ksajnok@cft.edu.pl

Operator Ordering in the Relativistic Quantization and its Thermodynamic Implications

Karol Sajnok [1, 2], Kacper Dębski [2]

[1] Institute of Theorethical Physics, University of Warsaw
[2] Center for Theorethical Physics, Polish Academy of Sciences

⟨IS|RQI⟩

RQI-N, Naples, 2025

ksajnok@cft.edu.pl

Motivation – ordering problem

$$E = \sqrt{g_{00}}\sqrt{m^2c^4 - c^2 g^{ij} p_i p_j}$$

Classical relativistically invariant energy: a function of the metric (position) and the momentum

VS

Quantum non-trivial commutation relation of position and momentum

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$$

Usually: arbitrary assumption of the Weyl ordering...

$$:\hat{x}\hat{p}^2:_{\mathrm{Weyl}} = \frac{1}{3}\hat{x}\hat{p}^2 + \frac{1}{3}\hat{p}\hat{x}\hat{p} + \frac{1}{3}\hat{p}^2\hat{x}$$

...but there is another general option

ksajnok@cft.edu.pl

Solution – ordering parameters

$$E = \sqrt{g_{00}}\sqrt{m^2c^4 - c^2g^{ij}p_ip_j}$$

Conditions:
* hermitian Hamiltonian
* classical (commuting) limit

1. Time-like ordering $\hat{H} = \gamma\sqrt{g_{00}(r+\hat{x})}\hat{H}_p + \gamma^*\hat{H}_p\sqrt{g_{00}(r+\hat{x})}$

$2\mathrm{Re}\gamma = 1$

ksajnok@cft.edu.pl

Solution – ordering parameters

$$E = \sqrt{g_{00}}\sqrt{m^2c^4 - c^2 g^{ij}p_ip_j}$$

Conditions:
* hermitian Hamiltonian
* classical (commuting) limit

1. Time-like ordering $\hat{H} = \gamma\sqrt{g_{00}(r+\hat{x})}\hat{H}_p + \gamma^*\hat{H}_p\sqrt{g_{00}(r+\hat{x})}$
$2\mathrm{Re}\gamma = 1$

2. Spatial ordering $\hat{H}_p = \sqrt{\hat{H}_0{}^2 - c^2 : g^{rr}(r+\hat{x})\hat{p}^2 :}$

ksajnok@cft.edu.pl

Solution – ordering parameters

$$E = \sqrt{g_{00}}\sqrt{m^2c^4 - c^2 g^{ij}p_ip_j}$$

Conditions:
* hermitian Hamiltonian
* classical (commuting) limit

1. Time-like ordering $\hat{H} = \gamma\sqrt{g_{00}(r+\hat{x})}\hat{H}_p + \gamma^*\hat{H}_p\sqrt{g_{00}(r+\hat{x})}$
   $2\mathrm{Re}\gamma = 1$

2. Spatial ordering $\hat{H}_p = \sqrt{\hat{H_0}^2 - c^2 : g^{rr}(r+\hat{x})\hat{p}^2 :}$

2.a. Taylor series (if necessary)

$$: g^{rr}(r+\hat{x})\hat{p}^2 : = g^{rr}(r)\hat{p}^2 + \sum_{k=1}^{\infty} \partial_r^k g^{rr}(r) : \hat{x}^k \hat{p}^2 :$$

ksajnok@cft.edu.pl

Solution – ordering parameters

$$E = \sqrt{g_{00}}\sqrt{m^2c^4 - c^2 g^{ij}p_i p_j}$$

Conditions:
* hermitian Hamiltonian
* classical (commuting) limit

1. Time-like ordering $\hat{H} = \gamma\sqrt{g_{00}(r+\hat{x})}\hat{H}_p + \gamma^*\hat{H}_p\sqrt{g_{00}(r+\hat{x})}$
   $2\mathrm{Re}\gamma = 1$

2. Spatial ordering $\hat{H}_p = \sqrt{\hat{H}_0{}^2 - c^2 :g^{rr}(r+\hat{x})\hat{p}^2:}$

2.a. Taylor series (if necessary)

$$:g^{rr}(r+\hat{x})\hat{p}^2: = g^{rr}(r)\hat{p}^2 + \sum_{k=1}^{\infty}\partial_r^k g^{rr}(r) :\hat{x}^k\hat{p}^2:$$

