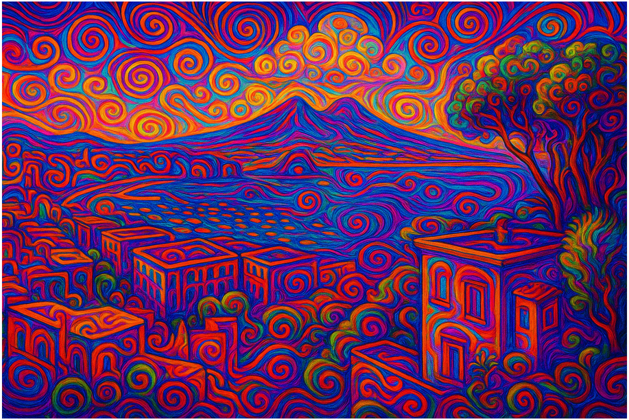
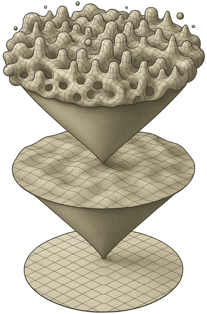
QUANTUM REFERENCE FRAMES IN QUANTUM SPACETIMES



Flavio Mercati - University of Burgos

Relativistic Quantum Information North - Naples, 25.6.2025

Planck-scale spacetime foam



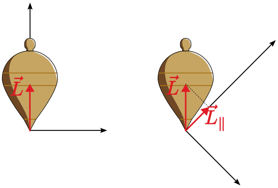
Quantum Gravity → spacetime foam?
[Wheeler, Ann. Phys. **2** (1957)]

Lorentz violation?
[Pavlopoulos, Phys. Rev. **159** (1967)]

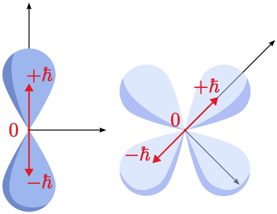
Deformation of Lorentz invariance without violation of Relativity Principle?
[Amelino-Camelia, Nature **418** (2002)]

[Rovelli–Speziale, PR**D67** (2003)]

Minimal length
$\not\Downarrow$
Lorentz violation?

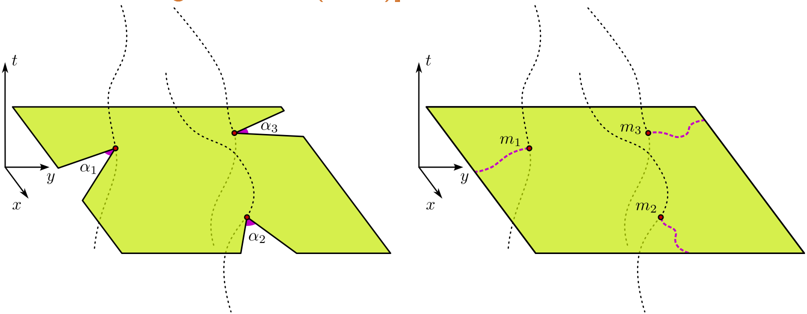


Classical angular momentum



Quantum angular momentum

[Matschull–Welling, CQG **15** (1998)]: 2+1D QG + matter

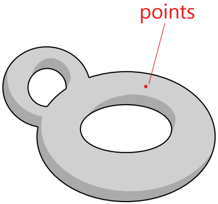


$$\int \mathcal{D}A\, e^{i \int_{M^3} \mathrm{tr}\left(A\wedge dA + \frac{2}{3} A\wedge A\wedge A\right) + \frac{\alpha}{2\pi}\int_\gamma \mathrm{tr}(gA)} = e^{i\, S_{\mathit{eff}}[\gamma]}\,,$$

$S_{\mathit{eff}}[\gamma] \;\Rightarrow\; [\hat{\boldsymbol{x}}^\mu, \hat{\boldsymbol{x}}^\nu] = i\,\ell\,\varepsilon^{\mu\nu}{}_\rho\, \hat{\boldsymbol{x}}^\rho\,,$ $\quad \ell =$ "Planck length" (really, just $\hbar$).

