Computational Quantum Field Theory

in curved spacetimes

Mohammed Alkhateeb

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CQFT for Schwinger effect and wave packet propagation

- Schwinger effect: Creation of particle-antiparticle pairs under strong electric fields.
- Klein tunneling: Undamped wave packet propagation through potential barriers higher than the incident wave energy by more than mc^2.
- CQFT has been successful in accounting for the Schwinger effect, Klein tunneling, and regular tunneling in relativistic quantum mechanics.
- Strong fields (including gravitational fields).
- Extend CQFT to curved spacetimes!

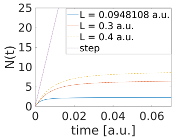

Figure: Fermion pair production in a barrier.

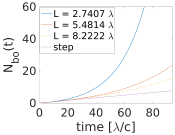

Figure: Boson pair production in a barrier.

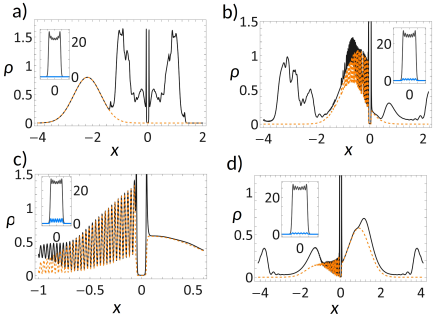

Figure: M. Alkhateeb and A. Matzkin, 2022.

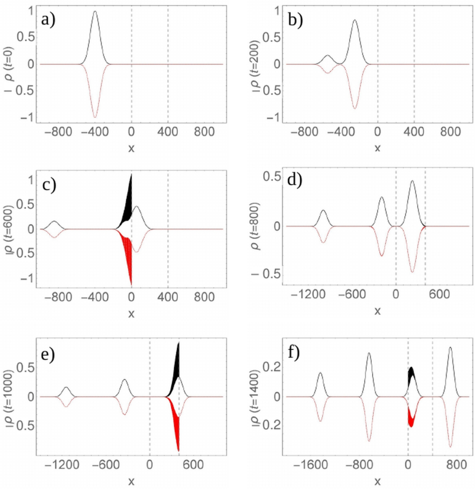

Figure: Klein tunneling of a Dirac wave packet. [M. Alkhateeb and Alexandre Matzkin, 2020]

Outline

1. CQFT for Schwinger effect and wave packet propagation
2. CQFT in curved spacetimes
   - Formalism
   - Observables
   - Numerical treatment
3. Illustrations
4. Conclusion

Formalism

- For spin-$\frac{1}{2}$ fermions (Dirac)

$$i\gamma^a \partial_a \psi - m\psi = 0$$

$$i\gamma^\nu \nabla_\nu \psi - m\psi = 0$$

or more explicitly:

$$i e^{\nu}{}_{a} \gamma^a (\partial_\nu + \Omega_\nu)\psi - m\psi = 0$$

$$g_{\mu\nu}(x) e^{\mu}{}_{a}(x) e^{\nu}{}_{b}(x) = \eta_{ab}$$
$$\eta_{ab} e^{a}{}_{\mu}(x) e^{b}{}_{\nu}(x) = g_{\mu\nu}(x)$$

$$\Omega_\nu(x) = -\frac{i}{4}\omega_{ab\nu}(x)\sigma^{ab}$$

$$\omega^a_{b\nu} = e^{a}{}_{\mu}(x)\partial_\nu e^{\mu}{}_{b}(x) + e^{a}{}_{\mu}(x) e^{\sigma}{}_{b}(x)\Gamma^\mu_{\nu\sigma}$$

$$i\hbar\frac{\partial}{\partial\tau}\psi = h_D\psi$$

- Canonical quantization of the free particle field

$$\hat{\psi}(\xi) = \sum_p \langle \xi | p \rangle \hat{b}_p + \sum_n \langle \xi | n \rangle \hat{d}_n^\dagger$$

- Different mode expansions give answers to different questions.
- The Hamiltonian:

$$\hat{\mathcal{H}}_D = \bar{\psi} h_D \psi$$

- Time dependent field operator

$$\hat{\psi}(\tau, \xi) = \sum_p \langle \xi | p \rangle \hat{b}_p(\tau) + \sum_n \langle \xi | n \rangle \hat{d}_n^\dagger(\tau)$$

