New results on fibers and bases of elliptic Calabi Yaus for F-theory

String Geometry 2025
ICTP, Trieste, Italy

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Washington (Wati) Taylor, MIT

Based on work (to appear) with

Fatima Abbasi

Richard Nally

Main Result

Complete classification of toric elliptic fibers and bases in the KS database

Outline:

1. Intro and CY basics

2. Ingredients: toric bases and fibers

3. Results: statistics and preliminary highlights

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Intro on Calabi-Yau threefolds: manifolds used in superstring compactification

physically:

- Ricci flat: $R_{\mu\nu} = 0$ (solve vacuum Einstein equations)
- Kähler manifolds (complex structure compatible with SUSY)

mathematically: trivial canonical class $K = 0$ (up to torsion)

Long studied by mathematicians and physicists

— Used in compactification of string theory: 10D $\rightarrow$ 4D, 6D, ...

Open Question:
Are there a finite number of topological types of Calabi-Yau threefolds?

Many large classes of CY3s have been constructed:

— CICY (complete intersection CYs): 7,890 [Candelas/Dale/Lutken/Schimmrigk]
— Toric hypersurface CY3s: 473.8M reflexive 4D polytopes [Kreuzer/Skarke]
— Generalized CICYs [Anderson/Apruzzi/Gao/Gray/Lee, ...]
— Elliptic CY3s [Grassi, Gross, ...]

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An *elliptic* or *genus one fibered* CY3 X:
A torus (fiber) at each $p \in B_2$
$\pi : X \to B_2$

Elliptic: $\exists$ section $\sigma : B_2 \to X, \pi\sigma = \mathrm{Id}$

- Elliptic Calabi-Yau threefold has Weierstrass model

$$y^2 = x^3 + fx + g, \quad f, g \text{ sections of } \mathcal{O}(-4K_B), \mathcal{O}(-6K_B)$$

- Elliptic CY3s have extra structure, more manageable mathematically
- Used for 6D F-theory construction (fiber τ = 10D axiodilaton)
- Evidence has been accumulating that
  most known Calabi-Yau threefolds have elliptic/g1 structure
  (i.e. birationally equivalent to an elliptic or genus one fibered CY3)
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Classifying elliptic Calabi-Yau threefolds

$\exists$ finite # of topological types of elliptic CY3s [Gross]

Combining math + physics (F-theory), can systematically construct ECY3s

- Bases B_2: must be $\mathbb{F}_m, \mathbb{P}^2$, Enriques or blow-ups [Grassi]
- Finite number of distinct strata in space of B_2 Weierstrass models [Gross, Kumar/Morrison/WT]

"Algorithm" for constructing all ECY3s:

I. Construct bases by iterative blow-ups of $\mathbb{F}_m, \mathbb{P}^2$ [large $h^{2,1}$ ✓]

II. Tune Weierstrass models

A. Codimension 1 singularities $\leftrightarrow$ nonabelian G (Kodaira) [mostly ✓]
B. Mordell-Weil rank $\leftrightarrow U(1)^k$ [$k \leq 2$ ✓]
C. Discrete G ($\sim$ multisections) [no systematics yet X]
D. Codim. 2 singularities $\leftrightarrow$ matter [generic NA ✓, exotic, $U(1) \times G'$?]

Question: how do these elliptic CY's fit with known general CY's

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Game plan

- Consider toric hypersurface constructions (KS)
- Toric ingredients:
  - — Toric fibers (16)
  - — Toric bases (61,539)

Identify all "obvious" toric elliptic fibrations in the KS database

$\rightarrow$ 2.2 billion fiber-base combinations

Gives us a huge range of interesting examples to explore!

