Quantum entanglement of two bosonic modes in de Sitter space

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Outline

- time evolution of Gaussian q. E. of two bosonic modes associated with a scalar q. field in de Sitter space and in interaction with a thermal reservoir (in th. of open ss based on completely positive q. dyn. semigs)
- q. E. strongly depends on squeezing of bimodal state, parameters characterizing the thermal env., curvature parameter of de Sitter space, and mass parameter
- thermal env. and curvature have a destructive influence on E., whose survival time depends on the competition between contrary effects provided by squeezing of bimodal state, curvature, and thermal bath
- E. is minimized for values 1/2 and 3/2 of mass parameter, corresp. to conformally coupled scalar field, resp. minimally coupled massless field

Quantum field theory in de Sitter space

- 4D Euclidean de Sitter space can be embedded in 5D Euclidean space $\rightarrow$ open charts in de Sitter space, def. by Hubble radius H^{-1}, have coordinate frames that can be derived through analytic continuation of Euclidean metric, and can be separated into 3 regions (R,C,L), each one with its own individual metric:

$$ds_R^2 = H^{-2}\left[-dt_R^2 + \sinh^2 t_R \left(dr_R^2 + \sinh^2 r_R d\Omega^2\right)\right]$$
$$ds_C^2 = H^{-2}\left[dt_C^2 + \cos^2 t_C \left(-dr_C^2 + \cosh^2 r_C d\Omega^2\right)\right]$$
$$ds_L^2 = H^{-2}\left[-dt_L^2 + \sinh^2 t_L \left(dr_L^2 + \sinh^2 r_L d\Omega^2\right)\right]$$

$d\Omega^2$ - metric on two-sphere

- regions R and L, described by coordinates (t_R, r_R) and (t_L, r_L), resp., are causally disconnected

- region C is described by coordinates (r_C, t_C)

- solutions of Klein-Gordon eq. for a free scalar field in de Sitter space:

$$u_{\sigma p\ell m}(t,r,\Omega) \sim \frac{H}{\sinh t}\chi_{p,\sigma}(t)\,Y_{p\ell m}(r,\Omega), \quad -\mathbf{L}^2 Y_{p\ell m} = \left(1+p^2\right) Y_{p\ell m}$$

$(t,r) \equiv (t_R, r_R)$ or (t_L, r_L)

$Y_{p\ell m}$ - harmonic fs. on 3D hyperbolic space

- positive frequency mode fs corresp. to Euclidean (Bunch-Davies) vacuum, in which the free scalar field is initially prepared, and which are supported in both regions R and L:

$$\chi_{p,\sigma}(t) = \begin{cases} \frac{e^{\pi p}-i\sigma e^{-i\pi\nu}}{\Gamma\left(\nu+ip+\frac{1}{2}\right)} P^{ip}_{\nu-\frac{1}{2}}(\cosh t_R) - \frac{e^{-\pi p}-i\sigma e^{-i\pi\nu}}{\Gamma\left(\nu-ip+\frac{1}{2}\right)} P^{-ip}_{\nu-\frac{1}{2}}(\cosh t_R) \\ \frac{\sigma e^{\pi p}-ie^{-i\pi\nu}}{\Gamma\left(\nu+ip+\frac{1}{2}\right)} P^{ip}_{\nu-\frac{1}{2}}(\cosh t_L) - \frac{\sigma e^{-\pi p}-ie^{-i\pi\nu}}{\Gamma\left(\nu-ip+\frac{1}{2}\right)} P^{-ip}_{\nu-\frac{1}{2}}(\cosh t_L) \end{cases}$$

$P^{\pm ip}_{\nu-\frac{1}{2}}$ - associated Legendre fs

- index σ takes values ± 1, corresp. to indep. sols in each open region R and L

- ν - mass parameter (m - mass of scalar field):

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$$

- value $\nu = 1/2$ corresponds to conformal coupled scalar field and $\nu = 3/2$ to minimally coupled massless scalar field

