Relativistic implications of entropy and purity

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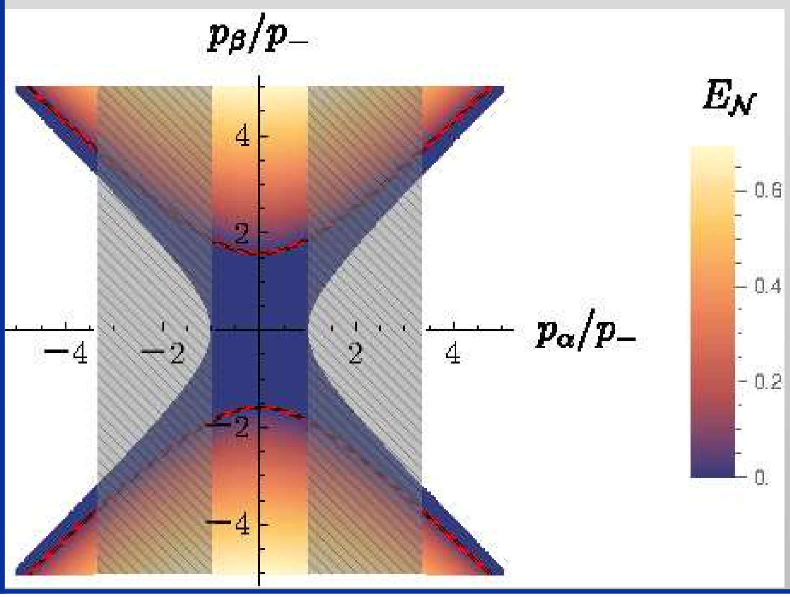

[with Joseph Balsells, arXiv:2506.14705]

Geodesic motion of quantum objects

Quantum object on classical Riemannian space-time, metric g_{ab}.

Extended by quantum fluctuations: Experiences tidal forces.

New formalism:

Geometric optics approximation of QFT on curved space-time.

Implications:

- $\rightarrow$ Entropy and purity acquire weight.
- $\rightarrow$ Complete description requires non-Riemannian geometry.

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Quantum proper time

$$\Delta\tau = \frac{1}{m^2c}\int\sqrt{-\langle\hat{g}^{ab}\hat{p}_a\hat{p}_b\rangle}\mathrm{d}\tau$$

Evaluated with quantum-information methods, focusing on time dilation.

[Smith, Ahmadi: Nat. Comm. 11 (2020) 5360;

Zych, Costa, Pikovski, Brukner: Nat. Comm. 2 (2022) 505]

New strategy:

Geometrical formulation of quantum mechanics combines quantum and space-time physics.

Geometric formulation

$$\frac{\mathrm{d}\langle\hat{O}\rangle}{\mathrm{d}t} = \frac{\langle[\hat{O},\hat{H}]\rangle}{i\hbar}$$

Equivalent to Hamiltonian dynamics:

→ Poisson bracket

$$\{\langle\hat{O}_1\rangle_\rho, \langle\hat{O}_2\rangle_\rho\} = \frac{\langle[\hat{O}_1,\hat{O}_2]\rangle_\rho}{i\hbar}$$

extended by Leibniz rule. Phase-space geometry.

→ Effective Hamiltonian $H_{\mathrm{eff}}(\rho) = \langle\hat{H}\rangle_\rho$ on state space.

→ Hamilton's equations

$$\frac{\mathrm{d}\langle\hat{O}\rangle_\rho}{\mathrm{d}t} = \{\langle\hat{O}\rangle_\rho, H_{\mathrm{eff}}(\rho)\}$$

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State space

Basic expectation values $q = \langle \hat{q} \rangle$ and $p = \langle \hat{p} \rangle$, central moments

$$\Delta(q^a p^b) = \langle (\hat{q} - q)^a (\hat{p} - p)^b \rangle_{\rm symm} \,,$$

in completely symmetric ordering: phase-space coordinates.

Not canonical, for instance
$$\begin{aligned} \{\Delta(q^2), \Delta(p^2)\} &= 4\Delta(qp), \\ \{\Delta(q^2), \Delta(qp)\} &= 2\Delta(q^2), \\ \{\Delta(qp), \Delta(p^2)\} &= 2\Delta(p^2). \end{aligned}$$

Casimir–Darboux theorem:

Every Poisson manifold has local coordinates q_i, p_i, C_I such that $\{q_i, p_j\} = \delta_{ij}$ and $\{q_i, C_I\} = 0 = \{p_i, C_I\}$.

Casimir functions C_I conserved for any Hamiltonian.