- Relevance for 3+1D QG: can be formulated as a topological BF theory + "simplicity" constraints [Plebanski, JMP **18** (1977)].
- [Freidel–Livine PRL **96** (2006)]: QG + scalar field → nonlocal effective field theory = scalar field on **noncommutative spacetime.**

[Gelfand–Naimark, Mat. Sbornik. **12** (1943)]



topological spaces

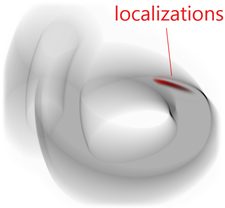
$\longleftrightarrow$

pure states: $\delta^{(n)}(x - x_0)$

$$f(x) \cdot g(x) = g(x) \cdot f(x)$$

Abelian algebras

$\uparrow$ *commutative limit*



noncommutative spaces

$\longleftrightarrow$

pure states:

$\varphi = a\,\varphi_1 + (1-a)\varphi_2 \;\Rightarrow\; \varphi_1 = \varphi_2$

$$[f(x), g(x)] \neq 0$$

non-Abelian algebras

[Connes, Noncommutative Geometry (1994)]

Quantum groups

$[\hat{x}^\mu, \hat{x}^\nu] = i\,\ell\,\varepsilon^{\mu\nu}{}_\rho\,\hat{x}^\rho$ does not break Poincaré invariance. It is invariant under a **quantum group/Hopf algebra** deformation of ISO(2,1).

Poincaré invariance = **comodule** property:

$$\hat{x}'^\mu = \Lambda^\mu{}_\nu \otimes \hat{x}^\nu + a^\mu \otimes \hat{1}\,, \qquad \boxed{[\hat{x}'^\mu, \hat{x}'^\nu] = i\,\ell\,\epsilon^{\mu\nu}{}_\rho\,\hat{x}'^\rho},$$

$$[\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] = [a^\mu, \Lambda^\rho{}_\sigma] = 0\,, \quad \boxed{[a^\mu, a^\nu] = i\,\ell\,\epsilon^{\mu\nu}{}_\rho\,a^\rho}.$$

$\Lambda^\mu{}_\nu$, $a^\mu \in$ noncomm. deformation of algebra of functions on ISO(2,1).

Group axioms (composition law, inverse, identity), must be **compatible** with $[a^\mu, a^\nu] \neq 0$ (*i.e.* homomorphisms), Then you have a **Hopf algebra.**

Examples of quantum groups

- $SU_q(2)$ [Woronowicz, Publ. Res. Inst. Math. Sci. **23** (1987)]
- $GL_q(N)$, $SL_q(N)$, $SU_q(N)$ $\boxed{\hat{T}^1{}_1\,\hat{T}^1{}_2 = q\,\hat{T}^1{}_2\,\hat{T}^1{}_1\,,\ldots}$
  [Faddeev *et al.*, Leningrad Math. J. **1** (1990)]
- $SO_q(p,q)$ [Aschieri, Lett. Math. Phys. **49** (1999)]
- **T-Poincaré:** 17 classes of models
  [Mercati, PTEP **2024** 073B06 & 123B05, arXiv:2404.08729, 2311.16249]
  - **θ-Poincaré** [Balachandran–Martone, MPLA **24**, 1811 (2009)]
  - **κ-Poincaré** [Lukierski *et al.* PLB **271**, 321 (1991)]
    [Ballesteros *et al.*, PLB **351**, 137 (1995)]
  - **ρ-Poincaré** [Lizzi–Scala–Vitale, PRD **106**, D106 (2022)]
  - **λ-Poincaré** [Gubitosi *et al.* PRD **105**, 126013 (2022)]

Works considered in this talk

1. Localization and reference frames in **κ-Minkowski/Poincaré**
   [Carotenuto-Lizzi-Mercati-Manfredonia, IJGMMP **19** (2022), arXiv:2011.10628]
   [Lizzi-Manfredonia-Mercati-Poulain, PRD **99** (2019), arXiv:1811.08409]

2. Localization and reference frames in **ρ-Minkowski/Poincaré**
   [Lizzi–Vitale, PLB **818** (2021), arXiv:2101.06633]
   [Lizzi–Scala–Vitale, PRD **106** (2022), arXiv:2205.10862]