2.a. Hermiticity

$$:\hat{x}^k\hat{p}^2: := \alpha\hat{p}^2\hat{x}^k + \beta\hat{p}\hat{x}^k\hat{p} + \gamma\hat{x}^k\hat{p}^2$$

$$:\hat{x}^k\hat{p}^2: = \hat{x}^k\hat{p}^2 - i(1 - 2i\mathrm{Im}\alpha)k\hbar\hat{x}^{k-1}\hat{p} - \alpha k(k-1)\hbar^2\hat{x}^{k-2}$$

ksajnok@cft.edu.pl

Relativistic Quantum Hamiltonian

In the low energy (momentum) & small perturbation (position) limit:

$$\hat{H} = \hat{H}_0 \sqrt{g_{00}} \Bigg[1 + \frac{g'_{00}}{2g_{00}} \hat{x}$$

linear potential

kinetic energy
+ curvature correction

$$- \frac{c^2}{2\hat{H}_0^{\,2}} \left(g^{rr} + \left(g'^{rr} + \frac{g'_{00} g^{rr}}{2g_{00}} \right) \hat{x} \right) \hat{p}^2$$

$$+ \frac{i\hbar c^2}{2\hat{H}_0^{\,2}} \left((1 - 2i\mathrm{Im}\alpha) g'^{rr} + \gamma \frac{g'_{00} g^{rr}}{g_{00}} \right) \hat{p} \Bigg]$$

purely quantum operator ordering correction

ksajnok@cft.edu.pl

Rindler Quantum Hamiltonian

In the conformally flat Rindler coordinates:

$$\sqrt{g_{00}(\hat{x})} = 1 + \frac{g\hat{x}}{c^2}$$ without reference to any central point, ie. $r = 0$

Hamiltonian without any approximations:

$$\hat{H} = \hat{H}_0 \left[\left(1 + \frac{g\hat{x}}{c^2}\right) \sqrt{1 + \frac{c^2\hat{p}^2}{\hat{H}_0^2}} - i\gamma \frac{\hbar g}{\hat{H}_0^2} \frac{\hat{p}}{\sqrt{1 + \frac{c^2\hat{p}^2}{\hat{H}_0^2}}} \right]$$

In the low energy (momentum) limit:

$$\hat{H} = mc^2 \left(1 + \frac{g\hat{x}}{c^2}\right)\left(1 + \frac{\hat{p}^2}{2m^2c^2}\right) - i\gamma \frac{\hbar g \hat{p}}{mc^2}$$

We can now study its spectrum, eigenfunctions and the specific heat!

ksajnok@cft.edu.pl

Perturbative eigenvalues (Rindler)

Airy Hamiltonian $\hat{H}_0 = mc^2 + \frac{\hat{p}^2}{2m} + mg\hat{x},$

Perturbation $\hat{H}' = \frac{g\hat{x}\hat{p}^2}{2mc^2} - i\gamma\frac{\hbar g\hat{p}}{mc^2}.$

$$L = \left(\frac{\hbar^2}{2m^2 g}\right)^{1/3}$$

$$\hat{\xi} = \frac{\hat{x}}{L}$$

$E_n^{(0)} = mc^2 + \epsilon_n$ rest energy + Airy zeros

$E_n^{(1)} = \frac{mg^2L^2}{c^2}\left(a_n\langle n|\hat{\xi}|n\rangle - \langle n|\hat{\xi}^2|n\rangle\right)$ ordering independent

$$E_n^{(2)} = \frac{mg^3L^3}{c^4}\sum_{k\neq n}\frac{\left|\left(a_n(1-\gamma)+\gamma a_k\right)\langle k|\hat{\xi}|n\rangle - \langle k|\hat{\xi}^2|n\rangle\right|^2}{a_n - a_k}$$

gamma “mixing energies”