- parameter p is normalized by H and it represents curvature parameter of de Sitter space: effect of curvature of 3D hyperbolic space appears when p approaches to 1, and gets stronger by decreasing p, so that the limit of infinite curvature is reached for $p \to 0$

- normalization factor for sols

$$N_p = \frac{4 \sinh \pi p \sqrt{\cosh \pi p - \sigma \sin \pi \nu}}{\sqrt{\pi} |\Gamma(\nu + ip + \frac{1}{2})|}$$

- scalar field (annihilation operator $a_{\sigma p \ell m}$ and creation operator $a^{\dagger}_{\sigma p \ell m}$)

$$\hat{\phi}(t, r, \Omega) = \frac{H}{\sinh t} \int dp \sum_{\sigma, \ell, m} \left[a_{\sigma p \ell m} \chi_{p,\sigma}(t) + a^{\dagger}_{\sigma p \ell - m} \chi^{*}_{p,\sigma}(t) \right] Y_{p \ell m}(r, \Omega)$$

$a_{\sigma p \ell m}|0>_{\mathrm{BD}} = 0$ defines Bunch-Davies vacuum

- Bunch-Davies vacuum for a global observer in de Sitter space connects R vacuum and L vacuum through Bogoliubov transf. and can be expressed as a two-mode squeezed state of R and L vacua in Fock space:

$$|0>_{\mathrm{BD}} = U_{R,L}(\gamma_p)\,|0>_R\,|0>_L = \sqrt{1-{\gamma_p}^2}\sum_{n=0}^{\infty}\gamma_p^n|n>_R\,|n>_L$$

$U_{R,L}(\gamma_p) = e^{\gamma_p(c_R^\dagger c_L^\dagger - c_R c_L)}$ - two-mode squeezing op. corresp. to a Gaussian channel

c_R and c_L - annihilation ops of R and L vacuum, resp.

$|n>_{R,L} = \frac{1}{\sqrt{n!}}(c_{R,L}^\dagger)^n|0>_{R,L}$

squeezing parameter:

$$\gamma_p = \mathrm{i}\frac{\sqrt{2}}{\sqrt{\cosh 2\pi p + \cos 2\pi\nu} + \sqrt{\cosh 2\pi p + \cos 2\pi\nu + 2}}$$

- for $\nu = 1/2$ and $\nu = 3/2$ simplifies to $|\gamma_p| = e^{-\pi p}$, which tends to 1 in the limit of small p (large curvature) and takes small values in the limit of large p

- squeezing operator $U_{R,L}(\gamma_p)$ is a Gaussian operation that preserves the Gaussianity of state, and in phase space it is represented by the curvature-induced symplectic op. acting on modes B and $\bar{B}$ observed by Bob and, resp. anti-Bob:

$$S_{B,\bar{B}}(\gamma_p) = \frac{1}{\sqrt{1-\gamma_p{}^2}} \begin{pmatrix} I_2 & |\gamma_p| Z_2 \\ |\gamma_p| Z_2 & I_2 \end{pmatrix}$$

I_2 - unity matrix in 2×2 space, Z_2 is σ_z Pauli matrix $\rightarrow$ under this transf. the single mode observed by Bob is converted to a two-mode in two open charts

- assume Alice is a global observer situated in Bunch-Davies vacuum and Bob is an observer who stays in region R of open charts of de Sitter space
- consider an initial entangled Gaussian two-mode SVS of the free scalar field shared by Alice and Bob
- in Bunch-Davies vacuum, this state, with squeezing parameter s, is described by

$$\sigma_{AB}^{\mathrm{in}}(s) = \begin{pmatrix} \cosh 2s\, I_2 & \sinh 2s\, Z_2 \\ \sinh 2s\, Z_2 & \cosh 2s\, I_2 \end{pmatrix} \tag{1}$$

- Bunch-Davies vacuum observed by global observer Alice can be expressed as a two-mode squeezed state of R vacuum observed by Bob and L vacuum observed by anti-Bob