Elliptic fibrations may be our best tool for organizing and understanding CY's

Toric ingredients: hypersurface construction [Batyrev, Kreuzer/Skarke]

Toric variety: characterized by toric divisors $D_i \leftrightarrow$ rays $v_i \in \mathbb{Z}^d$, fan (cones)

Anti-canonical class $-K = \sum_i D_i$ (never compact CY)

Anti-canonical hypersurface $\Rightarrow$ CY by adjunction

∇ polytope: convex hull of vertices v_i

{monomials} $\leftrightarrow$ lattice points in *dual polytope* $\Delta = \nabla^* = \{w : w \cdot v \geq -1\}$

Batyrev: $\nabla = \nabla^{**}$ reflexive $\leftrightarrow$ 1 interior point
$\leftrightarrow$ hypersurface CY generically smooth (avoids singularities)

Kreuzer-Skarke: constructed all 473.8M reflexive ∇_4

∇, Δ describe mirror Calabi-Yau threefolds $h^{1,1} \leftrightarrow h^{2,1}$

Symmetry in toric hypersurface construction early evidence for mirror symmetry

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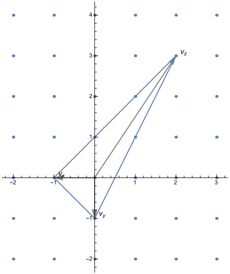
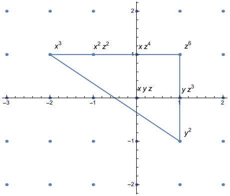
Example: Batyrev for generic elliptic curve in $P^{2,3,1}$

∇





Δ

Gives general Weierstrass ("Tate form") model for elliptic curve:

$$y^2 + a_1 yx + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

Completing square, cube $\to$ standard (short) Weierstrass form

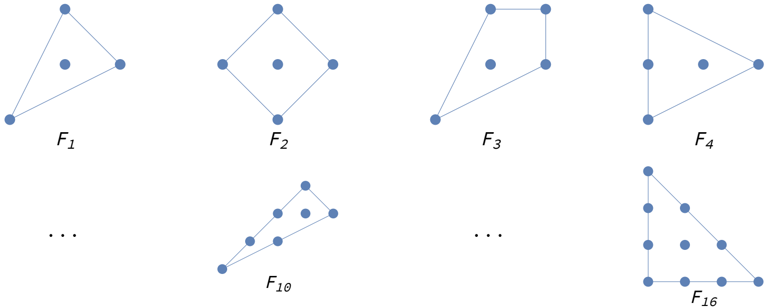
$$y^2 = x^3 + fx + g$$

More generally: a reflexive 2D subpolytope $\to$ elliptic fibration (in some – possibly vex – flop phase) (fibration $\leftrightarrow$ projection)

Ingredients: toric fibers

$\nabla_2 \subset \nabla$, ∇_2 reflexive

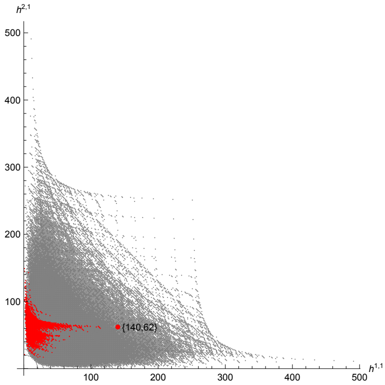
Only 16 reflexive ∇_2's; can do Kodaira/Nagel/Weierstrass on each
[Braun, Braun/Grimm/Keitel, Klevers/Mayorga Pena/Oehlmann/Piragua/Reuter])



-1 curve $C = D_i^{(2)}$: satisfies $-K \cdot C = C \cdot C + 2 = 1$, $v_i = v_{i-1} + v_{i+1}$

All but $F_1 = \mathbb{P}^2, F_2 = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1, F_4 = \mathbb{F}_2$ have -1 curves $\Rightarrow$ toric sections
Some fibers have multiple sections $\Rightarrow$ U(1)'s in F-theory

Huang/WT 2019: all but 29,223 4D reflexive polytopes have 2D fiber $\nabla_2 \subset \nabla$



Largest $h^{2,1}$ without toric fiber: $(140, 62)$

Similar results for CICYs [Anderson/Gao/Gray/Lee]:

- 99.3% (7837/7890) of CICYs have "obvious" elliptic/g1 fibration

Ingredients: toric bases

Complete classification of toric bases [Morrison/WT]

- Finite # bases, all from blowing up $\mathbb{P}^2$ or Hirzebruch $\mathbb{F}_m \leq 12$ [Grassi, Gross]
- Bases characterized by cone of "effective" curves C_i w/ intersections $C_i \cdot C_j$

Blow-up sequence of toric B

- Construction terminates with -13 curves (non-minimal)

Result: 61,539 toric bases (including -9, -10, -11 curves)

