Classification of Minimal Coulomb Branches

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Based on work with
M. Sperling & Z. Zhong [2312.05304]
Q. Lamouret, S. Moura Soysüren & M. Sperling [2504.05373]
and J. Grimminger, A. Hanany, S. Giacomelli, ...

Outline

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Inventiones mathematicae
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Symplectic singularities

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Introduction

We introduce in this paper a particular class of rational singularities, which we call *symplectic*, and classify the simplest ones. Our motivation comes from the analogy between rational Gorenstein singularities and Calabi-Yau manifolds: a compact, Kähler manifold of dimension n is a Calabi-Yau manifold if it admits a nowhere vanishing n-form, while a normal variety V of dimension n has rational Gorenstein singularities[1] if its smooth part V_{reg} carries a nowhere vanishing n-form, with the extra property that its pull-back in any resolution $X \to V$ extends to a holomorphic form on X. Among Calabi-Yau manifolds an important role is played by the symplectic (or hyperkähler) manifolds, which admit a holomorphic, everywhere non-degenerate 2-form; by analogy we say that a normal variety V has *symplectic singularities* if V_{reg} carries a closed symplectic 2-form whose pull-back in any resolution $X \to V$ extends to a holomorphic 2-form on X.

We discuss in §4 whether a classification of isolated symplectic singularities makes sense. Each such singularity gives rise to many others by considering its quotient by a finite group; to get rid of those we propose to consider only isolated symplectic singularities with trivial local fundamental group. The singularities $(\overline{\mathcal{O}}_{\min}, 0)$ have this property when the Lie algebra is not of type C_l; it is certainly desirable to find more examples.

(4.3) It would be interesting to find more examples of isolated symplectic singularities with trivial local fundamental group, and also examples with *infinite* local fundamental group.

Outline

SCFT
(with 8 susy)

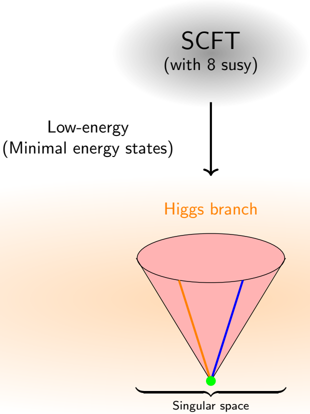
SCFT
(with 8 susy)

Low-energy
(Minimal energy states)

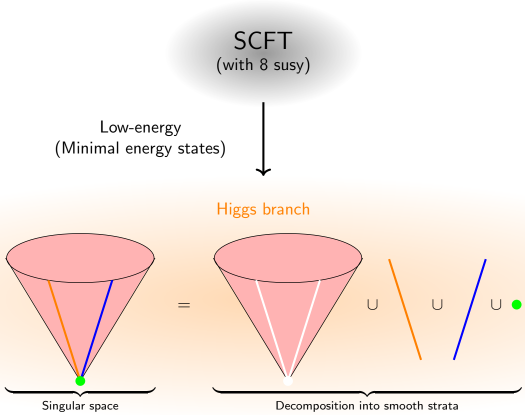
Higgs branch

IR geometry

IR geometry


SCFT
(with 8 susy)
Low-energy
(Minimal energy states)
Higgs branch
Singular space