  - considering that there are no initial correlations between the state described by the covariance matrix (1) and the subs. observed by anti-Bob, the initial covariance matrix of entire system is $\sigma^{\mathrm{in}}_{AB}(s) \oplus I_{\bar{B}}$

  $\rightarrow$ a full description of s. involves three modes: mode A observed by a global observer Alice, mode B observed by Bob in region of open charts R and mode $\bar{B}$ observed by anti-Bob in region L of open charts

  - CM of three-mode Gaussian state describing complete s.:

$$\sigma_{AB\bar{B}}(s, \gamma_p) = \left[I_A \oplus S_{B,\bar{B}}(\gamma_p) \right] \left[\sigma^{\mathrm{in}}_{AB}(s) \oplus I_{\bar{B}} \right] \left[I_A \oplus S_{B,\bar{B}}(\gamma_p) \right]^{\mathrm{T}}$$

- since an observer living in region L is causally disconnected from region R, the physically accessible information is encoded in mode A of Alice and mode B of Bob

  - by performing partial trace over $\bar{B}$ in tripartite covariance matrix we will get rid of mode $\bar{B}$ associated with anti-Bob, obtaining initial CM for Alice and Bob, associated with region R:

$$\sigma_{AB}(s, \gamma_p) = \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{C}^{\mathrm{T}} & \mathcal{B} \end{pmatrix}$$

$$\mathcal{A} = \cosh 2s \; I_2, \;\; \mathcal{B} = \frac{{\gamma_p}^2 + \cosh 2s}{1 - {\gamma_p}^2} I_2, \;\; \mathcal{C} = \frac{\sinh 2s}{\sqrt{1 - {\gamma_p}^2}} Z_2$$

Time evolution of q. E. in de Sitter space

- time evolution of Gaussian q. E. of two bosonic modes in de Sitter space, interacting with a thermal env.

  - dynamics is studied by employing the formalism of OQSs, based on completely positive dyn. semigs

  - Markovian Kossakowski-Lindblad master eq. for density operator $\rho(t)$, describing irreversible time evolution of an open s. ($\hbar = 1$)

$$\frac{d\rho(t)}{dt} = -\mathrm{i}[\mathcal{H}, \rho(t)] + \frac{1}{2}\sum_k \left(2L_k\rho(t)L_k^\dagger - \left\{\rho(t), L_k^\dagger L_k\right\}_+\right)$$

  - Hamiltonian of two bosonic modes (with identical frequencies $\omega = 1$)

$$\mathcal{H} = \frac{1}{2}(x^2 + p_x^2 + y^2 + p_y^2)$$

  - ops. $L_k, L_k^\dagger$ describe interaction of s. with a general env.

- if initial states are Gaussian and ops L_k are chosen polynomials of first degree in canonically conjugated quadrature ops x, p_x, y, p_y of two bosonic modes, then, due to the l. character of dynamics, Gaussianity is preserved in time

  - time evolution of corresponding bimodal CM $\sigma(t)$ is given by Lyapunov eq. of motion:

$$\frac{d\sigma(t)}{dt} = Y\sigma(t) + \sigma(t)Y^{\mathrm{T}} + 2D$$

  - drift matrix

$$Y = \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ -1 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & -1 & -\lambda \end{pmatrix}$$

  λ - dissipation rate, and we assume diffusion matrix

$$D = \mathrm{diag}\{\lambda \coth \frac{1}{2k_B T}, \lambda \coth \frac{1}{2k_B T}, \lambda \coth \frac{1}{2k_B T}, \lambda \coth \frac{1}{2k_B T}\}$$

  (k_B - Boltzmann const., T - temperature of thermal bath)

- solution

$$\sigma(t) = \Gamma(t)[\sigma_{AB}(s,\gamma_p) - \sigma_T]\Gamma^{\mathrm{T}}(t) + \sigma_T$$