Second-order moments

$$\begin{aligned}\{\Delta(q^2),\Delta(p^2)\} &= 4\Delta(qp)\\ \{\Delta(q^2),\Delta(qp)\} &= 2\Delta(q^2)\\ \{\Delta(qp),\Delta(p^2)\} &= 2\Delta(p^2)\end{aligned}$$

Casimir–Darboux variables:

$$\Delta(q^2)=s^2 \quad , \quad \Delta(qp)=sp_s \quad , \quad \Delta(p^2)=p_s^2+\frac{C}{s^2}$$

with canonical $\{s,p_s\}=1$ and $\{s,C\}=0=\{p_s,C\}$.

$$C=\Delta(q^2)\Delta(p^2)-\Delta(qp)^2\geq\hbar^2/4$$

[Jackiw, Kerman: PLA 71 (1979) 158;
Jalabert, Pastawski: PRL 86 (2001) 2490;
Prezhdo: Theo Chem Acc 116 (2006) 206;
Vachaspati, Zahariade: PRD 98 (2018) 065002]

Two classical degrees of freedom

[Baytaş, MB, Crowe, arXiv:1810.12127, arXiv:1811.00505]

$$\Delta(x_1^2) = s_1^2 \quad , \quad \Delta(x_2^2) = s_2^2 \quad , \quad \Delta(x_1 x_2) = s_1 s_2 \cos\beta$$

with new canonical variable β: Correlation as an angle.

Momentum p_β appears in $\Delta(x_1 p_2)$ and in

$$\Delta(p_1^2) = p_{s_1}^2 + \frac{U_1}{s_1^2}$$

where

$$\begin{aligned} U_1 &= (p_\alpha - p_\beta)^2 \\ &+ \frac{1}{2\sin^2\beta}\left((C_1^2 - 4p_\alpha^2) - \sqrt{C_2^4 - C_1^4 + (C_1^2 - 4p_\alpha^2)^2}\sin(\alpha+\beta)\right) \end{aligned}$$

with additional canonical pair (α, p_α) and Casimirs C_1, C_2.

Purity $\mathrm{Tr}(\rho^2)$ conserved for any Hamiltonian: Function of C_1, C_2.

Hamiltonian proper time

[Balsells, MB, arXiv:2410.08156]

Canonical coordinates $\{x^a,\pi_b\}=\delta^a_b$. Hamiltonian

$$H=-\frac{1}{2m}\left(g^{ab}(x)\pi_a\pi_b+m^2c^2\right)$$

implies geodesic motion, proper time $\Delta\tau=\int\sqrt{1+2H/(mc^2)}\,\mathrm{d}\tau$.

$$\begin{aligned}
H_{\mathrm{eff}} &= -\frac{1}{2m}\left(\langle g^{ab}(\hat{x})\hat{\pi}_a\hat{\pi}_b\rangle+m^2c^2\right)\\
&= -\frac{1}{2m}\Big(g^{ab}\pi_a\pi_b+m^2c^2\\
&\quad +g^{ij}\Delta(p_ip_j)+\frac{\partial g^{aj}}{\partial x^k}\pi_a\Delta(x^kp_j)+\frac{1}{2}\frac{\partial^2 g^{ab}}{\partial x^k\partial x^l}\pi_a\pi_b\Delta(x^kx^l)\Big)
\end{aligned}$$

Quantize position $\hat{x}$ in g^{ab}, but no quantization of gravity.

Finsler geometry

[Balsells, MB, arXiv:2503.06667]

Inserting canonical variables, crucial expression

$$\sqrt{C_2^4 - C_1^4 + (C_1^2 - 4p_\alpha^2)^2}$$

from U_1 appears in effective Hamiltonian/proper time.

→ In general, non-quadratic in momenta:
Non-Riemannian (Finsler) geometry in combined space-time/state space.

→ Quadratic if

$$C_1 = C_2 \qquad \text{forbidden}$$
$$p_\alpha = 0 \qquad \text{Gaussian}$$

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The weight of purity

C_1 and C_2 conserved: $\dot{C}_{1,2} = \{C_{1,2}, H\} = 0$ for any H.

Related to symplectic eigenvalues $\nu_\pm$ of covariance matrix $\sigma_{ij} = \Delta(x_i x_j)$:

$$C_1^2 = \nu_+^2 + \nu_-^2 \qquad , \qquad C_2^2 = \nu_+^2 - \nu_-^2 \, .$$

Uncertainty relation implies $\nu_+, \nu_- \geq \hbar/2$, thus $C_1 \neq C_2$.

Purity

$$\mu(\sigma) = \frac{\hbar^2}{4\sqrt{\det(\sigma)}} = \frac{\hbar^2}{4\nu_+\nu_-} = \frac{\hbar^2}{2\sqrt{C_1^4 - C_2^4}}$$

appears in $\sqrt{C_2^4 - C_1^4 + (C_1^2 - 4p_\alpha^2)^2}$.