Note: also substantial progress on classifying non-toric bases
[Martini/WT, WT/Wang, Kim/Vafa/Xu]

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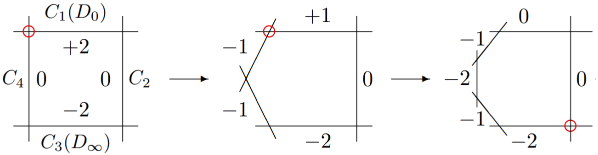

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Classification of all KS fibers + bases

With Fatima Abbasi, Richard Nally:

Complete classification of all 2D subpolytopes $\nabla_2 \subset \nabla$ [cf. Braun 2011] and all associated bases

483 M KS polytopes $\rightarrow$ 2.25 B fibration structures (2.26B w/ multiplicities)

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(including singular versions with rays missing)

Full results only two weeks old, today: some initial highlights

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Distribution of fibrations

- Fibrations per polytope on average: ~ 4.7
  (note 9.85 average obvious fibrations for CICYs [AGGL])
- Most fibrations: 362 (Hodge numbers (68, 4); 10 distinct bases)
- Most fibrations at small $h^{2,1}(X)$

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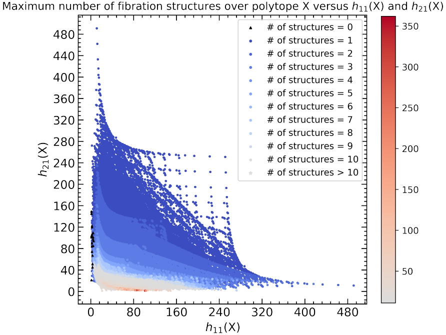

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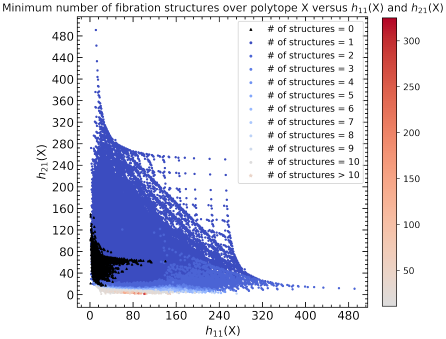

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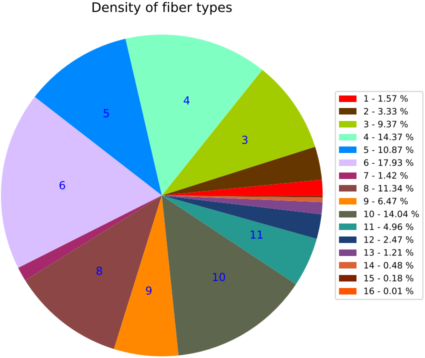

- Most common fibers:

F_6 : 406 M fibers (generic extra U(1))
F_4 : 325 M fibers (no section, $\mathbb{Z}_4$ symmetry)
F_{10} : 318 M fibers (standard Tate + generalizations)

- Least common fiber: F_{16} : 185 K fibers (more later)

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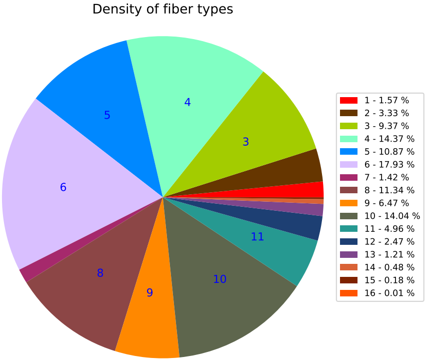