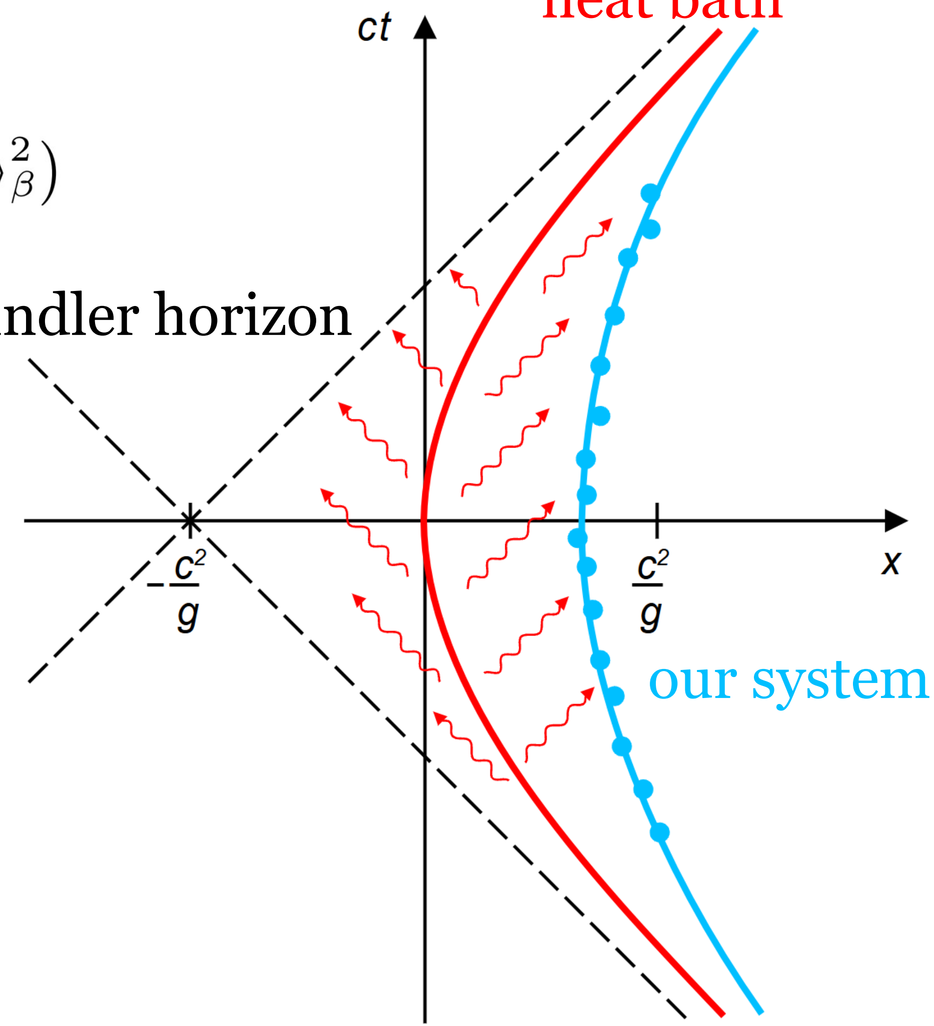
ksajnok@cft.edu.pl

Specific Heat - Setup

$$\frac{C_V}{N} = \frac{\partial}{\partial T}\left(\frac{\langle E\rangle_\beta}{Z}\right) = \frac{1}{k_B T^2}\left(\langle E^2\rangle_\beta - \langle E\rangle_\beta^2\right)$$