$\sigma_{AB}(s,\gamma_p)$ - initial CM of observers Alice and Bob

$\Gamma(t) = \exp(Yt)$, with $\Gamma(t) \to 0$ when $t \to \infty$

- evolution generated by the Gaussian completely positive map is det. by two 4 x 4 real matrices Γ and $A = \sigma_T - \Gamma\sigma_T\Gamma^{\mathrm{T}}$, which satisfy

$A + \mathrm{i}\Omega_{AB} \geq \mathrm{i}\Gamma\Omega_{AB}\Gamma^{\mathrm{T}}$, $\Omega_{AB} = \oplus_1^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ - symplectic form

- CM corresp. to asymptotic Gibbs state of the s. of two bosonic modes, interacting with thermal reservoir of temperature T (Boltzmann constant $k_B = 1$):

$$\sigma_T = \begin{pmatrix} \coth\frac{1}{2T} & 0 & 0 & 0 \\ 0 & \coth\frac{1}{2T} & 0 & 0 \\ 0 & 0 & \coth\frac{1}{2T} & 0 \\ 0 & 0 & 0 & \coth\frac{1}{2T} \end{pmatrix}$$

- logarithmic negativity as a measure to quantify q. E., defined in terms of symplectic invariants of CM

$$\sigma(t) = \begin{pmatrix} \mathcal{A}(t) & \mathcal{C}(t) \\ \mathcal{C}^{\mathrm{T}}(t) & \mathcal{B}(t) \end{pmatrix}$$

$$E_N = -\log_2 g(\sigma(t)), \tag{2}$$

$$g(\sigma(t)) = \frac{1}{\sqrt{2}}\sqrt{\Delta(t) - \sqrt{\Delta^2(t) - 4\det\sigma(t)}}$$

- seralian

$$\Delta(t) \equiv \det\mathcal{A}(t) + \det\mathcal{B}(t) - 2\det\mathcal{C}(t)$$

- for $E_N \leq 0$ the state is separable and $E_N > 0$ determines the strength of E

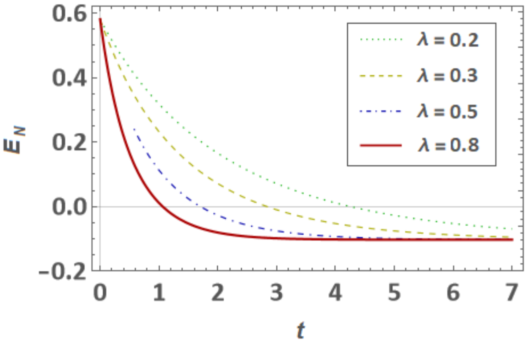

Figure 1: Alice-Bob Gaussian quantum entanglement E_N versus time t for different values of dissipation rate λ ($p = 0.1, s = 0.3, T = 0.3, \nu = 1$).

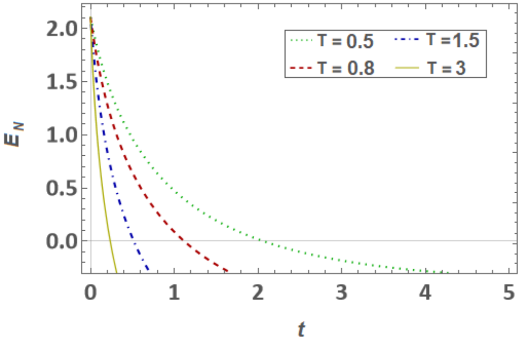

Figure 2: Alice-Bob Gaussian quantum entanglement E_N versus time t for different values of temperature T of the bath ($p = 0.1, \lambda = 0.2, \nu = s = 1$).