3. Quantum Euler angles, quantum alignment protocols and "doubly quantum mechanics" in $\mathbf{SU_q(2)}$
   [Amelino-Camelia–D'Esposito–Fabiano–Frattulillo–Hoehn–Mercati, PTEP **2024** (2024), arXiv:2211.11347]
   [D'Esposito–Fabiano–Frattulillo–Mercati, Quantum **9** (2025), arXiv:2412.05997]

4. Hopf algebra associated to quantum reference frame transformations
   [Ballesteros–Giacomini–Gubitosi, Quantum **5** (2021), arXiv:2012.15769]
   [Ballesteros–Fernandez-Silvestre–Giacomini–Gubitosi, arXiv:2504.00569]

1. κ-localization & reference frames

"Timelike" κ-Minkowski spacetime:

$$[\hat{\boldsymbol{x}}^0, \hat{\boldsymbol{x}}^i] = \frac{i}{\boldsymbol{\kappa}}\,\hat{\boldsymbol{x}}^i\,, \quad [\hat{\boldsymbol{x}}^i, \hat{\boldsymbol{x}}^j] = 0\,, \qquad \boldsymbol{\kappa} \in \mathbb{R}\,.$$

- Representation: on $\mathcal{H}_{\kappa-\mathrm{Mink}} = L^2(\mathbb{R}^3)$:

$$\hat{\boldsymbol{x}}^i \to x^i\,, \qquad \hat{\boldsymbol{x}}^0 \to \frac{i}{\boldsymbol{\kappa}}\left(\vec{x}\cdot\vec{\nabla} + \tfrac{3}{2}\right) = \frac{i}{\boldsymbol{\kappa}}\left(r\,\partial_r + \tfrac{3}{2}\right)\,,$$

- Spectra: $\sigma(\hat{\boldsymbol{x}}^\mu) = \mathbb{R}\ \ \forall\mu$. No point spectrum.
- Improper eigenfunctions of $\hat{\boldsymbol{x}}^0$: $\quad \varphi_\tau(\vec{x}) = \boldsymbol{\kappa}^{-i\,\tau}\,|\vec{x}|^{-\frac{3}{2}-i\,\tau}$ complete basis ($\hat{\boldsymbol{x}}^0$ is selfadjoint).
- (r,θ,ϕ) and (τ,θ,ϕ) are two complete sets of commuting "observables". Mellin transform relates the two bases.

[Carotenuto-Lizzi-Mercati-Manfredonia, IJGMMP **19** (2022), arXiv:2011.10628]

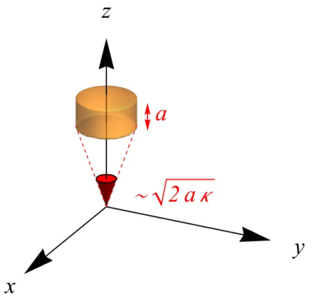
- **Log-Gaussians:** $e^{-\frac{\log^2(r/r_0)}{\sigma^2}} r^{i\tau_0}$ saturate uncertainty $\Delta\hat{\boldsymbol{x}}^0\,\Delta\hat{\boldsymbol{r}} \geq \frac{\langle\hat{\boldsymbol{r}}\rangle}{2\boldsymbol{\kappa}}$.

- In the limit $r_0 \to 0$ and $\sigma \to \infty$, while $r_0\,e^{-\sigma^2} \to 0$, the state becomes **perfectly localized** on the **temporal axis,** $r = 0$, $\tau = \tau_0$.

- "Cilindretto" state centred at $(0, 0, z_0)$.

  Mellin transform: $\Delta\tau \sim \frac{z_0}{2\,a}$.

  $a \to 0$ or $z_0 \to \infty$ yield a non-normalizable Mellin transform (infinitely delocalized in time).