$C_V^0 \leftrightarrow E_n^{(0)}$ and $E_n^{(1)}$

$C_V \leftrightarrow E_n^{(0)}$, $E_n^{(1)}$, and $E_n^{(2)}$



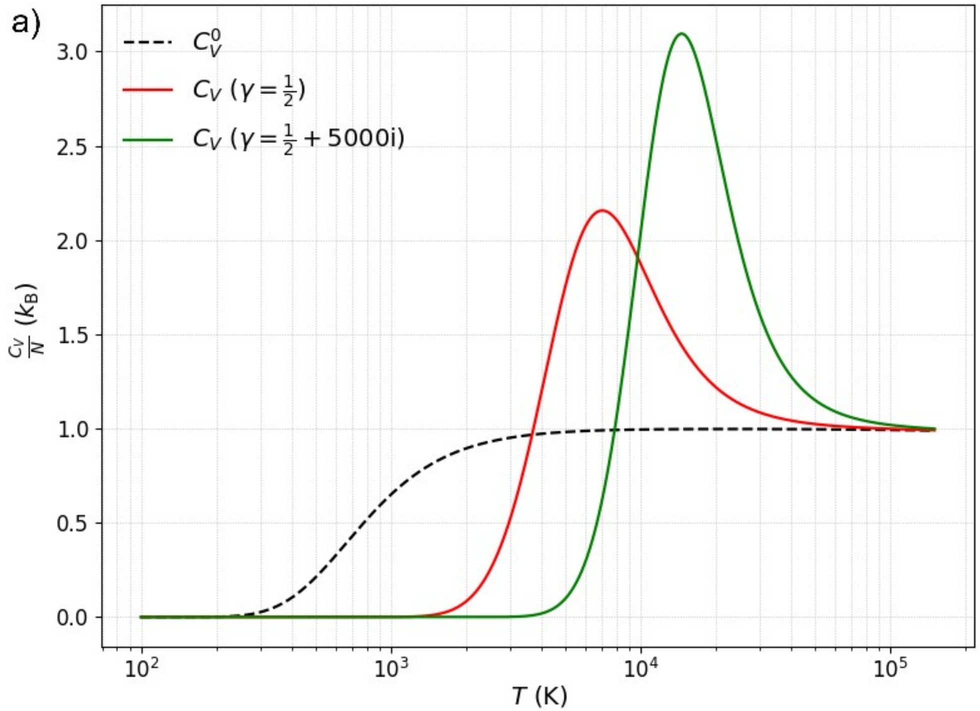
We always measure local temperature!
Tolman-Ehrenfest law

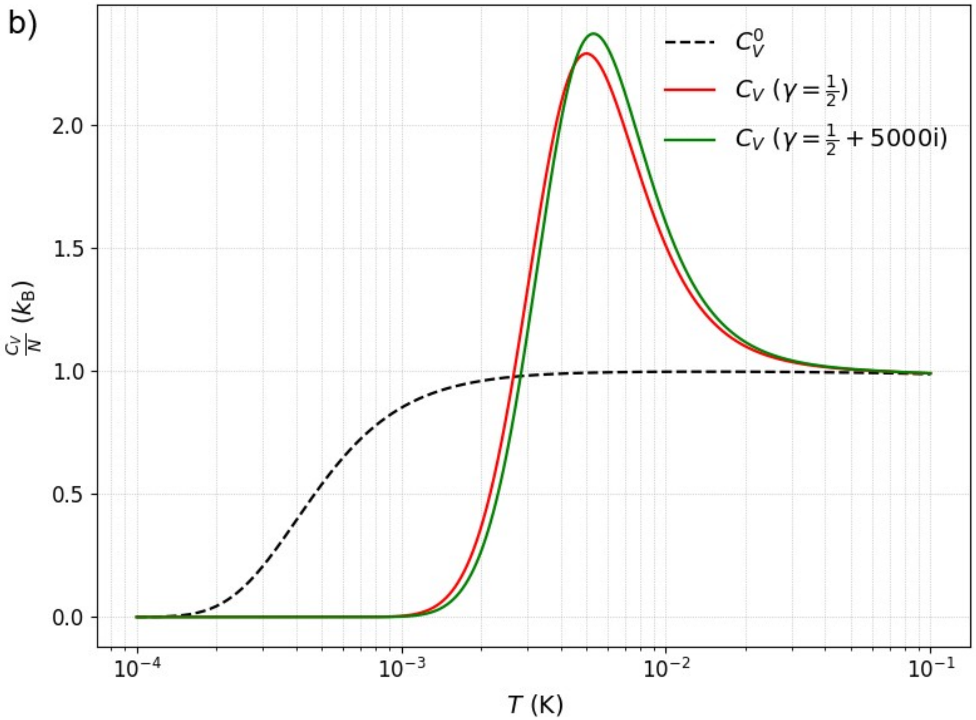
$$T(x)\sqrt{g_{00}(x)} = \Theta_{\text{hb}}$$