IR geometry


SCFT
(with 8 susy)
Low-energy
(Minimal energy states)
Higgs branch
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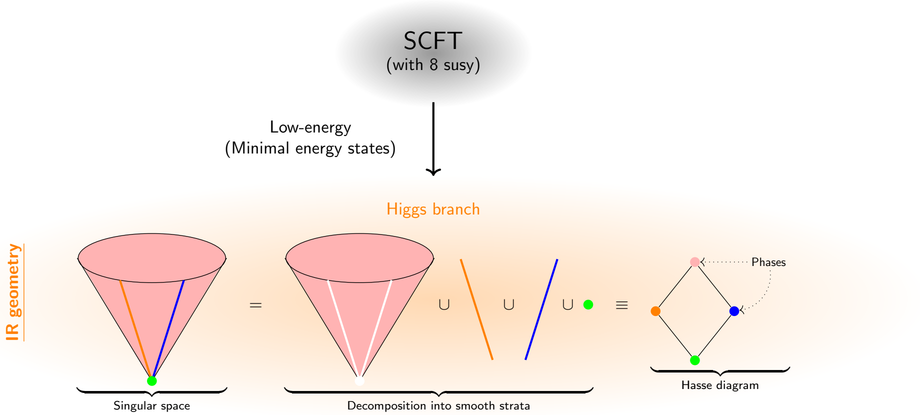
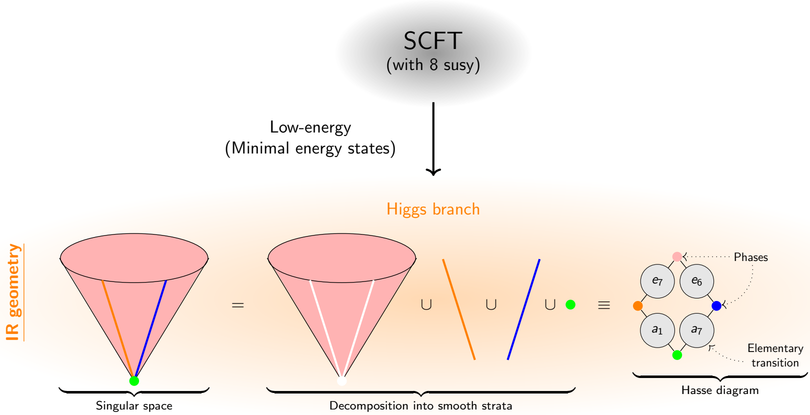
Singular space
Decomposition into smooth strata


SCFT
(with 8 susy)
Low-energy
(Minimal energy states)
Higgs branch
IR geometry
Phases
Singular space
Decomposition into smooth strata
Hasse diagram


SCFT
(with 8 susy)
Low-energy
(Minimal energy states)
Higgs branch
IR geometry
=
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∪
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≡
e_7
e_6
a_1
a_7
Phases
Elementary transition
Singular space
Decomposition into smooth strata
Hasse diagram

The question:

What are the elementary building blocks (=atoms) of Higgs branches (=molecules)?

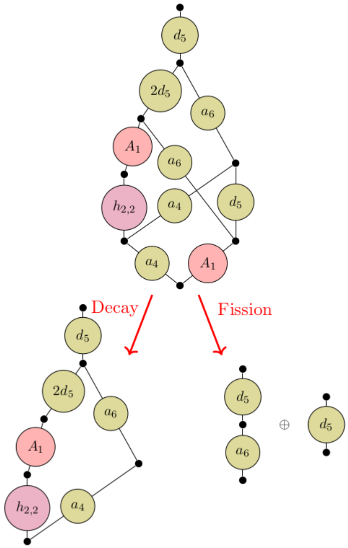
The strategy:

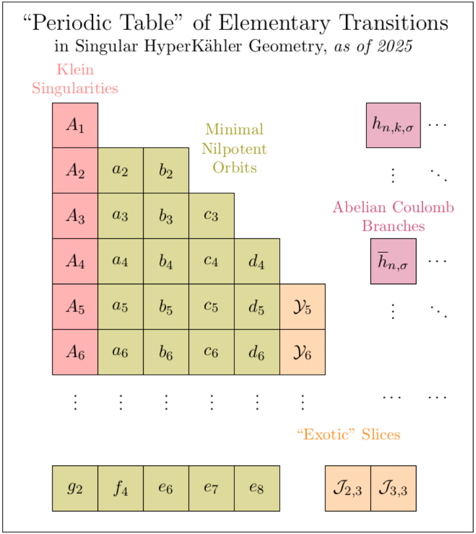
In order to find atoms, try to break into pieces until it is not possible.