[Lizzi–Manfredonia–Mercati–Poulain, PRD **99** (2019)]

κ-Poincaré group

$$[\hat{\boldsymbol{a}}^\mu, \hat{\boldsymbol{a}}^\nu] = \tfrac{i}{\boldsymbol{\kappa}} \left(\delta^\mu{}_0\, \hat{\boldsymbol{a}}^\nu - \delta^\nu{}_0\, \hat{\boldsymbol{a}}^\mu\right) , \qquad [\hat{\boldsymbol{\Lambda}}^\mu{}_\nu, \hat{\boldsymbol{\Lambda}}^\rho{}_\sigma] = 0 \,,$$

$$[\hat{\boldsymbol{\Lambda}}^\mu{}_\nu, \hat{\boldsymbol{a}}^\rho] = \tfrac{i}{\boldsymbol{\kappa}} \left[\hat{\boldsymbol{\Lambda}}^\mu{}_0\, \hat{\boldsymbol{\Lambda}}^\rho{}_\nu + \eta^{\mu\rho}\, \hat{\boldsymbol{\Lambda}}^0{}_\nu - \delta^\mu{}_0\, \hat{\boldsymbol{\Lambda}}^\rho{}_\nu - \delta^0{}_\nu\, \eta^{\mu\rho}\right] .$$

Representation: $\mathcal{H}_{\kappa-\text{Poinc}} = L^2[SO(3,1) \times \mathbb{R}^3]$. In 1+1 dimensions:

$$\hat{\boldsymbol{a}}^0 \to \frac{i}{\boldsymbol{\kappa}} \left(\frac{1}{2} + q^1 \frac{\partial}{\partial q^1}\right) + \frac{i}{\boldsymbol{\kappa}} \left(\frac{1}{2} \cosh\xi + \sinh\xi \frac{\partial}{\partial \xi}\right) ,$$

$$\hat{\boldsymbol{a}}^1 \to q^1 + \frac{i}{\boldsymbol{\kappa}} \left(\frac{1}{2} \sinh\xi + (\cosh\xi - 1) \frac{\partial}{\partial \xi}\right) ,$$

$$\hat{\boldsymbol{\Lambda}}^\mu{}_\nu \to \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} .$$

Proposed physical interpretation

Alice's coordinates: $\hat{\boldsymbol{x}}^\mu$, Bob's coordinates: $\hat{\boldsymbol{x}}'^\mu = \boldsymbol{\Lambda}^\mu{}_\nu \otimes \hat{\boldsymbol{x}}^\nu + \boldsymbol{a}^\mu \otimes \hat{\mathbf{1}}$.

A spacetime event (*e.g.* a detector clicking) will be described in terms of expectation values and higher momenta:

$$\text{Alice: } \langle \hat{\boldsymbol{x}}^\mu \rangle, \langle \hat{\boldsymbol{x}}^\mu \hat{\boldsymbol{x}}^\nu \rangle, \langle \hat{\boldsymbol{x}}^\mu \hat{\boldsymbol{x}}^\nu \hat{\boldsymbol{x}}^\rho \rangle \dots$$

$$\text{Bob: } \langle \hat{\boldsymbol{x}}'^\mu \rangle, \langle \hat{\boldsymbol{x}}'^\mu \hat{\boldsymbol{x}}'^\nu \rangle, \langle \hat{\boldsymbol{x}}'^\mu \hat{\boldsymbol{x}}'^\nu \hat{\boldsymbol{x}}'^\rho \rangle \dots$$

Bob's expectation values are taken on $\mathcal{H}_{\kappa-\text{Poinc}} \otimes \mathcal{H}_{\kappa-\text{Mink}}$.
Separable states on the $\hat{\boldsymbol{x}}'^\mu$ algebra:

$$|g\rangle \otimes |\psi\rangle \in \mathcal{H}_{\kappa-\text{Poinc}} \otimes \mathcal{H}_{\kappa-\text{Mink}}$$

represent transformed reference frames: no *a priori* reason to entangle states of transformation and of coordinates. Dynamics might change that.