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Logarithmic negativity

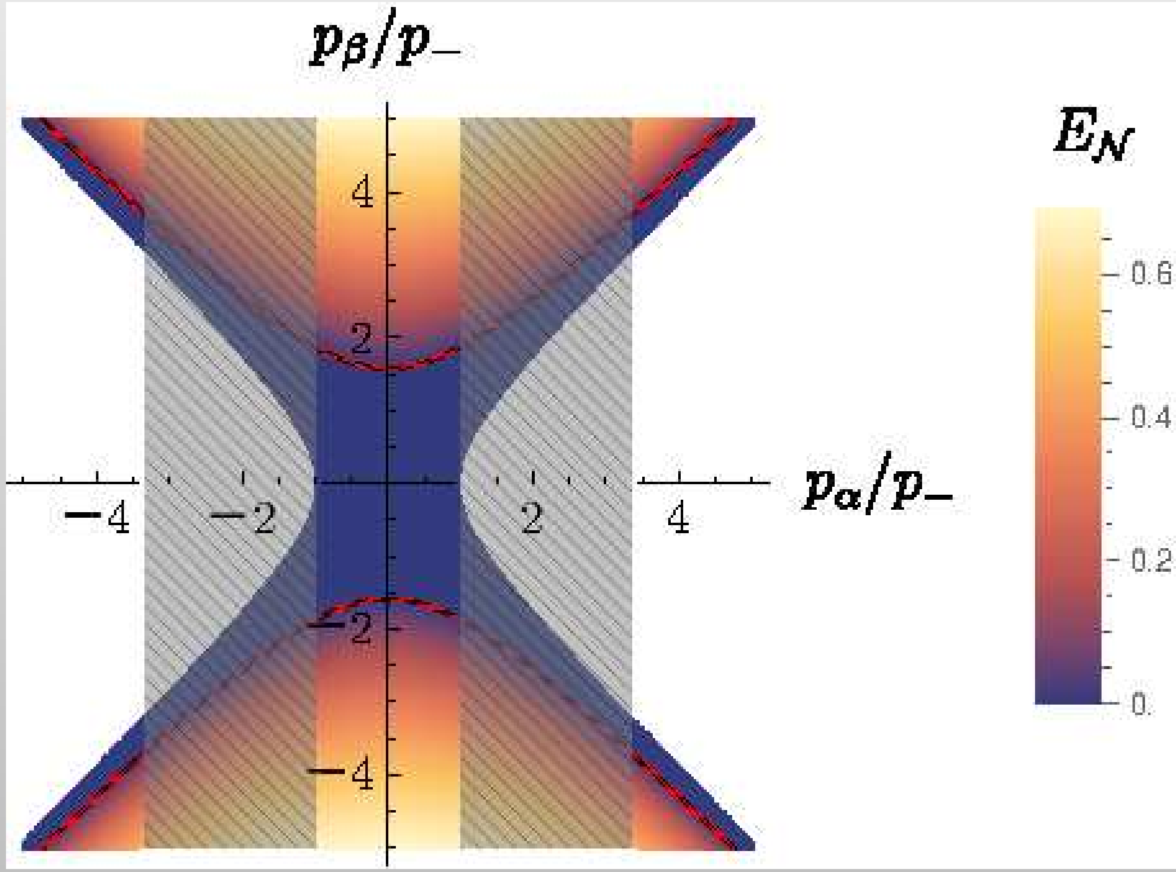
[Balsells, MB, arXiv:2506.14705]

Complicated expression simplifies for $\beta = \pi/2$:

$$e^{-2E_{\mathcal{N}}}|_{\beta=\pi/2} = C_1^2 + 4p_\beta^2 - 4p_\alpha^2 - \sqrt{(C_1^2 + 4p_\beta^2 - 4p_\alpha^2)^2 - C_1^4 + C_2^4}$$

$p_- = \frac{1}{2}(\nu_+ - \nu_-)$:
limits regions forbidden by reality conditions

Entangled states above and below red hyperbolae



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Summary

- → Detailed relationship between time dilation effects and quantum-information parameters.
- → Also applicable to tidal forces and geodesic deflection.
- → Specific indications for modified space-time structure, related to entropy and purity.
- → Remaining canonical parameter

$$\cos\alpha = -\sin\beta \frac{(\Delta^{x_1x_2}_{x_2p_2})^2 - (\Delta^{x_1x_2}_{x_1p_2})^2 + \Delta^{x_1x_2}_{x_1x_2}\Delta^{x_1p_1}_{x_1p_1} - \Delta^{x_1x_2}_{x_1x_2}\Delta^{x_2p_2}_{x_2p_2}}{\Delta^{x_1x_2}_{x_1x_2}\sqrt{(C_1^2 - 4p_\alpha^2)^2 - C_1^4 + C_2^4}}$$

where

$$\Delta^{B_1B_2}_{A_1A_2} = \Delta(A_1B_1)\Delta(A_2B_2) - \Delta(A_1B_2)\Delta(A_2B_1)$$

Distinguished by canonical nature of variables.
Specific quantum-information meaning?