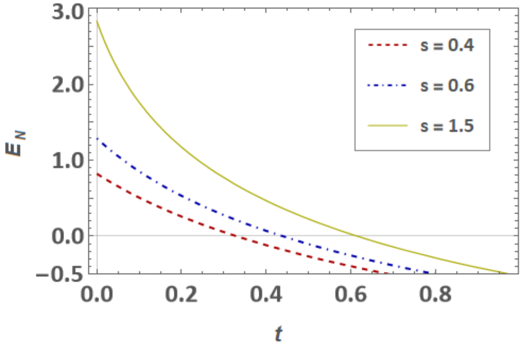

Figure 3: Alice-Bob Gaussian quantum entanglement E_N versus time t for different values of squeezing s ($p = 0.1, \lambda = 0.2$, $T = 2, \nu = 1$).

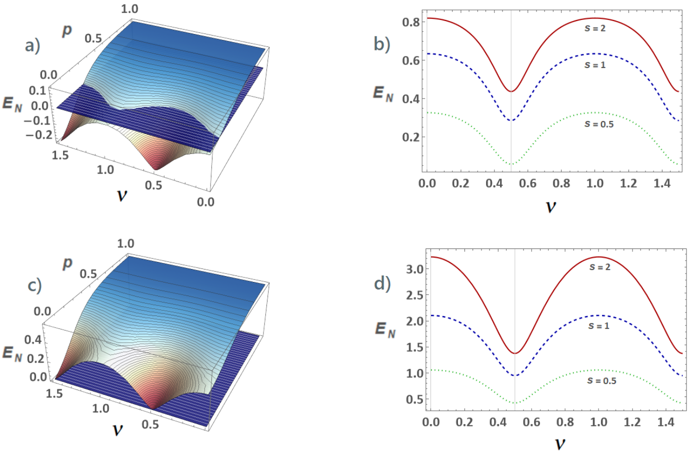

Figure 4: Left: Alice-Bob Gaussian quantum entanglement E_N versus curvature parameter p and mass parameter ν for a) - $(t = s = 0.5, T = 0.8, \lambda = 0.3)$ and c) - $(t = 0, s = 0.5, T = 0.8, \lambda = 0.3)$; Right: Alice-Bob Gaussian quantum entanglement E_N versus mass parameter ν for different values of squeezing s for b) - $(t = 0.5, T = 0.8, p = 0.1, \lambda = 0.3)$ and d) - $(t = 0, T = 0.8, p = 0.1, \lambda = 0.3)$.

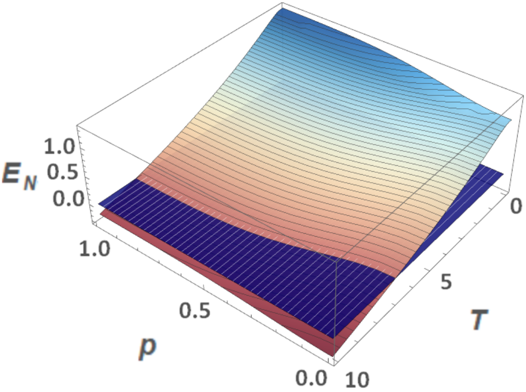

Figure 5: Alice-Bob Gaussian quantum entanglement E_N versus curvature parameter p and temperature T at a given moment of time ($t = 0.1, \nu = 1, \lambda = 0.2, s = 0.5$).

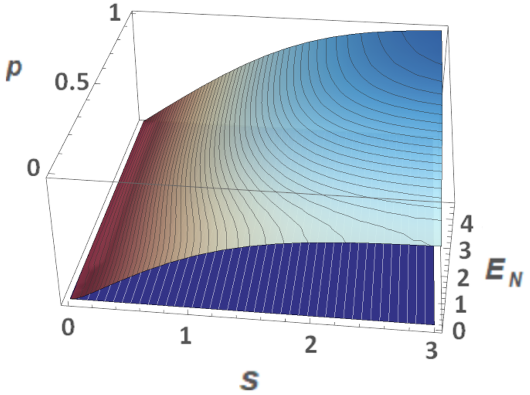

Figure 6: Alice-Bob Gaussian quantum entanglement E_N versus squeezing s and curvature parameter p at a given moment of time ($t = 0.1, \lambda = 0.2,\ T = 0.3, \nu = 1$).