- “Identity state” $|e\rangle$, perfectly localized at $\hat{\boldsymbol{a}}^\mu = \hat{\xi} = 0$. It connects two **coincident reference frames.**

- Cannot localize translations around $\hat{\boldsymbol{a}}^\mu = 0$, unless $\hat{\xi} = 0$ also. In 3+1D, the only “pure Lorentz transformations” with $\langle \hat{\boldsymbol{a}}^\mu \rangle = \Delta \hat{\boldsymbol{a}}^\mu = 0$ states are **pure spatial rotations.** No “pure boost” states. First observed in [Amelino-Camelia *et al.*, PLB **671** (2009)]

- Poincaré-transforming the κ-Minkowski spacetime origin state $|o\rangle$ as $|g\rangle \otimes |o\rangle$, all statistical properties of the translation operators transfer to the coordinates: $\langle g, o | \hat{\boldsymbol{x}}'^\mu \hat{\boldsymbol{x}}'^\nu \dots | g, o \rangle = \langle g | \hat{\boldsymbol{a}}^\mu \hat{\boldsymbol{a}}^\nu \dots | g \rangle$

- Poincaré-transforming with the identity state: Alice and Bob agree on all localization and statistical properties of the event $\langle e, \psi | \hat{\boldsymbol{x}}'^\mu \hat{\boldsymbol{x}}'^\nu \dots | e, \psi \rangle = \langle \psi | \hat{\boldsymbol{x}}^\mu \hat{\boldsymbol{x}}^\nu \dots | \psi \rangle$,

- “Pure translation” states, localized around $\hat{\xi} = 0$, exist and are identical to the κ-Minkowski states, as far as $\hat{\boldsymbol{a}}^\mu$ are concerned.

Uncertainty from transformed reference frames

A **pure translation** state always increases the variance of $\hat{\boldsymbol{x}}^\mu$:

$$\Delta(\hat{\boldsymbol{x}}'^\mu)^2 = \Delta(\hat{\boldsymbol{x}}^\mu)^2 + \Delta(\hat{\boldsymbol{a}}^\mu)^2 \geq \Delta(\hat{\boldsymbol{x}}^\mu)^2 \,,$$

all translations, *except perfectly-localized purely temporal ones* $\Delta\hat{\boldsymbol{a}}^\mu = 0$, increase the fuzzyness of the event.

You cannot undo uncertainty: cannot translate from Bob to a third observer who agrees with Alice on all her measurements.

When boosts are involved, one can decrease the uncertainty of **one** coordinate though Lorentz contractions.

No invariant notion of locality: the "sharpness" of an event depends on the reference frame.

[Lizzi–Manfredonia–Mercati–Poulain, PRD **99** (2019)]

2. ρ-localization & reference frames

[Lizzi–Vitale, PLB **818** (2021)] [Lizzi–Scala–Vitale, PRD **106** (2022)]

$$[\hat{\boldsymbol{x}}^0, \hat{\boldsymbol{x}}^1] = i\,\varrho\,\hat{\boldsymbol{x}}^2\,, \qquad [\hat{\boldsymbol{x}}^0, \hat{\boldsymbol{x}}^2] = -i\,\varrho\,\hat{\boldsymbol{x}}^1\,, \qquad [\hat{\boldsymbol{x}}^3, \cdot\,] = [\hat{\boldsymbol{x}}^1, \hat{\boldsymbol{x}}^2] = 0\,,$$

cylindrical coordinates: $[\hat{\boldsymbol{x}}^0, e^{i\,\hat{\boldsymbol{\varphi}}}] = \varrho\, e^{i\,\hat{\boldsymbol{\varphi}}}\,, \qquad [\hat{\boldsymbol{r}}, \cdot\,] = [\hat{\boldsymbol{x}}^3, \cdot\,] = 0\,.$

- Complete set of commuting operators: $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{x}}^3, \hat{\boldsymbol{\varphi}})$ or $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{x}}^3, \hat{\boldsymbol{x}}^0)$.
- $\hat{\boldsymbol{\varphi}}$ has compact spectrum and $\sigma(\hat{\boldsymbol{x}}^0) = \varrho(\mathbb{Z} + \alpha)$, $\alpha \in (0, 2\pi)$.
- **Have to choose a self-adjoint extension for $\hat{\boldsymbol{x}}^0$**, *e.g.* $\alpha = 0$.
- **Uncertainty bounds** $\rightarrow$ the only sharply localized states are on the $z - x^0$ plane.