Here – classical, but in the future…

ksajnok@cft.edu.pl

Specific Heat vs Temperature





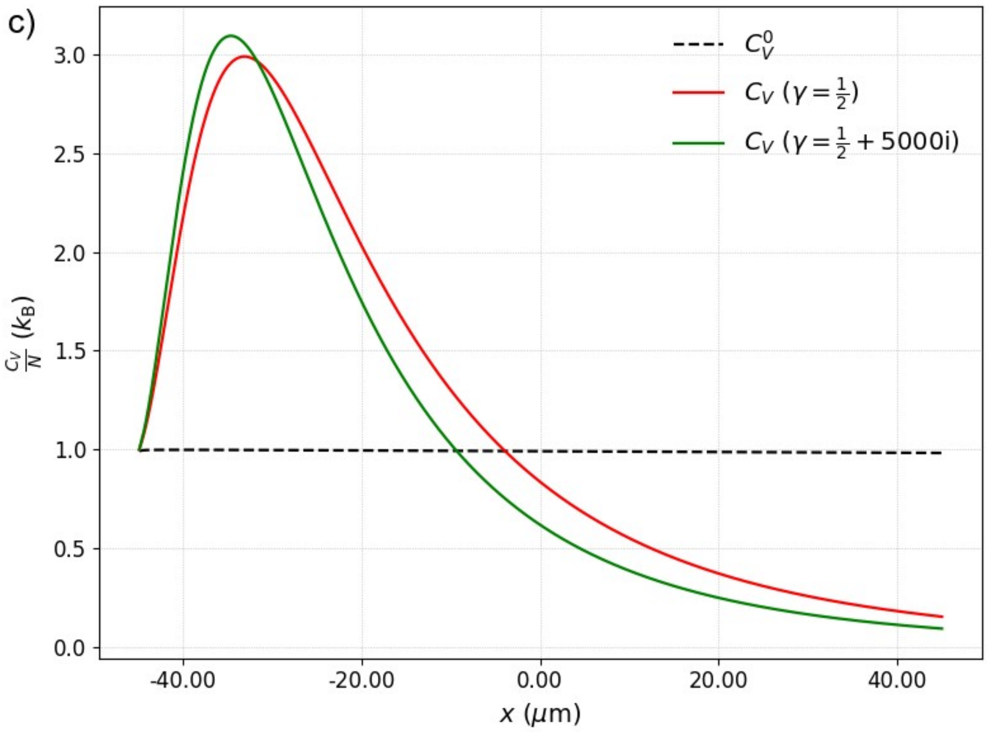
a) electrons accelerated by electric field
$m = 9 \times 10^{-31}$ kg $\quad a = 2 \times 10^{21}$ m/s^2

b) neutrinos near extreme black hole horizon
$m = 8 \times 10^{-37}$ kg $\quad a = 10^{15}$ m/s^2

Analogical peak-dip-plateau curve in: T. I. Rouabhia, et al. (2023) Phys. Part. Nucl. Lett. **20**, 112.

ksajnok@cft.edu.pl

Specific Heat vs distance



c) electrons accelerated by electric field
$m = 9 \times 10^{-31}$ kg $a = 2 \times 10^{21}$ m/s²

local temperature = Tolman-Ehrenfest law

$$T(x)\sqrt{g_{00}(x)} = \Theta_{\text{hb}}$$

the heat bath at $x = 0$ μm
$T_{hb} = 7 \times 10^3$ K

the event horizon at $x \approx -45$ μm

Experimental analysis of the specific heat under intermediate temperatures, small masses and extreme accelerations

↓

Physical values of ordering parameters

ksajnok@cft.edu.pl

Take-away

1) Operator ordering exposed: two free parameters govern spatial and temporal ordering in the relativistic Hamiltonian.

2) Semiclassical fails: a quantum perturbative spectrum is needed to reveal ordering effects.

3) Thermal signature: ordering shifts the specific-heat curve.

4) Numerical evidence: accelerated electrons or ultra-light particles in gravitational field.

5) Tolman–Ehrenfest law modulates the effect with distance.

6) Prospects: precision calorimetry under acceleration or in analogue-gravity setups could measure the physical values of ordering parameters.