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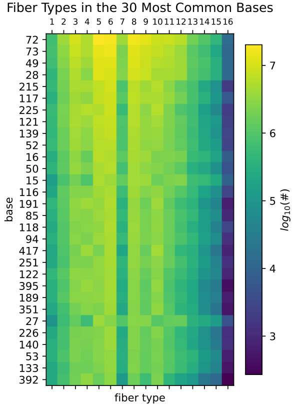
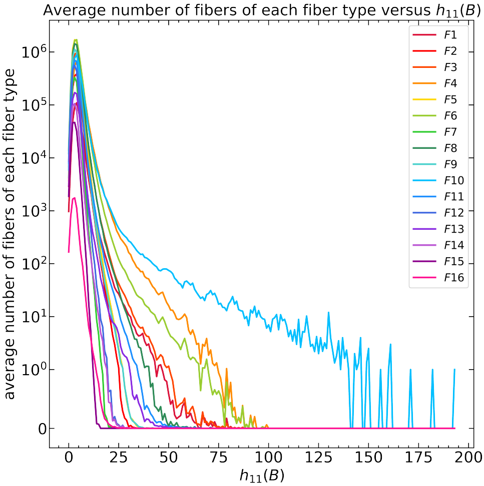
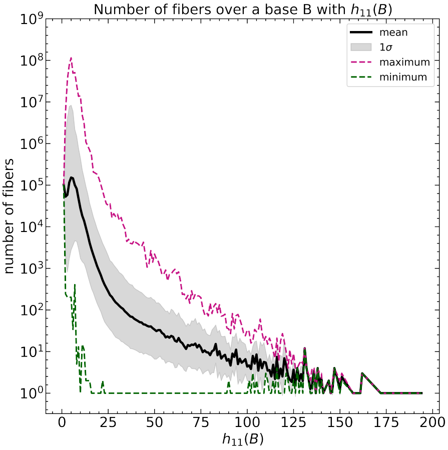
Distribution of fibers, continued





Distribution of bases: Number of fibrations greatest for small $h^{1,1}(B)$



Makes sense: expect bases with small $h^{1,1}(B)$ have most moduli ($h^{2,1}(X)$ for generic fibration), admit most tuning options

Examples I: popular bases

• Consider the base $B = \mathbb{P}^2$

Studied by [Braun 2011]

Braun: 102,581 distinct fibrations.
We find: 102,565 structures, 102,603 w/multiplicities (compatible)

• The base with the most fibrations:

Base 72: a gdP4 (-2, -1, -2, -1, -1, -1), 68 M structures, 115 M w/multiplicities

Other bases with O(50 M) fibrations are similar.

• Bases with e.g. -12 curves $\rightarrow$ rigid E_8's: fewer fibrations

e.g. $\mathbb{F}_{12}$: 210 structures, 242 fibrations (w/multiplicities)

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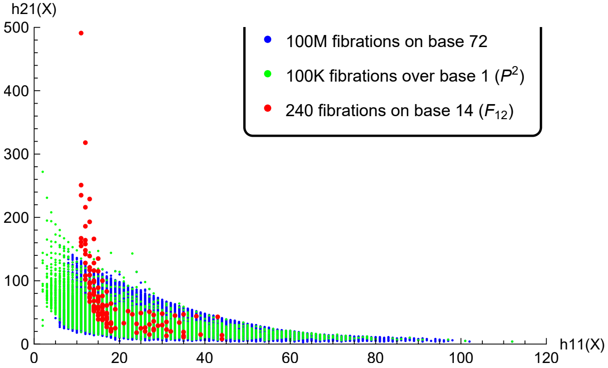
Hodge numbers of fibrations over selected bases



Generic fibration over $\mathbb{P}^2$: Hodge numbers (2, 272)
Generic fibration over $\mathbb{F}_{12}$: Hodge numbers (11, 491)
Generic fibration over base 72: Hodge numbers (6, 156)
Note: examples with $h^{1,1}(X) = 4, 5$ over singular reduced bases

Examples II: SCFTs (cf. talks by Ami, Craig, etc.)

In many cases, there are non-flat fibrations, where divisors $\to$ base points.

In general these are SCFT's [Heckman/Morrison/Rudelius/Vafa]
In these cases there are no toric resolutions on the base.

Simple example: over -9, -10, -11 curves there are 3, 2, 1 (4, 6) points on an E_8 divisor

In general, Shioda-Tate-Wazir says

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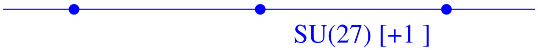
In general, Shioda-Tate-Wazir says (incorporating SCFT's)

$$h^{1,1}(X) = h^{1,1}(B) + \mathrm{rk}(G) + 1 + \Delta h^{1,1}(\mathrm{SCFT})$$

Interesting SCFT example: largest $h^{1,1}(X)$ fibration on $\mathbb{P}^2$: (112, 4) [Braun]

SU(27) on a line (+1 curve) has 3 non-flat fibers with 28 divisors.