$$[\hat{\boldsymbol{a}}^\mu, \hat{\boldsymbol{a}}^\nu] = 2\, i \varrho \left(\delta^\nu{}_0 \hat{\boldsymbol{a}}_{[1} \delta^\mu{}_{2]} - \delta^\mu{}_0 \hat{\boldsymbol{a}}_{[1} \delta^\nu{}_{2]} \right), \qquad [\hat{\boldsymbol{\Lambda}}^\mu{}_\nu, \hat{\boldsymbol{\Lambda}}^\rho{}_\sigma] = 0,$$

$$[\hat{\boldsymbol{\Lambda}}^\mu{}_\nu, \hat{\boldsymbol{a}}^\rho] = -2\, i \varrho \left(\delta^\rho{}_0 \delta^\mu{}_{[1} \hat{\boldsymbol{\Lambda}}_{2]\nu} - \hat{\boldsymbol{\Lambda}}^\rho{}_0 \hat{\boldsymbol{\Lambda}}^\mu{}_{[1} \eta_{2]\nu} \right),$$

representation of $\hat{\boldsymbol{a}}^\mu$ as ρ-Minkowski ⊕ vector fields on SO(3,1):

$$\hat{\boldsymbol{a}}^\rho \to i \varrho \left(\delta^\rho{}_0 \delta^\mu{}_{[1}\, \Lambda_{2]\nu} - \Lambda^\rho{}_0\, \Lambda^\mu{}_{[1}\, \eta_{2]\nu} \right) \frac{\partial}{\partial \Lambda^\mu{}_\nu} + i \varrho \left(\delta^\rho{}_i q^i - 2 \delta^\rho{}_0 q^{[1} \frac{\partial}{\partial q^{2]}} \right) + h.c.\,,$$

- ∃ identity state s.t. $\langle e | f(\hat{\boldsymbol{\Lambda}}^\mu{}_\nu, \hat{\boldsymbol{a}}^\rho) | e \rangle = f(\delta^\mu{}_\nu, 0)$,
- Unlike κ, pure Lorentz transformation states localized at $\hat{\boldsymbol{a}}^\mu = 0$ exist only for **pure rotations around the z axis.**
- Like κ, pure-translation states exist and match ρ-Minkowski states.
- The only perfectly localized states are **pure time translations** and pure **translations along the z axis.**
- Combinations of the three can be sharply localized too.
- Regarding uncertainty growth, **everything we found in κ holds here too.**

3. SUq(2) quantum Euler angles

Recall $\mathbf{SU_q(2)}$ (here we assume $q \in \mathbb{R}$):

$$\hat{U} = \begin{pmatrix} \hat{a} & -q\,\hat{c}^* \\ \hat{c} & \hat{a}^* \end{pmatrix}, \quad \hat{a}\,\hat{c} = q\,\hat{c}\,\hat{a} \qquad \hat{a}\,\hat{c}^* = q\,\hat{c}^*\,\hat{a} \qquad \hat{c}\,\hat{c}^* = \hat{c}^*\,\hat{c}$$

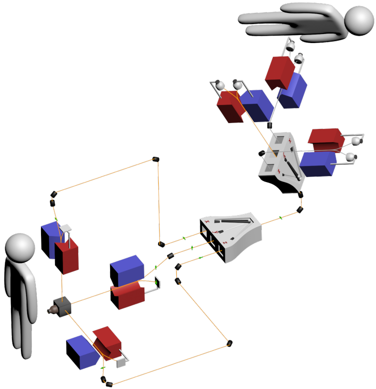
$$\hat{c}^*\,\hat{c} + \hat{a}^*\hat{a} = \hat{\mathbf{1}} \qquad \hat{a}, \hat{a}^* - \hat{a}^*\hat{a} = (1 - q^2)\hat{c}^*\hat{a}\,.$$

[Amelino-Camelia *et al.*, PTEP **2024** (2024), arXiv:2211.11347]:

Spin-1 (co-)representation: $\hat{\boldsymbol{R}}_{ij} = \frac{1}{2}\mathrm{tr}\left(\sigma_j\,\hat{U}^\dagger\sigma_i\hat{U}\right)$.