- Commutation (anti-commutation) relations :

$$\{\hat{\psi}^\dagger(x), \hat{\psi}(y)\} = \delta(x - y) \Leftrightarrow \{\hat{c}_p^\dagger, \hat{c}_q\} = \delta(p - q)$$

Time evolution

- The idea is to calculate the time evolution of QFT states in terms of that of the first quantized states.
- Time evolution of the quantum field operator in terms of 1st quantized Hamiltonian $\hat{h}$ and QFT Hamiltonian $\hat{\mathcal{H}}$:

$$[\hat{\mathcal{H}}, \hat{\psi}] = i\frac{\partial}{\partial t}\hat{\psi} = \hat{h}\hat{\psi}$$

$$i\frac{\partial}{\partial t}\hat{\psi} = \sum_p \hat{h}|p\rangle \hat{b}_p + \sum_n \hat{h}|n\rangle \hat{d}_n^\dagger$$

- Time evolution of creation and annihilation operators can be calculated in terms of time evolution of 1st quantized states

$$\hat{b}_p(\tau) = \sum_{p'} \langle p'|\hat{U}|p\rangle \hat{b}_p + \sum_{n'} \langle n'|\hat{U}|p\rangle \hat{d}_n^\dagger$$

$$\hat{d}_n^\dagger(\tau) = \sum_{p'} \langle p'|\hat{U}|n\rangle \hat{b}_p + \sum_{n'} \langle n'|\hat{U}|n\rangle \hat{d}_n^\dagger$$

where $\hat{U}$ Iis the time evolution operator. $\hat{U} = e^{-i\hat{h}\tau}$

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Observables

- Using

$$\hat{\psi}(\tau,\xi) = \sum_p \langle x|p\rangle \hat{b}_p(\tau) + \sum_n \langle x|n\rangle \hat{d}_n^\dagger(\tau)$$
$$= \hat{\psi}_+(\tau,\xi) + \hat{\psi}_-(\tau,\xi)$$

- Charge density:

$$\hat{\rho}_{ch}(\tau,\xi) = \hat{\psi}^\dagger(\tau,\xi)\hat{\psi}(\tau,\xi)$$

- Number densities

$$\hat{\rho}_+ = \hat{\psi}_+^\dagger(\tau,\xi)\hat{\psi}_+(\tau,\xi)$$

$$\hat{\rho}_- = \hat{\psi}_-^\dagger(\tau,\xi)\hat{\psi}_-(\tau,\xi)$$

- Wave packets

$$\|\chi\rangle\rangle = \sum_p g_+(p) b_p^\dagger \|0\rangle\rangle.$$

- All expectation values are evaluated in terms of the modes, the amplitudes $< n|\hat{U}|p >$, $< n|\hat{U}|p >$ and so on after applying the operators algebra.

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Numerical treatment

- One-dimensional position space lattice of width Λ:

$$X = \{-\frac{N}{2}\delta x, (-\frac{N}{2}+1)\delta x, ... \frac{N}{2}\delta x\}$$

  where $\delta x = \frac{\Lambda}{N+1}$.
- Reciprocal space lattice

$$P = \{-\frac{N}{2}\delta p, (-\frac{N}{2}+1)\delta p, ... \frac{N}{2}\delta p\}$$

  where $\delta p = \frac{2\pi}{\Lambda}$.
- Evaluate $V(X)$ on X and h_k on P.
- Evaluate the basis vectors for each value $p = n\delta p$ were $n \in \{-\frac{N}{2}, -\frac{N}{2}+1, ..., \frac{N}{2}\}$

$$|p\rangle = \begin{pmatrix} \psi_p(p) \\ \chi_p(p) \end{pmatrix} = N(p) \begin{pmatrix} 1 \\ \frac{cp}{mc^2+E(p)} \end{pmatrix}$$

$$|n\rangle = \begin{pmatrix} \psi_n(p) \\ \chi_n(p) \end{pmatrix} = N(p) \begin{pmatrix} -\frac{cp}{mc^2+E(p)} \\ 1 \end{pmatrix}$$

- Time evolution applied alternatively in x and p spaces:

$$\hat{U}(\delta t) = e^{-i(h_k+V)\delta t/\hbar} \approx e^{-iV\frac{\delta t}{2\hbar}} e^{-ih_k\delta t/\hbar} e^{-iV\frac{\delta t}{2\hbar}}$$
$$= \hat{U}_V \hat{U}_k \hat{U}_V$$

- Calculate the amplitudes $< n|\hat{U}|p >$ and $< n|\hat{U}|n' >$ and so on.