(A related but distinct question: classify rank-1 theories)

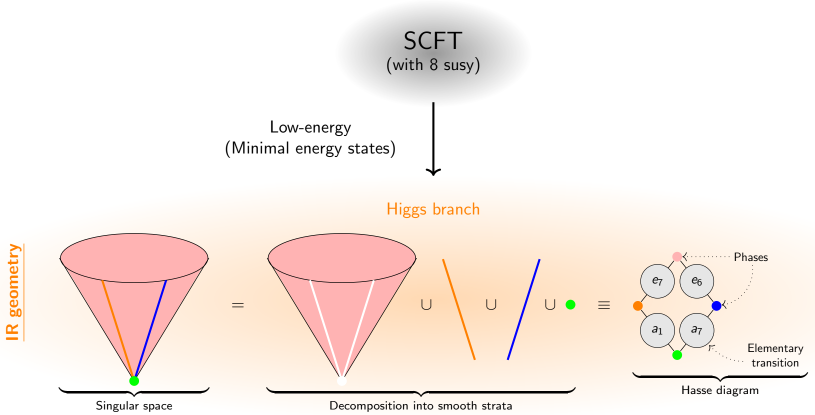
Outline

Breaking a symplectic singularity into pieces:





How to obtain this without knowing the stratification a priori?


SCFT
(with 8 susy)
Low-energy
(Minimal energy states)
Higgs branch
IR geometry
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Phases
e_7
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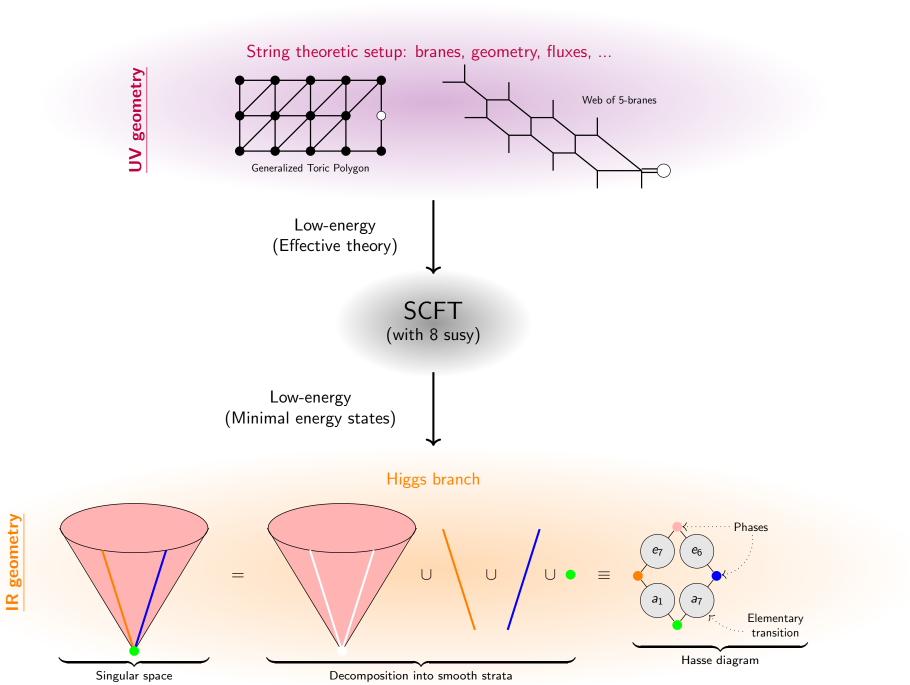
Elementary
transition
Singular space
Decomposition into smooth strata
Hasse diagram