Resolution: 3 SCFT's with SU(18) $\times$ SU(9) on a (non-toric) string (-2, -2, -1)
Satisfies anomaly cancellation, etc. (SU(N) on -2: $2N$ fundamentals)

Lesson: many toric hypersurface CY's have SCFT singularities,
resolution $\rightarrow$ non-toric bases

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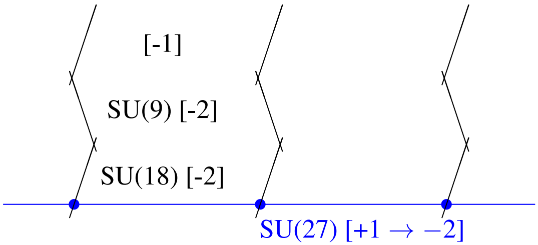

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Examples III: Largest $h^{1,1}$, (491, 11) polytope

We find two fibrations, matching known results:

Fibration A: standard F_{10} stacking, $h^{1,1}(B) = 193$

Base: ([-12, -1, -2, -2, -3, -1, -5, -1, -3, -2, -2, -1]^{16}, -12, 0)

(second -12 really -11 w/ a non-toric (4, 6) SCFT point)

$G = E_8^{17} \times F_4^{16} \times G_2^{32} \times SU(2)^{32}$
[Candelas/Perevalov/Rajesh, Morrison/WT, Kim/Vafa/Xu]

Fibration B: Fiber 13, $h^{1,1}(B) = 10$

Base: (-4, -1, <u>-3</u>, -1, -4, -1, -4, -1, -4, 0)

(<u>-3</u> has a non-toric (4, 6) SCFT point.)

$G = SO(64) \times Sp(56) \times (SO(176) \times Sp(40)) \times Sp(72) \times SO(128) \times Sp(48) \times SO(80) \times Sp(24) \times SO(32)$
[Aspinwall/Morrison, Kim/Vafa/Xu]

Examples III: Largest $h^{1,1}$, (491, 11) polytope

We find two fibrations, matching known results:

Fibration A: standard F_{10} stacking, $h^{1,1}(B) = 193$

Base: ([-12, -1, -2, -2, -3, -1, -5, -1, -3, -2, -2, -1]^{16}, -12, 0)

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Examples IV: tuning on non-toric divisors (beyond toric Tate)

Simplest tunings (e.g. Tate from standard $\mathbb{P}^{2,3,1}$ stacking) $\rightarrow G$ on toric D.

More general fibers, stacking $\rightarrow G$ on non-toric divisors
(implicit in [Klevers/Mayorga Pena/Oehlmann/Piragua/Reuter])

Example:

β

Δ

α

Gives general Weierstrass ("Tate form") model for elliptic curve:

$$\alpha y^2 + a_1 yx + a_3 y = \beta x^3 + a_2 x^2 + a_4 x + a_6$$

$\{\alpha = 0\} \rightarrow SU(2)$, $\{\beta = 0\} \rightarrow SU(3)$ $\Rightarrow$ e.g. SU(2) on $2H$ on $\mathbb{P}^2$

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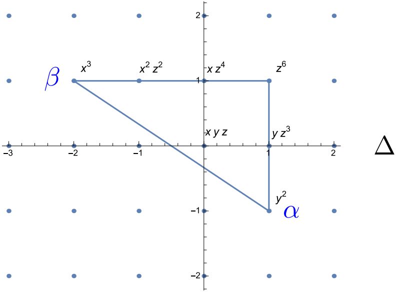

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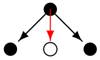
About 11 M ($\sim 0.5\%$) of fibrations have singularity in base



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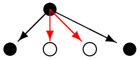

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Multi-sections $\rightarrow$ discrete G through Tate-Shafarevich/Weil-Chatalet [Braun/Morrison, Morrison/WT, . . .]

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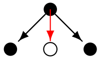

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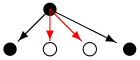

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Conclusions

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• The toric hypersurface construction provides abundant information about gauge groups on non-toric divisors and non-toric bases through SCFTs

Some further directions and open questions:

• How complete is the KS database for simple/complex tunings (nonabelian, abelian, discrete G, non-toric divisors; exotic matter)? For non-toric bases via SCFTs?

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Thank You!

We hope to see you in Boston for String Pheno 2025, July 7-11