Components of 3D rotation matrix do not commute with each other,
$[\hat{\boldsymbol{R}}_{xx}, \hat{\boldsymbol{R}}_{xy}] \neq 0\,, \quad [\hat{\boldsymbol{R}}_{zy}, \hat{\boldsymbol{R}}_{xy}] \neq 0\,, \ldots$

Two rotated labs can exchange N electrons in eigenstates of σ_x, σ_y and σ_z in order to determine their relative orientation. In commutative spacetime, in the large-N limit, their relative Euler angles can be determined exactly. If spacetime is noncommutative, these angles are incompatible observables



Thought experiment is completely independent of the energy of the electrons. N is the multiplier that magnifies the noncommutativity effects (q could be related to Λ/M_P^2 ratio, [Major–Smolin, NPB **473** (1996)]).

[D'Esposito *et al.*, Quantum **9** (2025), arXiv:2412.05997]

To have a consistent framework, spinors describing spin-1/2 states need to live on a noncommutative generalization of $\mathbb{C}^2$:

$$\hat{\boldsymbol{\psi}} = \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \end{pmatrix}, \qquad \hat{\boldsymbol{x}}\,\hat{\boldsymbol{y}} = \boldsymbol{q}\,\hat{\boldsymbol{y}}\,\hat{\boldsymbol{x}}\,, \qquad \hat{\boldsymbol{\psi}}'_\alpha = \hat{\boldsymbol{U}}_{\alpha\beta} \otimes \hat{\boldsymbol{\psi}}_\beta\,.$$

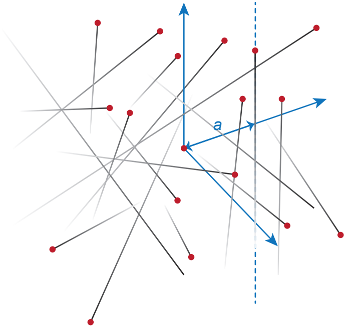
- Following this logic, what in commutative spacetime one would call the probability of outcomes of Stern–Gerlach experiments, has to be promoted to self-adjoint operator on Hilbert space, $P(\uparrow) \to \hat{P}(\uparrow)$.

- Framework admits semiclassical superpositions of "probability eigenstates": **Doubly quantum mechanics.**

Implications

- **Framework is Covariant & relational:**
  All predictions are $\mathbf{SU_q(2)}$-covariant. Reference frame changes are genuine quantum operations.
- **No classical background:**
  Removes the need for any classical geometry in the description—apparatus and observer are fully quantum.
- **Probabilities acquire quantum uncertainty:**
  Even "probability" is observer- and context-dependent, with its own quantum fluctuations.
- **Test-bed for quantum gravity:**
  This framework explicitly realizes a quantum theory where *all* physical reference structures are quantized.

4. Quantum "Taitian" reference frames



"Taitian" reference frames:

[Tait, Proc. R. Soc. Edinb. **11** (1884)]

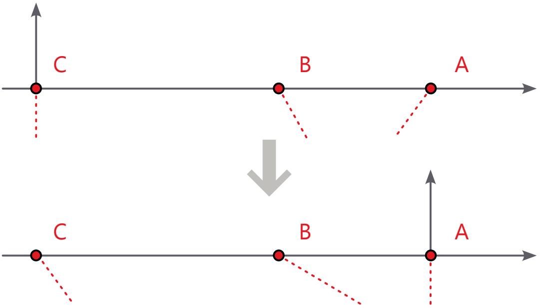
Inertial reference frames attached to **physical degrees of freedom:** two inertial particles, one acting as origin and the other providing orientation and acting as Neumann's inertial clock.

Assuming a relational / perspective neutral framework, one finds surprising features: entanglement and superposition are frame-dependent, causal order can be in superposition, etc.
[Giacomini–Castro-Ruiz–Brukner, NJP **18** (2016)]
[Vanrietvelde–Hoehn–Giacomini–Castro-Ruiz, Quantum **4** (2020)].
...