UV geometry
String theoretic setup: branes, geometry, fluxes, ...
Generalized Toric Polygon
Web of 5-branes
Low-energy
(Effective theory)
SCFT
(with 8 susy)
Low-energy
(Minimal energy states)
IR geometry
Higgs branch
=
∪
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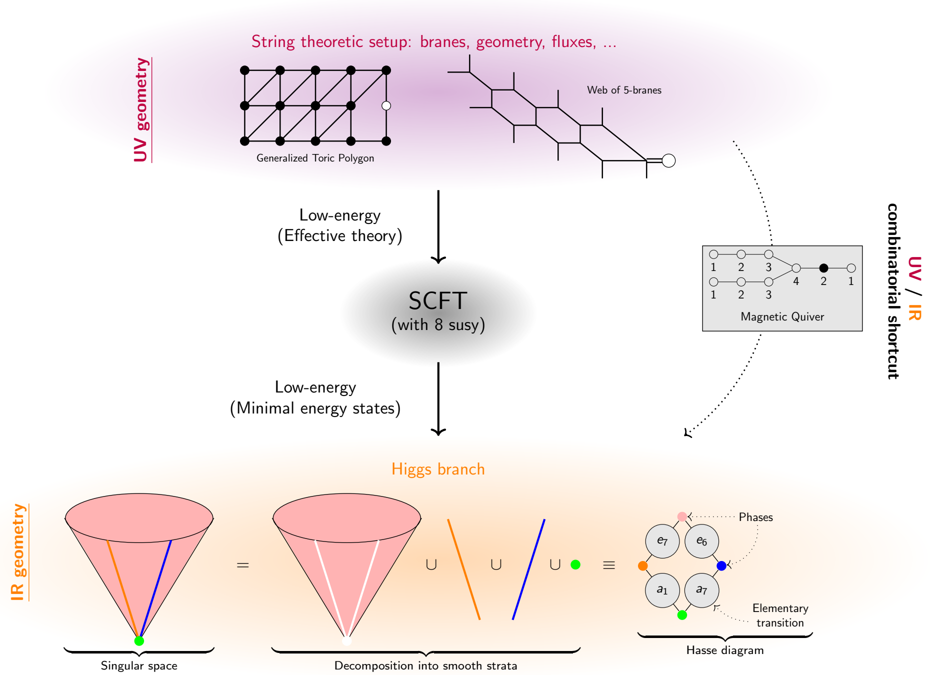
Phases
e_7
e_6
a_1
a_7
Elementary transition
Singular space
Decomposition into smooth strata
Hasse diagram



String theoretic setup: branes, geometry, fluxes, ...
UV geometry
Generalized Toric Polygon
Web of 5-branes
Low-energy
(Effective theory)
SCFT
(with 8 susy)
Low-energy
(Minimal energy states)
Magnetic Quiver
1 2 3 4 2 1
1 2 3
UV / IR
combinatorial shortcut
Higgs branch
IR geometry
=
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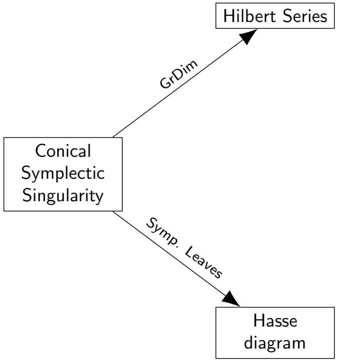
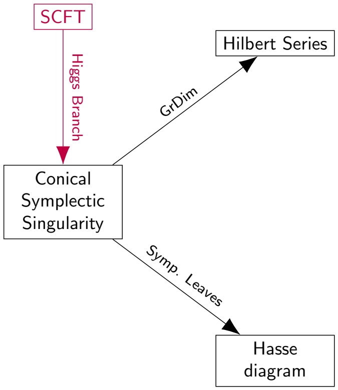
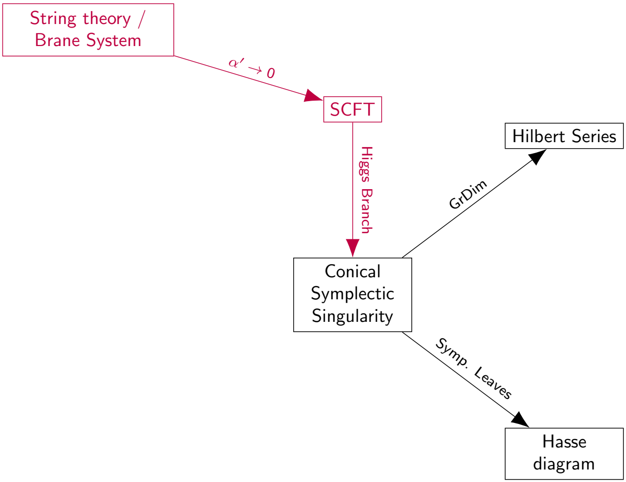
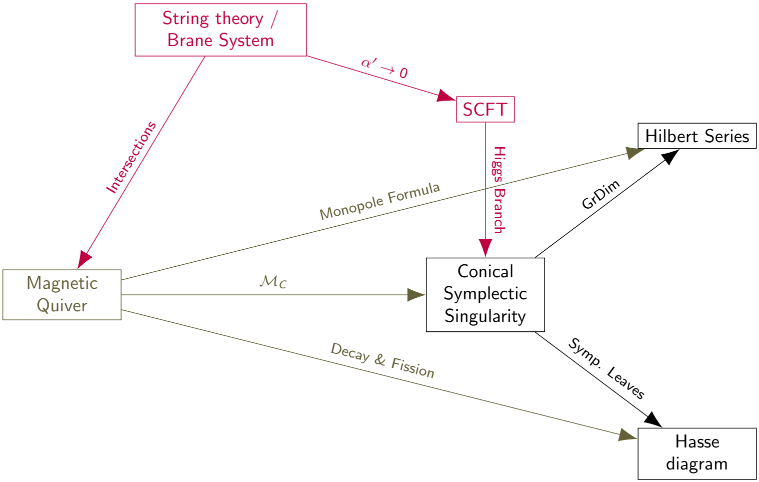
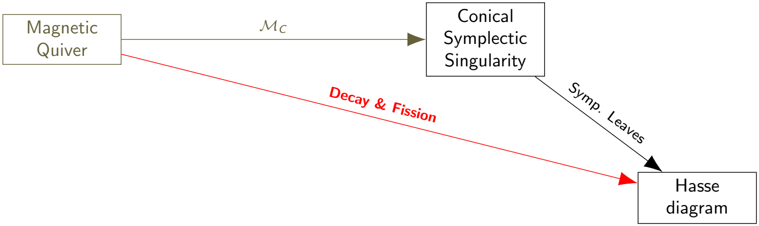
e_7
e_6
a_1
a_7
Phases
Elementary
transition
Singular space
Decomposition into smooth strata
Hasse diagram