Illustrations

- The metric

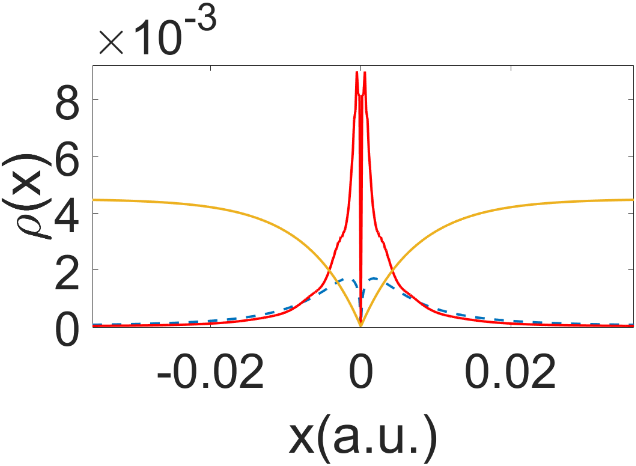
$$ds^2 = \alpha(x)d\tau^2 - \frac{1}{\alpha(x)}d\xi^2$$

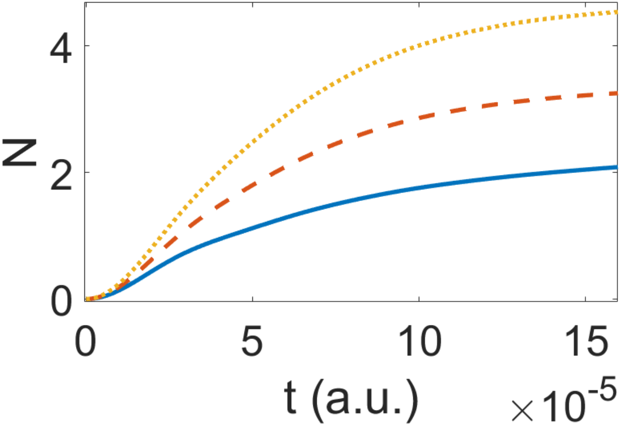
- Calculate the Christoffel sympols, the tetrads and the spin connections.
- The Dirac equation

$$\frac{i}{\sqrt{\alpha(x)}}\sigma_z\left(\partial_t - \frac{\alpha'(x)}{8}\sigma_x\right)\psi(\tau,\xi) - \left(\sqrt{\alpha(x)}\sigma_y\partial_x - mc^2\right)\psi(\tau,\xi) = 0$$

- The Hamiltonian:

$$\hat{H} = \hat{H}_{fr} + i(\alpha(x)-1)\sigma_x c\hat{p} + (\sqrt{\alpha(x)}-1)mc^2\sigma_z + i\frac{\alpha'(x)}{8}\sigma_x$$


×10^-3
8
6
4
2
0
ρ(x)
-0.02
0
0.02
x(a.u.)


4
2
0
N
0
5
10
15
t (a.u.)
×10-5

Conclusion

- We extended the frame work of CQFT to curved spacetimes.
- We are emplying the too in investigating the dynamics of wave packet propagation and pair creation.
- Investigating other quantum phenomena in curved spacetimes!

References

- M. Alkhateeb, X. Gutierrez de la Cal, M. Pons, D. Sokolovski, and A. Matzkin. Phys. Rev. A 111, 012222 – Published 29 January, 2025.
- M. Alkhateeb and A. Matzkin. Phys. Rev. A 109, 062223 – Published 20 June, 2024
- M. Alkhateeb and A. Matzkin, Phys. Rev. A 106, L060202 – Published 19 December, 2022.

Thanks!