Linux.

- **Modular Algebraic Quantum Gravity**

Basic Ideology of Modular Algebraic Quantum Gravity ⋆

Modular Algebraic Quantum Gravity is based on these four points:

- *space-time (NC) geometry is induced by (covariant) states*

  covariant states ⇒ *NC-geometry*

- *modular theory takes the role of dynamical constraint*

  phase-space Einstein equation ≃ *Tomita-Takesaki theory*

- *quantum geometry is spectrally reconstructed (spaceoids)*

  nc C-algebras, spectral triples* ←(Gel'fand-Naĭmark duality)→ *nc spaceoids ??*

- *covariance is enforced by categorical principles.*

  covariant transport ≃ *modular holonomy*

Proposed Strategy of MAQG ⋆

Starting from a pair $(\mathscr{A}, \omega)$ algebra/state, define:

1. via **modular localization** a notion of local subsystem

$$(\mathscr{A}, \omega) \longmapsto \mathscr{A}^\omega := \{\mathcal{A} \subset \mathscr{A} \mid \omega|_\mathcal{A} \text{ KMS}\}$$

2. a **spectral modular geometry** on each local subsystem

$$\mathscr{A}^\omega \longmapsto \mathscr{M}^\omega_\mathscr{A} := \{(\mathcal{A}_\omega, \mathcal{H}_\omega, K_\omega, J_\omega) \mid \mathcal{A} \in \mathscr{A}^\omega\}$$

3. a **modular holonomy groupoid**, from geometries in $\mathscr{M}^\omega_\mathscr{A}$,

$$\mathscr{M}^\omega_\mathscr{A} \longmapsto \mathscr{H}^\omega_\mathscr{A}$$

   inducing **modular fluctuations** of modular generators K_ω

4. **a phase-space/space-time non-commutative geometry** from the modular data $(\mathscr{M}^\omega_\mathscr{A}, \mathscr{H}^\omega_\mathscr{A})$.

- 1. Modular Localization

Modular Localization in QG

(B.Schroer has been the first to suggest a modular localization in AQFT)

- **In QG, we propose to reverse the AQFT functor**:

$$(\mathscr{A}, \omega) \rightsquigarrow \mathscr{A}^\omega := \{\mathcal{A} \subset \mathscr{A} \mid \mathcal{A} \text{ is modular for } \omega\}$$

- no a-priori assumed geometry;
- for a given *covariant vacuum state* ω
- over an *algebra of partial observables* $\mathcal{A}$,

  we extract a **modular-local net** $\mathscr{A}^\omega$ consisting of **all** unital subalgebras $\mathcal{A} \subset \mathscr{A}$ that are dynamically modularly stable:

  the covariant vacuum ω is cyclic-separating in the ω-GNS representation of the subalgebra and hence it is an equilibrium KMS-state for its induced modular automorphism group.
- Is this modular-net describing a QFT on a new geometry?

- **2. Modular Spectral Geometries**

Modular Local Spectral Geometries (arXiv:1007.4094v1)

Modular theory associates to every KMS-state ω on a modular local algebra $\mathcal{A}$ a **modular spectral non-commutative geometry**

$$(\mathscr{A}, \omega) \longmapsto (\mathcal{A}_\omega, \mathcal{H}_\omega, \xi_\omega, K_\omega, J_\omega)_{\mathcal{A}}$$

- $\mathcal{H}_\omega$ is the Hilbert space of the GNS representation π_ω induced by $\omega|_{\mathcal{A}}$, with cyclic separating unit vector $\xi_\omega \in \mathcal{H}_\omega$,
- $K_\omega := \log \Delta_\omega$ is the generator of the one-parameter unitary group $t \mapsto \Delta_\omega^{it}$ spatially implementing the modular one-parameter group of $*$-automorphisms $\sigma_t^\omega \in \mathrm{Aut}(\pi_\omega(\mathcal{A})'')$,
- J_ω is the conjugate-linear operator spatially implementing the modular conjugation anti-isomorphism $\gamma_\omega : \pi_\omega(\mathcal{A})'' \to \pi_\omega(\mathcal{A})'$,
- $\mathcal{A}_\omega := \{a \in \mathcal{A} \mid [K_\omega, \pi_\omega(a)] \in \pi_\omega(\mathcal{A})''\}$.

