Cyclic quantum causal modelling with a graph separation theorem

Carla Ferradini Victor Gitton V. Vilasini

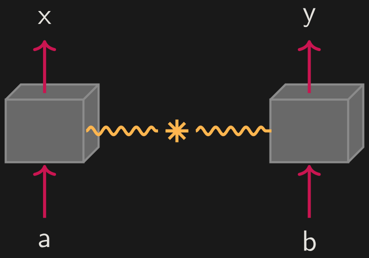
(arXiv:2502.04168)

ETH zürich Inria

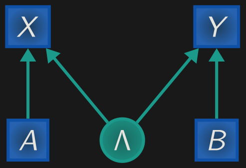
25 June, 2025

Quantum causal modelling

Quantum circuit



Causal model



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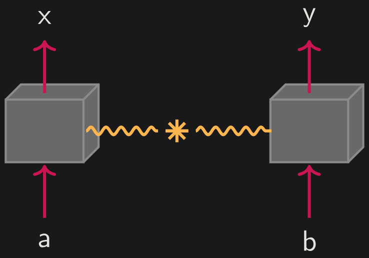

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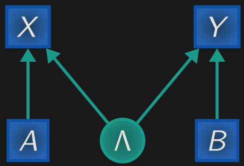


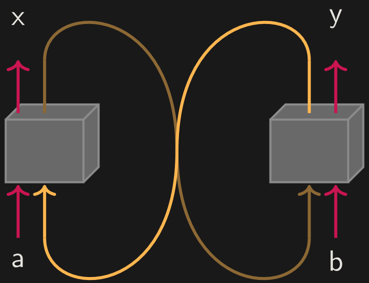
Probability rule

Graph separation theorem

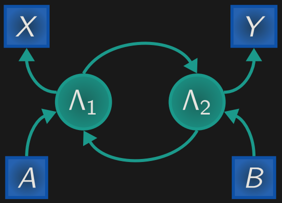
...

Cyclic quantum causal modelling

Quantum circuit



Causal model



Cyclic quantum causal modelling

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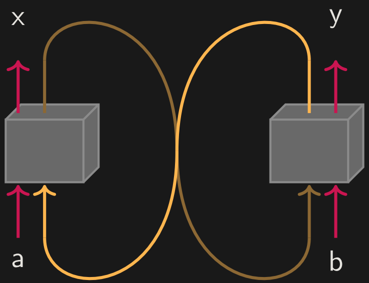

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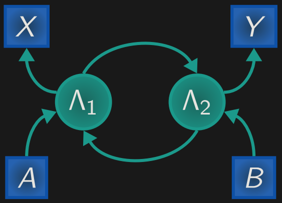


Probability rule?

Graph separation theorem?

...

Cyclic quantum causal modelling

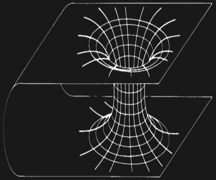
Motivation

Cyclic quantum causal modelling

Motivation

Closed timelike curves

Solutions of GR [1]



[1] Lloyd et al. *Phys Rev Lett* (2011)

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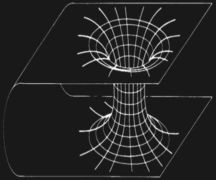
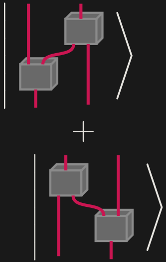


Indefinite causality

[2,3]



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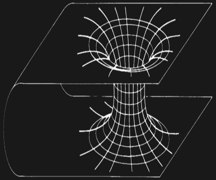

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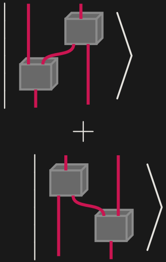
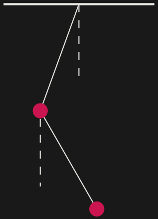


Feedback processes

[4]



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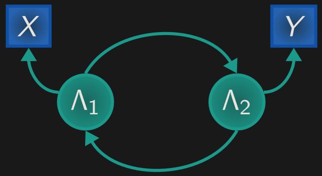
[4] Pearl et al. arXiv:1302.3595 (2013)

Probability rule

Evaluating probabilities



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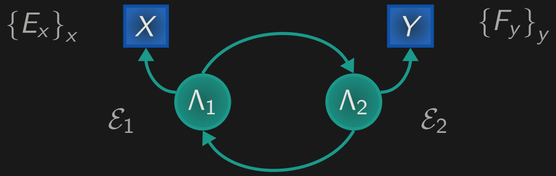

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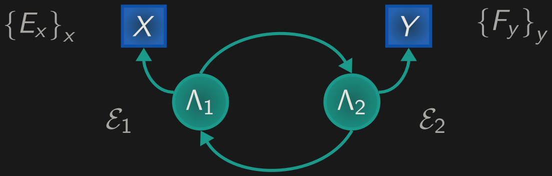
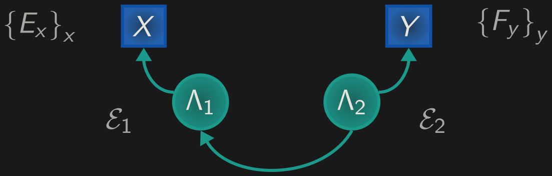


$\Pr(x, y)_G$?