Hopf algebra from Taitian reference frames

[Ballesteros–Giacomini–Gubitosi, Quantum **5** (2021)]

1+1D non-relativistic point particles (commutative time):



to go from C's reference frame to A's, need to translate/boost B with:

$$\hat{U}_{\mathsf{transl}} = \exp\left(\frac{i}{\hbar}\hat{x}_A \otimes \hat{p}_B\right), \qquad \hat{U}_{\mathsf{boost}} = \exp\left(\frac{i}{\hbar}\frac{\hat{p}_A}{m_A} \otimes \hat{K}_B\right),$$

generalizes Galilean transformations to operator-valued transformation parameters. If A is in a superposition, so is the transformation.

Recursively commuting $\hat{x}_A \otimes \hat{p}_B$ and $\frac{\hat{p}_A}{m_A} \otimes \hat{K}_B$, one is able to close a **1-parameter family of 7D Lie algebras** $\mathcal{D}(7)$, parametrized by t.

[Opanowicz, J. Phys. **A** 31(1998)]

There are 26 quantum group deformations of the 1+1D Galilei group.

[Ballesteros *et al.*, arXiv:2504.00569]

One of these deformations with commutative time, at first order in the deformation parameter $\boldsymbol{\alpha}$ (the noncommutativity length scale) reproduces the Lie algebra $\mathcal{D}(7)$.

Quantum group parameters		**Dual Hopf algebra generators**	
$\hat{b}$	time translations	$\hat{P}_0$	0-momentum
$\hat{a}$	spatial translations	$\hat{P}_1$	1-momentum
$\hat{v}$	Galilean rapidity	$\hat{K}$	Galilean boost
$\hat{\theta}$	Bargmann phase	$\hat{M}$	mass central extension

represented as $f(\hat{q}_A, \hat{p}_A)$ (left columns); represented as $f(\hat{q}_B, \hat{p}_B)$ (right columns)

Quantum group parameters		Dual Hopf algebra generators	
$\hat{b}$	time translations	$\hat{P}_0$	0-momentum
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$\hat{\theta}$	Bargmann phase	$\hat{M}$	mass central extension

represented as $f(\hat{q}_A,\hat{p}_A)$ — represented as $f(\hat{q}_B,\hat{p}_B)$

The exponents in the quantum group exponential formula: $e^{\hat{\theta}\otimes\hat{M}}e^{\hat{b}\otimes\hat{P}_0}e^{\hat{a}\otimes\hat{P}_1}e^{\hat{v}\otimes\hat{K}}$ close the $\mathcal{D}(7)$ algebra, if expanded at first order in the noncommutativity length scale $\boldsymbol{\alpha}$.

The correspondence depends on assuming $\boldsymbol{\alpha} \propto \frac{1}{m_A}$. The Galilei algebra is recovered in the limit $m_A \to \infty$ (no "backreaction").

The full, all orders in $\boldsymbol{\alpha}$, quantum group seems to provide deformations of standard quantum mechanics suppressed by m_A^{-1}.

References

- Quantum groups (in particular $\mathbf{SU_q(2)}$)
[Chari–Pressley, *A Guide to Quantum Groups*, CUP (1994)]
- T-Minkowski spacetimes & T-Poincaré quantum groups
[Mercati, PTEP **2024** 073B06, arXiv:2404.08729]
[Mercati, PTEP **2024** 123B05, arXiv:2311.16249]
- Localization and reference frames in $\boldsymbol{\kappa}$ and $\boldsymbol{\rho}$
[Carotenuto-Lizzi-Mercati-Manfredonia, IJGMMP **19** (2022), arXiv:2011.10628]
[Lizzi-Manfredonia-Mercati-Poulain, PRD **99** (2019), arXiv:1811.08409]
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[Lizzi–Scala–Vitale, PRD **106** (2022), arXiv:2205.10862]
- Quantum alignment protocols and "doubly quantum mechanics" in $\mathbf{SU_q(2)}$
[Amelino-Camelia–D'Esposito–Fabiano–Frattulillo–Hoehn–Mercati, PTEP **2024** (2024), arXiv:2211.11347]
[D'Esposito–Fabiano–Frattulillo–Mercati, Quantum **9** (2025), arXiv:2412.05997]
- Hopf algebra associated to quantum reference frame transformations
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