Notice that K_ω is a **first-order operator**: $[[K_\omega, \mathcal{A}_\omega], J_\omega \mathcal{A}_\omega J_\omega] = 0$.

Modular Local Covariant Quantum Theories

(forthcoming work with T.Suwannapeng)

Considering the previous modular-net of spectral geometries,

$$(\mathcal{A}, \omega) \mapsto (\mathcal{A}_\omega, \mathcal{H}_\omega, \xi_\omega, K_\omega, J_\omega), \qquad \mathcal{A} \subset \mathscr{A}.$$

in the language of **covariant quantum theory** we suggest that:

- $\mathcal{A}_\omega$ is a **local algebra of** bounded **partial observables**,
- ξ_ω is a **covariant vacuum**,
- $\mathcal{H}_\omega$ is a **boundary Hilbert space**,
- K_ω is a **covariant Wheeler-DeWitt constraint**,
- J_ω is a **generalized CPT operator**.

Invariant observables: $T \in \pi_\omega(\mathcal{A}_\omega)''$ such that $[K_\omega, T] = 0$.

Covariant observables: $\pi_\omega(\mathcal{A}_\omega)'' \rtimes_{\sigma^\omega} \mathbb{R}$ (crossed product).

- 3. Modular Covariance Groupoid

Modular Covariance Holonomy Groupoid (arXiv:1705.06711)

The pair $(\mathscr{A}, \omega)$ determines a **groupoid $\mathscr{H}^\omega_\mathscr{A}$ of modular covariance morphisms** between modular spectral geometries.

For any inclusion $\mathcal{A}_1 \subset \mathcal{A} \supset \mathcal{A}_2$ with $\mathcal{A}_1, \mathcal{A}, \mathcal{A}_2 \in \mathscr{A}^\omega$, we have an

- inclusion of ω-GNS spaces: $H_{\omega|_{\mathcal{A}_1}} \subset H_{\omega|_{\mathcal{A}}} \supset H_{\omega|_{\mathcal{A}_2}}$

For any $\mathcal{A}_1, \mathcal{A}_2 \in \mathscr{A}^\omega$, we have a (possibly empty) family

- $\mathscr{H}^\omega_\mathscr{A}(\mathcal{A}_1; \mathcal{A}_2)$ of **modular holonomies** from $\mathcal{A}_1$ to $\mathcal{A}_2$: unitary operators $U \in \mathcal{B}(H_{\omega|_{\mathcal{A}}})$, where $\mathcal{A} \in \mathscr{A}^\omega$, with $\mathcal{A}_1 \subset \mathcal{A} \supset \mathcal{A}_2$ and $[K_{\omega|_{\mathcal{A}}}, U] = 0$, $\mathrm{Ad}_U(\mathcal{A}_1) = \mathcal{A}_2$, $U(H_{\omega|_{\mathcal{A}_1}}) = H_{\omega|_{\mathcal{A}_2}}$.

In the special case of local C*-algebras in AQFT, every unitary operator U_g, $g \in \mathcal{P}^\uparrow_+$ implementing Poincaré covariance provides a modular covariance operator. All local modular flows (not only those associated with space-like wedge-algebras) provide one-parameter groups of covariance morphisms (hence generalizing Bisognano-Wichmann theorem).

What kind of space-time induces this type of covariance?

Modular Fluctuations

The modular holonomy groupoid induces, for every $\mathcal{A} \in \mathscr{A}^\omega$:

- modular fluctuations of the modular generator

$$K_\omega \mapsto U \circ K_\omega \circ U^*, \quad \mathcal{A} \in \mathscr{A}^\omega, \quad U \in \mathscr{H}^\omega_{\mathscr{A}}(\mathcal{A}; \mathcal{A}).$$

- modular fluctuations of the covariant vacuum $\omega|_{\mathcal{A}}$:

$$\omega|_{\mathcal{A}} \mapsto \omega|_{\mathcal{A}} \circ \mathrm{Ad}_U, \quad \xi_\omega \mapsto U(\xi_\omega) \quad \mathcal{A} \in \mathscr{A}^\omega, \quad U \in \mathscr{H}^\omega_{\mathscr{A}}(\mathcal{A}; \mathcal{A}).$$