Evaluating probabilities

For any causal model on acyclic G

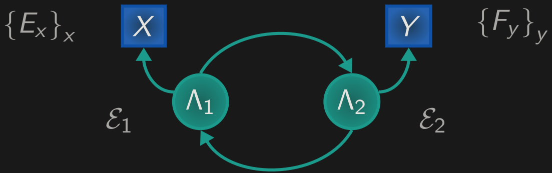


$$\Pr_{\mathrm{acyc}}(x, y)_G = \operatorname{Tr}\left[(E_x \otimes F_y)(\mathcal{E}_1 \circ \mathcal{E}_2)\right]$$

Henson et al. *New J Phys* (2014)
Barrett et al. arXiv:1906.10726 (2019)

Evaluating probabilities

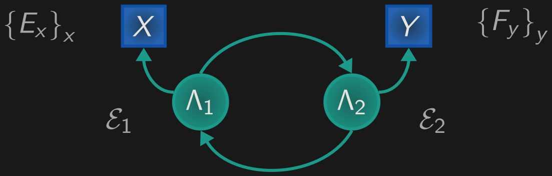
For any causal model on arbitrary G



$\Pr(x, y)_G$?

Evaluating probabilities

For a subset of cyclic causal models on G

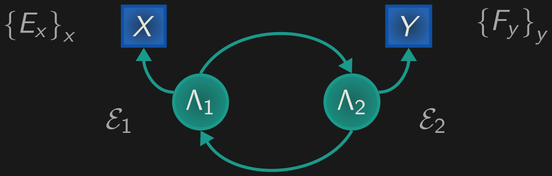


$$\Pr(x, y)_G$$

Forré et al. arXiv:1710.08775 (2017)
Bongers et al. *Ann Statist* (2021)
Barrett et al. *Nat Commun* (2021)

Evaluating probabilities

For any causal model* on arbitray G

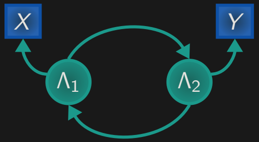


$$\Pr(x, y)_G$$

*with finite dimensional $\mathcal{H}$ and finite-cardinality random variables

Idea: Cyclic $\mapsto$ acyclic with postselection

G :



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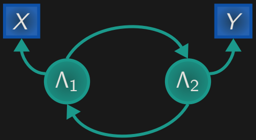
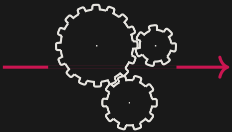




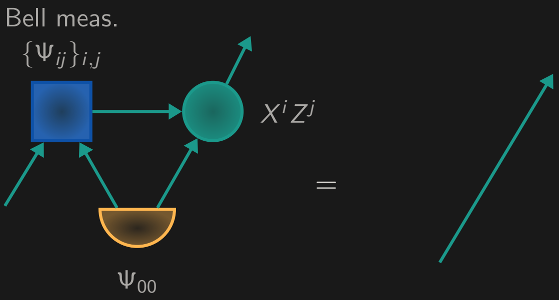
G_{acyc}

$\Pr(x, y)_G$?

$\Pr_{\text{acyc}}$

How: post selected teleportation

Goal: simulate an identity channel



Teleportation protocol

How: post selected teleportation

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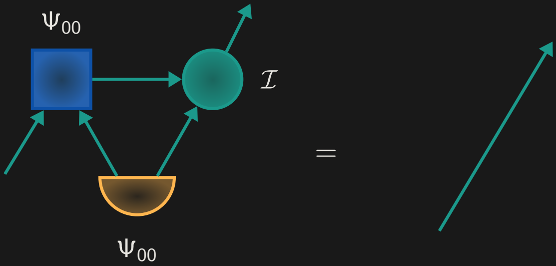