Hilbert Series
GrDim
Conical Symplectic Singularity
Symp. Leaves
Hasse diagram

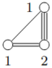

SCFT
Higgs Branch
Hilbert Series
GrDim
Conical Symplectic Singularity
Symp. Leaves
Hasse diagram


String theory / Brane System
$\alpha' \to 0$
SCFT
Higgs Branch
Conical Symplectic Singularity
GrDim
Hilbert Series
Symp. Leaves
Hasse diagram

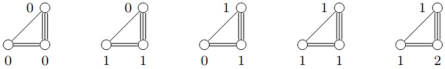

String theory / Brane System
$\alpha' \to 0$
SCFT
Intersections
Higgs Branch
Hilbert Series
Monopole Formula
GrDim
Magnetic Quiver
$\mathcal{M}_C$
Conical Symplectic Singularity
Decay & Fission
Symp. Leaves
Hasse diagram

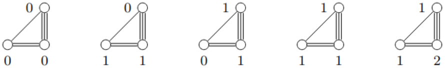

Magnetic Quiver
$\mathcal{M}_C$
Conical Symplectic Singularity
Decay & Fission
Symp. Leaves
Hasse diagram

Decay and Fission



Decay and Fission

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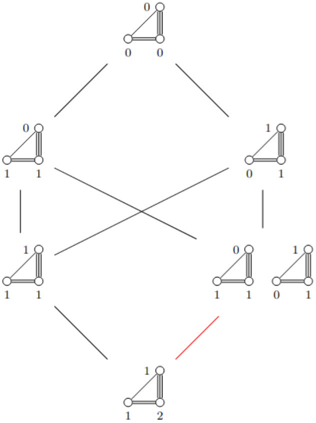
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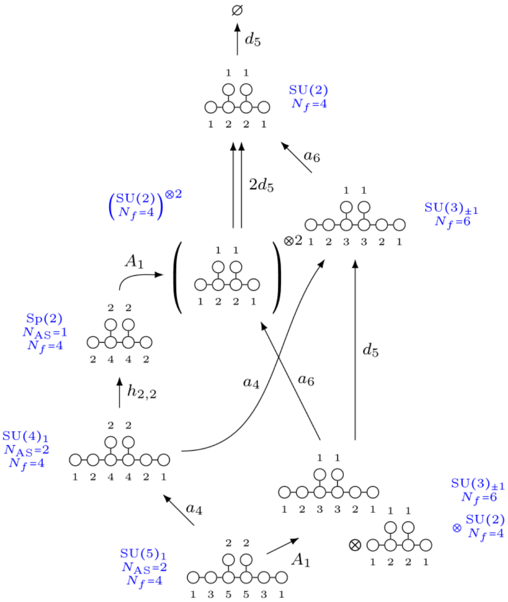
Decay and Fission



Sharp question:

What are the unitary magnetic quivers whose $\mathcal{M}_C$ has an isolated singularity?