- a modular 1-parameter group of $*$-functors on a W*-category:

$$\begin{bmatrix} \pi_\omega(\mathcal{A})'' & \dots & \pi_\omega(\mathcal{A})'' \\ \vdots & \ddots & \vdots \\ \pi_\omega(\mathcal{A})'' & \dots & \pi_\omega(\mathcal{A})'' \end{bmatrix} \qquad t \mapsto \begin{bmatrix} \Delta^{it}_{\xi_\omega} & \dots & \Delta^{it}_{\xi_\omega, U\xi_\omega} \\ \vdots & \ddots & \vdots \\ \Delta^{it}_{U\xi_\omega, \xi_\omega} & \dots & \Delta^{it}_{U\xi_\omega} \end{bmatrix}$$

Non-commutative Klein-Cartan Geometries? (conjectural)

- Klein's Erlangen program characterizes geometry from its group of symmetries:
  Klein's geometries are homogeneous spaces.[12]
- Cartan dealt with local symmetries: *Cartan's geometries are bundles of homogeneus spaces with a connection.*[13]

Conjecture: we are here dealing with a non-commutative version of Cartan's geometries, where the modular covariance between local modular spectral geometries take the place of connections and local symmetries. Space-time will emerge as a spaceoid equipped with such "modular connection".

More general notions of "holonomy bimodules" might be necessary.

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[12] F.Klein (1872) arXiv:0807.3161.

[13] See the book: R.W.Sharpe (1997) Differential Geometry: Cartan's Generalization of Klein's Erlangen Program, Springer.

Phase-space Quantum Gravity? (very conjectural)

- Tomita-Takesaki modular theory is here taking, at the level of phase-space, the role of the quantum version of Einstein's equation associating "geometries" to "matter content":

$$(\mathscr{A}, \omega) \longmapsto (\mathscr{M}^{\omega}_{\mathscr{A}}, \mathscr{H}^{\omega}_{\mathscr{A}})$$

local Tomita-Takesaki modular theory	$\simeq$	phase-space quantum Einstein equation

$K_{\omega|_{\mathcal{A}}} \simeq$ quantum $T_{\mu\nu} \simeq$ local matter content

modular holonomies $\simeq$ quantum $G_{\mu\nu} \simeq$ local curvature

- The **global phase-space geometries** $(\mathscr{M}^{\omega}_{\mathscr{A}}, \mathscr{H}^{\omega}_{\mathscr{A}})$ induced by the pair $(\mathscr{A}, \omega)$ are similar in spirit to those described by A.Corichi-M.Ryan-D.Sudarsky arXiv:gr-qc/0203072.

- 4. Space-Time Modular NC Geometry
(Extremely Speculative Roadmap)

Modular Filters and Correlations (partially motivated by Rasaakka)

In AQFT, points in Minkowski space can be reconstructed from maximal filters of local observable algebras (Bannier)

- similarly, one can construct a family of points $\mathcal{F}$ corresponding to maximal modular filters in $\mathscr{A}^\omega$.
- two points $\mathcal{F}_1, \mathcal{F}_2$ are not connected iff it is possible to find local algebras $\mathcal{A}_j \in \mathcal{F}_j$, $j = 1, 2$ and $\mathcal{A} \in \mathscr{A}^\omega$, with $\mathcal{A}_1 \subset \mathcal{A} \supset \mathcal{A}_2$ such that the modular flow of $\mathcal{A}$ restricts, via conditional expectations (Tomiyama-Takesaki theorem), to the modular flows of $\mathcal{A}_j$
- any modular covariance unitary between $\mathcal{A}_1$ and $\mathcal{A}_2$ gives a "1-arrow linking from the point $\mathcal{F}_1$ to the point $\mathcal{F}_2''$.
- the non-commutative convolution algebra of the groupoid of modular filters above, *might be* a candidate for a non-commutative algebra of space-time.

How to find a "spectral geometry" over this convolution algebra?

Space-Time NC-geometry from Local Modular Data

(only physical motivation)

It is still premature to attempt an answer but these are indications:

- The strength of modular fluctuations between local modular geometries encodes the metric data between spacetime points: in conformal AQFT modifying the proximity of double cones produces perturbations of the local modular flows.
- There is superposition of different metrics: the strength of the previous fluctuations depends not only on the relative position, but also on the "dimension of the double cones".
- As conceptual experiment we might try to recover a "space-time spectral spaceoid" from the relative geometric position of first-quantization modular theory inside local algebras of free fields (or even QM) on Minkoswki space.