Teleportation protocol

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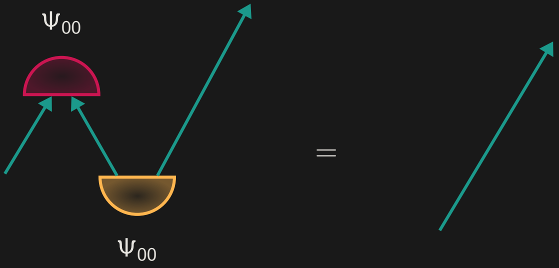
Postselected teleportation protocol ✓

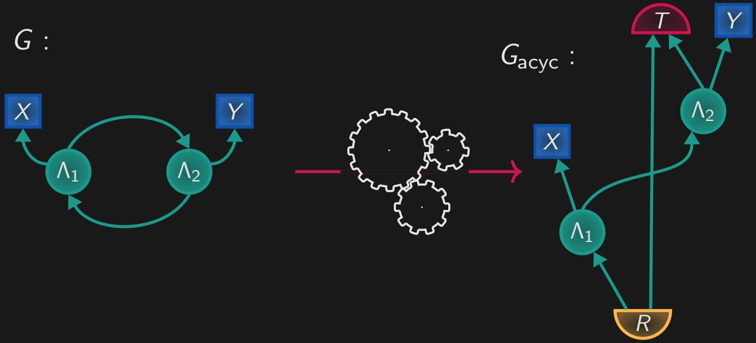


$$\mathrm{Tr}_{AB}\left[|\Psi_{00}\rangle\langle\Psi_{00}|_{AB}\,\rho_A\,|\Psi_{00}\rangle\langle\Psi_{00}|_{BC}\right] = p_{\checkmark}\,\rho_C$$

Lloyd et al. *Phys Rev Lett* (2011)

Lloyd et al. *Phys Rev D* (2011)

Idea: Cyclic $\mapsto$ acyclic with postselection



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Idea: Cyclic $\mapsto$ acyclic with postselection

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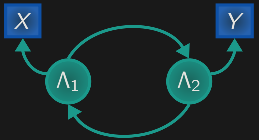

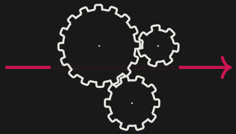

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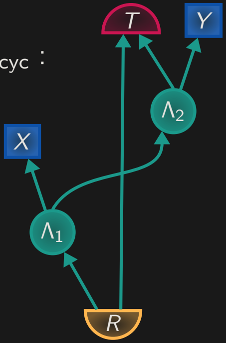

$\Pr(x, y)_G$?

$\Pr_{\mathrm{acyc}}(x, y, t = \checkmark)_{G_{\text{acyc}}}$

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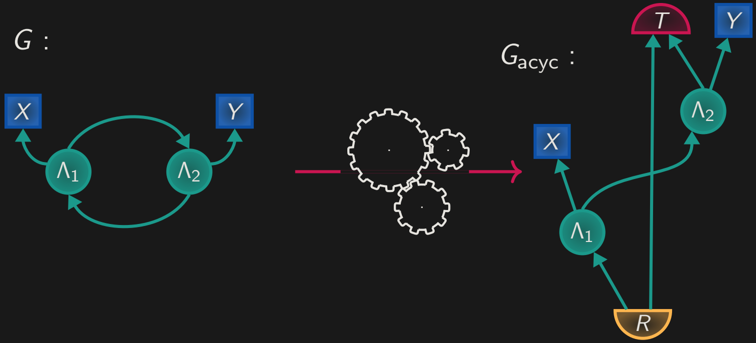

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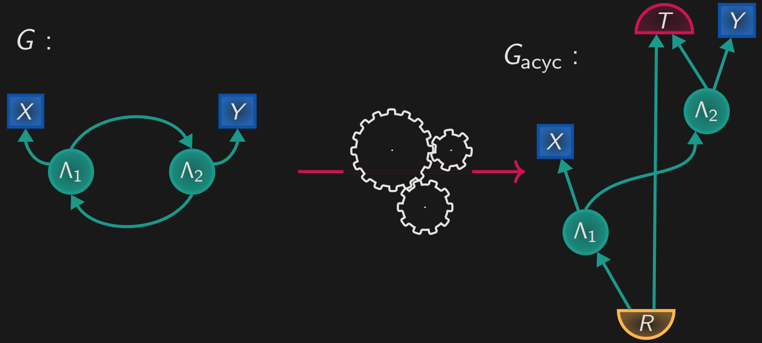

$$\Pr(x, y)_G \overset{\text{def}}{:=} \Pr_{\mathrm{acyc}}(x, y | t = \checkmark)_{G_{\text{acyc}}}$$

Graph separation theorem

Graph separation theorem

Given a directed graph G and a compatible probability Pr

Pearl. *Cambridge University Press* (2009)