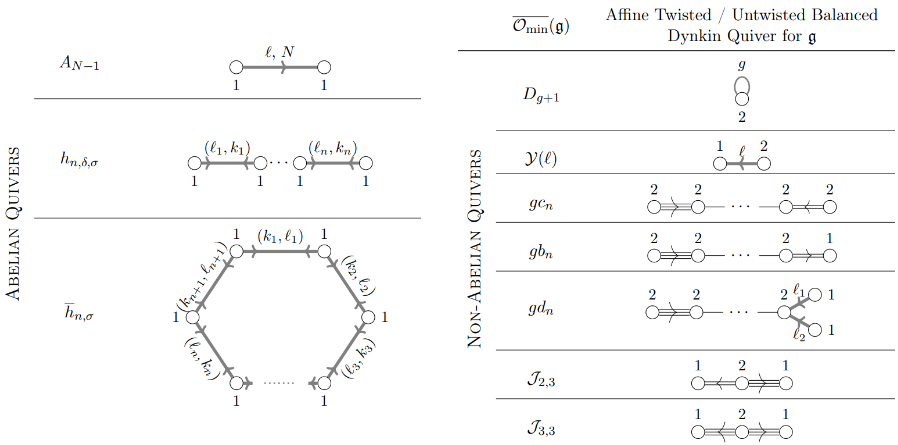
Using the Fission & Decay algorithm:

What are the stable unitary magnetic quivers ?

Outline

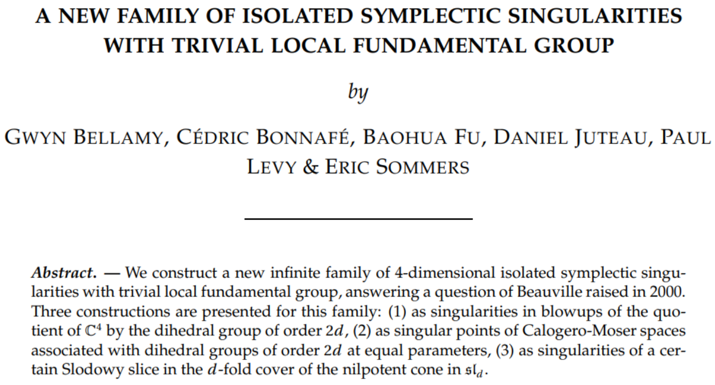
Main Result

Theorem: The full list of stable unitary quivers is given in the table below.



The Theorem combined with the Decay & Fission Conjecture imply the classification of ICSS which are $\mathcal{M}_C$ of a unitary quiver.

Comment on $\mathcal{Y}(d)$



A NEW FAMILY OF ISOLATED SYMPLECTIC SINGULARITIES WITH TRIVIAL LOCAL FUNDAMENTAL GROUP

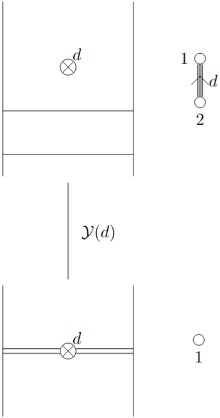
by

GWYN BELLAMY, CÉDRIC BONNAFÉ, BAOHUA FU, DANIEL JUTEAU, PAUL LEVY & ERIC SOMMERS

Abstract. — We construct a new infinite family of 4-dimensional isolated symplectic singularities with trivial local fundamental group, answering a question of Beauville raised in 2000. Three constructions are presented for this family: (1) as singularities in blowups of the quotient of $\mathbb{C}^4$ by the dihedral group of order $2d$, (2) as singular points of Calogero-Moser spaces associated with dihedral groups of order $2d$ at equal parameters, (3) as singularities of a certain Slodowy slice in the d-fold cover of the nilpotent cone in $\mathfrak{sl}_d$.