From Local Modular Data to Local Gauge Theory?

(very speculative and vastly incomplete)

Will any reasonable local physics (gauge theory, Dirac) appear?

1-st order local phase-space spectral geometries $(\mathscr{M}^{\omega}_{\mathscr{A}}, \mathscr{H}^{\omega}_{\mathscr{A}})$

modular ⇝ polarization

2-nd order local configuration-space spectral geometries $(\hat{\mathscr{M}}^{\omega}_{\mathscr{A}}, \hat{\mathscr{H}}^{\omega}_{\mathscr{A}})$

Dirac "square root" ⇝ Gauge$\otimes$Gauge$^*\simeq$Gravity

1-st order local "gauge" spectral geometries

From Local to Global? (speculative)

The "analytical" rigidity of modular local data seem to force the adoption of techniques typical of algebraic (NC)-geometry!

Descent How to "globally glue" together local spectral geometries?

- We probably need a generalization of descent theory for sheaves/stacks to the non-commutative case.

In this direction, I am only aware of the work of Flori Fritz on the notion of **gleaves**.

Transport How to treat holonomy transport in the global case?

- Dealing with dynamical systems, either **invariance** or, much more likely, **covariance** constructions are viable.

One might need to pass to **crossed products holonomy algebras** of $(\mathscr{M}^{\omega}_{\mathscr{A}}, \mathscr{H}^{\omega}_{\mathscr{A}})$ as pioneered in the work of Astrup Grimstrup on quantum holonomy algebras.

A theory of non-commutative descent and transport is still missing!

From 1-categorical to ∞-categorical Holonomies? (speculative)

- It is a common trend in algebraic geometry to generalize the notion of spaces with singularities via vertical categorification: **derived geometries** (see for example Eugster Pridham).
- In non-commutative (algebraic) geometry, the "vertical escape to ∞" is known to be essentially unavoidable (see for example Kontsevich Soibelman, Ginzburg).

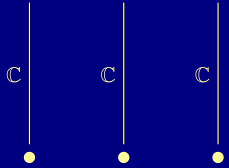
One might need to consider **derived NC-geometries** associated to higher-holonomies of our modular data.

Unfotunately, we are not aware of treatments of **spectral derived NC-geometry** and it is not even clear what will characterize commutative spaces there: we conjectured that a modification to the exchange property might be relevant (Cahiers paper)

Spectrum: Classical Space $\leftrightarrow$ Quantum Space

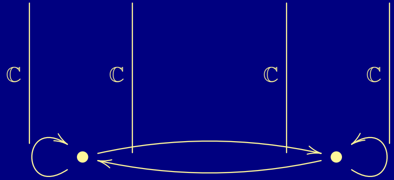
(motivated by work on "quantum spaceoids" with R.Conti, N.Pitiwan)

$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$



$$\begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

$\mathbb{M}_2(\mathbb{C})$

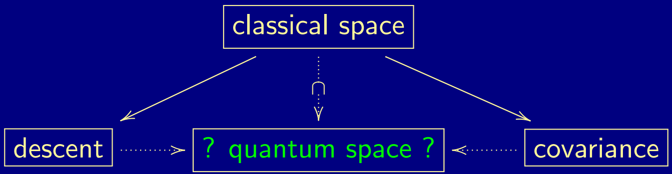


$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

Conjecture [proved in the discrete case]: There is a duality between unital C*-algebras and non-commutative spaceoids: (Fell line-bundles over transition amplitude spaces satisfying certain uniformity and saturation conditions).

The Two "Souls" of (Non-commutative) Geometry ⋆

(motivated by R.Conti question: "What is a Geometry?")



Descent (localization/gluing)	Covariance (transport/holonomy)
sheaves/stacks	Klein-Cartan geometries
Grothendieck topoi	Ehresmann connections
Grothendieck categories	category theory
	(higher) homotopy/holonomy

- **Warnings and Disclaimers**

Why Non-commutative Space-Time?

classical physics ⟷ classical geometry

quantum physics ⟷ NC geometry ?