Graph separation theorem

Given a directed graph G and a compatible probability Pr

Graph structure property

$\downarrow$

d-separation in G

$\perp^{d}$

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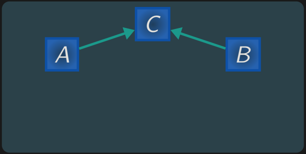

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$A \perp^d B$ and $A \not\perp^d B|C$

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Causal mechanisms

$\downarrow$

independencies in Pr

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Graph structure property

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$$\begin{tikzcd} A \arrow[r] & C & B \arrow[l] \end{tikzcd}$$

$A \perp^d B$ and $A \not\perp^d B|C$

$A \perp\!\!\!\perp B|C$ if

$$\Pr(a, b|c) = \Pr(a|c)\Pr(b|c)$$

Pearl. *Cambridge University Press* (2009)

Graph separation theorem

Given a directed graph G and a compatible probability $\Pr$

Graph structure property		Causal mechanisms
$\downarrow$	$\xrightleftharpoons{\text{acyclic } G}$	$\downarrow$
d-separation in G		*independencies* in $\Pr$
$\perp^d$		$\perp\!\!\!\perp$

$A \perp^d B$ and $A \not\perp^d B|C$

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d-separation theorem

G is acyclic

Pearl. *Cambridge University Press* (2009)

d-separation theorem

$\underline{G \text{ is acyclic}}$

Soundness: If $A \perp^d B|C$ in G, then <u>for all</u> causal models $A \perp\!\!\!\perp B|C$ in $\mathrm{Pr}_{\mathrm{acyc}}$

Pearl. *Cambridge University Press* (2009)

d-separation theorem

<u>G is acyclic</u>

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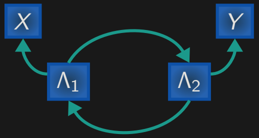
Fails for arbitrary cyclic G!

Pearl. *Cambridge University Press* (2009)

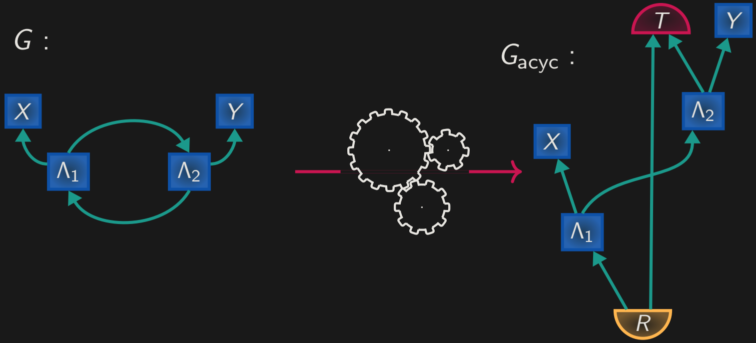
Idea behind p-separation

G :



$(X \perp^p Y)_G$

Idea behind *p*-separation



$(X \perp^p Y)_G$ $(X \perp^d Y | T)_G$

p-separation theorem

for any G

p-separation theorem

<u>for any G</u>

Soundness: If $A \perp^p B|C$ in G, then <u>for all</u> causal models $A \perp\!\!\!\perp B|C$ in Pr

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For causal models* on arbitrary G

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Further results

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- Companion paper for classical causal models (arXiv:2502.04171)

$\longrightarrow$ Markovianity, number and existence of solutions

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→ Markovianity, number and existence of solutions

- Detailed relations between causal modelling frameworks

→ characterising properties and inclusions between frameworks

Henson et al. *New J Phys* (2014)
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Costa et al. *New J Phys* (2016)

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- Mapping tensor networks to cyclic causal models (in preparation)

→ emergence of space-time and notion of causality

Thank you

Outlook

Causal discovery algorithms use *d*-separation for acyclic G

$\longrightarrow$ use *p*-separation in the cyclic case [1,2]

Indefinite causal structures are described with cyclic models

$\longrightarrow$ characterise special subsets, e.g., violating causal inequalities or causally non-separable [3,4]

Causal compatibility problems compatibility of Pr with G

$\longrightarrow$ extend known tecniques for acyclic G, e.g., inflation, to cyclic mapping them to acyclic with postselection [5]

Studying spacetime emergence using tensor networks

$\longrightarrow$ emergence of space time geometry from operational properties of causal models [6]

[1] Spirtes et al. *Appl Inform* (2016) [2] Giarmatzi et al. *npj Quantum Inf* (2018)
[3] Oreshkov et al. *Nat Commun* (2012) [4] Chiribella et al. *Phys Rev A* (2013)
[5] Wolfe et al. *J Causal Inference* (2019)
[6] Cotler et al. *J High Energ Phys* (2019)