[BBFJLS, 21]

Comment on $\mathcal{Y}(d)$



[AB, Grimminger, 22]

New families of ICSS

ICSS	Symmetry	$PL(\mathrm{HWG})$
gb_n	$\mathfrak{so}_{2n+1}$	$\mu_2 t^2 + (1 + \mu_1^2 + \mu_1^3) t^4 + \mu_1^3 t^6 - \mu_1^6 t^{12}$
gd_n	$\mathfrak{so}_{2n}$	$\mu_2 t^2 + (1 + \mu_1^2 + \mu_1^3) t^4 + \mu_1^3 t^6 - \mu_1^6 t^{12}$
gc_n	$\mathfrak{u}_1 \oplus \mathfrak{su}_{2n+1}$	$(1 + \mu_1 \mu_{2n}) t^2 + (q \mu_1^3 + q^{-1} \mu_{2n}^3) t^4 - \mu_1^3 \mu_{2n}^3 t^8$

Proof (1/2)

Definition. A **quiver** Q is a triple (V, A, K) where V is a finite set, A a function $V \times V \to \mathbb{Z}$ and K a function $V \to \mathbb{Z}_{>0}$, such that

(i) for all $x \in V$, $A(x,x) = -2 + 2g_x$, for some $g_x \in \mathbb{Z}_{\geq 0}$. If $K(x) = 1$, then $A(x,x) = -2$.

(ii) for all $x \neq y \in V$, $A(x,y) = 0$ if and only if $A(y,x) = 0$. If they are non-zero, then both are positive and one is a divisor of the other.

(iii) there exists a function $L : V \to \mathbb{Z}_{>0}$ such that for every $x, y \in V$, $A(x,y)L(y) = A(y,x)L(x)$.

Definition. A connected quiver Q is **good** if it is non-empty and one of the following holds true:

- Its Hilbert Series converges, is non-constant, and has in its series expansion no coefficient at order t.
- Q is $N \cdot A_0^{(1)}$ for $N \geq 2$ or $N \cdot X_n^{(r)}$ for $N \geq 1$.

Proof (2/2)

Definition. Let $Q = (V, A, K)$ be a quiver. A **fission product** of Q is a multiset $\{\!\{Q_1, \dots, Q_n\}\!\}$ with $n \geq 0$, where $Q_i = (V_i, A_i, K_i)$ are quivers such that :

1. for each $1 \leq i \leq n$, Q_i is a good connected subquiver of Q;
2. $\sum_{i=1}^{n} K_i \leq K$, where we define each K_i to vanish on $V \backslash V_i$

We call $\mathcal{L}(Q)$ the set of fission products of Q.

Definition. Let Q be a quiver and $\{\!\{Q_1, \dots, Q_n\}\!\}, \{\!\{Q'_1, \dots, Q'_m\}\!\}$ two fission products of Q. We define a **partial order** on $\mathcal{L}(Q)$ by $\{\!\{Q_1, \dots, Q_n\}\!\} \preccurlyeq \{\!\{Q'_1, \dots, Q'_m\}\!\}$ if there exists a partition $\{1, \dots, n\} = \bigsqcup_{j=1}^{m} I_j$, with the I_j possibly empty, such that for every $1 \leq j \leq m$, $\{\!\{Q_i : i \in I_j\}\!\} \in \mathcal{L}(Q'_j)$.

Conjecture. There exists a 1-to-1 correspondence between the poset of symplectic leaves of the Coulomb branch of a good 3d $\mathcal{N} = 4$ quiver theory Q and the poset $(\mathcal{L}, \succcurlyeq)$ of decay and fission products of the good quiver Q.

Theorem. Let Q be a good quiver. Then Q admits a decay product isomorphic to one of the quivers listed above.

Proof. On the board?

Conclusion

In a nutshell:

- We recover *all ICSS from the math and physics literature*,
- make *new additions*,
- and formulate a *formal framework*
- which enables us to prove a *completeness result*.

Open questions:

- String theory realization of the new slices
- Yet other ICSSs from other constructions?
- Reasoning à la Robertson-Seymour for more general cases?
- How to combine these building blocks to form general molecules?
- Proof of the Decay & Fission Conjecture?

Conclusion

Thank you!