- removal of QFT infinities and GR singularities
- Kaluza-Klein geometrical description of interations (nc-SM)
- quantum gravity (quantize: GR $\sim$ space-time geometry)

Disclaimer: Space-Time in Quantum Gravity (?)

There are two main (in principle not necessarily opposite) points of view on the role of space-time/geometry in quantum gravity:

- space-time/geometry exists microscopically
- space-time/geometry emerges macroscopically

Today it is almost universally accepted in most of the approaches to QG that space-time (classical or even quantum) is:[14]

- not fundamental: (Wheeler) pre-geometries at high energies
- it emerges at low energies: phase transition from a fundamental theory that is not geometrical.

In this talk, we take the (very controversial) road less traveled:

generalized geometry might exist also at the quantum level.

[14] Armas J (2021) **Conversations on Quantum Gravity** Cambridge U.P.

Warning: EQFT + Emergentism / NP Operationalism

The emergentism of (classical/quantum) geometrical properties (also in A.Connes' NCG) has very solid roots:

- the success of perturbative renormalization has drastically changed theoretical physics forever (Wilson, Feynman, Dyson, Weinberg, . . .)
- Effective Quantum Field Theory reductionism describes physical regimes emerging on top of each other: **no more need for "quantum geometry" to address singularity (GR) and divergence (QFT) issues**.

Here we adopt an *extremely controversial obsolete point of view*:

- there is an operational justification for the introduction of quantum space-time geometry at the non-perturbative level.

Classical Geometric < − > Quantum Algebraic

There is a "Cartesian discrasy" rather than a "Cartesian duality":

- traditionally geometry is indissoluble from classical physics,
- quantum physics uses the language of operator algebras.

Two different paradigmatic views exist on this issue:

(von Neumann) QT is not necessarily geometric	(Vinogradov) QT must be recasted in geometric language

Our point of view is intermediate:

- operator algebraic language is right for QT
- *generalized geometry* (NCG) can be adapted to QT
- the loss of space-time geometric degrees of freedom ($\mathcal{B}(\mathcal{H}_1) \simeq \mathcal{B}(\mathcal{H}_2)$) should be cured (missing ingredients).

Other Geometric Roles of Modular Theory

As a disclaimer, we recall that modular theory already counts several important interactions with (non-commutative) geometry:

- theory of foliations and type III: (Connes)
- modular class of Poisson manifolds: (Weinstein)
- real spectral triples: (Connes)
- modular thermal time: (Connes Rovelli)
- modular twisting of spectral triples: (Connes Moscovici)
- modular spectral triples: (Carey Phillips Rennie)
- modular curvature: (Connes Moscovici Khalkhali Fatizadeh)

Our point of view here is ideologically quite different and points to a full modular reconstruction of (some aspects of) geometry of space-time, more in line with the tradition in algebraic quantum field theory (see for example: Bisognano Wichmann, Schroer Wiesbrock, Borchers, Summers White, Morinelli Neeb Ólaffson).

Non-standard Claims

We question certain now standard assertions in NCG.

non-commutative topology $\overset{?}{\longleftrightarrow}$ C*-algebras

- C*-algebras are spectrally described by **spaceoids**: certain uniform bundles of transition amplitudes over pair groupoids (alternatively -see R.Cirelli- certain uniform Kähler bundles) and these are not barely topological structures!

non-commutative measure theory $\overset{?}{\longleftrightarrow}$ W*-algebras

- Again W*-algebras do not only contain (non-commutative) measure theoretic information, but also formalize (certain bundles of) complexified orthosymplectic spaces (hence their intrinsic modular dynamical character)!

Reasonable Objections / Answers

- Isn't background geometry already implicit in the choice of the local algebras of observables?

  our algebras consist of *partial observables* and are supposed to be *off-shell*: free algebras could be used (see Rasaakka).

- Should different algebras and "correlations between them" replace states?

  this might be possible, states will be just a very special choice of such "correlations".

- Why should the geometry reconstructed from modular theory be non-commutative?

  the modular fluxes of different local regions "interfere".

- Should it be a higher geometry as well?

  very likely yes: vertical categorification is already necessary in non-commutative (algebraic) geometry.