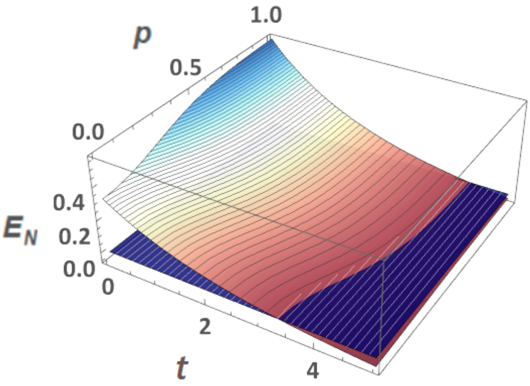

Figure 7: Alice-Bob Gaussian quantum entanglement E_N versus curvature parameter p and time t ($T = 0.3, \nu = 1, \lambda = 0.2$, $s = 0.2$).

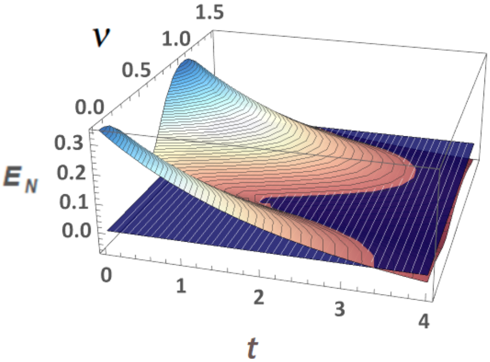

Figure 8: Alice-Bob Gaussian quantum entanglement E_N versus mass parameter ν and time t ($T = 0.3$, $p = 0.1, \lambda = 0.2$, $s = 0.2$).

Summary and Conclusions

- the initially existing q. E. of two bosonic modes gradually weakens over time, and disappears eventually at a finite moment of time - entanglement sudden death phenomenon (ESD); E. intensity, and also survival time of E., decrease by increasing temperature and dissipation parameter
- ESD impedes the realization of QIP (such tasks, like q. teleportation and q. cryptography, require existence of E.)
- when E. survives, its intensity increases with squeezing of modes
- thermal env. has a destroying influence on E., whose survival time depends on the competition between the contrary effects provided by the thermal bath and the squeezing of the bimodal state

Summary and Conclusions

- as de Sitter parameter p decreases, E. also decreases: effect of space curvature reduces E., the state becomes less entangled as curvature of open chart becomes larger; when $p \to 1$, for which effect of curvature appears, $E_N \to$ a finite value that depends practically only on squeezing s
- E. has an oscillatory behavior w.r.t. mass parameter ν, with period 1, which comes from $\cos 2\pi\nu$, however, in the flat space limit $p \to 1$ E. is almost not influenced by ν
- E. is minimized when $\nu = 1/2$ and $\nu = 3/2$, corresponding to conformal coupled scalar field, resp. minimally coupled massless field $\to$ a massive field preserves more E. than a massless field, i.e. E. of a massive scalar field is more robust than that of a massless field in de Sitter space
- for $t = 0$, $E_N = 0$ only when $p = 0$, $\nu = 1/2$ and $\nu = 3/2$, while for $t \neq 0$, $E_N = 0$ for definite values of curvature p; however, E. can survive for definite values of times and in presence of curvature, even in the limit of infinite curvature with $p \to 0$, but only for a mass parameter other than $\nu = 1/2$ and $\nu = 3/2$

Summary and Conclusions

- there is a critical value of p beyond which the amount of E. saturates; at a given moment of time and for a given temperature of the thermal env., the Gaussian q. E. can survive only for a sufficiently large value of the parameter p, that is for a sufficiently small curvature
- survival time of E. decreases as we make the curvature parameter p smaller
- the obtained results illustrate the role that the curvature of space and the parameters describing the squeezing and thermal env. play in order to ensure the preservation of E. over time for practical implementation of q. communication protocols that rely on E.

Thank you!