Non-toric brane webs, Calabi–Yau 3-folds, and 5d SCFTs

Pierrick Bousseau

University of Georgia

Strings and Geometry 2025
Trieste
10 April 2025

Introduction

- 5d SCFTs:
  - Geometric constructions
  - Brane web constructions
- Review of a known story:
  - Toric Calabi–Yau 3-folds
  - Webs of 5-branes.
- Recent development:
  - What are the non-toric Calabi–Yau 3-folds dual to webs of 5-branes with 7-branes? Alexeev–Argüz–B, 2410.04714
  - Use of a series of recent mathematical advances in mirror symmetry

    Gross–Siebert, Gross–Hacking–Keel, Engel–Friedman, Hacking–Keel–Yu, Alexeev–Argüz–B

5d SCFTs

- Predictions of string/M-theory: existence of interacting 5-dimensional quantum field theories
  - 5-dimensional $\mathcal{N} = 1$ SCFTs (8 supercharges) Seiberg 96
- Two main constructions:
  1) Singular geometry: M-theory on

$$\mathbb{R}^5 \times \overline{\mathcal{X}}$$

  - $\overline{\mathcal{X}}$: singular non-compact Calabi–Yau 3-fold.
  - Algebro-geometrically: canonical 3-fold singularity.
  - Differential geometric: complete Ricci-flat metric.

Morrison–Seiberg 96, Douglas–Katz–Vafa 96, Intriligator–Morrison–Seiberg 97, ..., Xie–Yau 17, Jefferson–Katz–Kim–Vafa 18, Bhardwaj–Jefferson 18, Closset–Del Zotto–Saxena 18, Bhardwaj 19, Closset–Nameki–Wang 20, Closset–Giacomelli–Schafer-Nameki–Wang 20, Collinucci–De Marco–Sangiovanni–Valandro 21, ..., Mu–Wang–Zhang 23, Acharya 24, ...

5d SCFTs

2) Intersecting branes in IIB string theory on $\mathbb{R}^{10}$:

$$\mathbb{R}^5 \times \mathbb{R}^5$$

Several possible ingredients: 5-branes, 7-branes, orientifolds, S-folds.

Aharony, Hanany 97, Aharony, Hanany, Kol 97, DeWolfe, Hanany, Iqbal, Katz, 99,..., Benini, Benvenuti, Tachikawa 09, ..., Bergman, Zafrir 15, ..., Acharya–Lambert–Najjar–Svanes– Tian 22, Kim–Kim–Lee 22,...

- Other construction: 5d SCFT from 6d SCFT on S^1.

- Coulomb branch
  - M-theory on $\overline{\mathcal{X}}$: Kähler moduli of crepant resolutions $\mathcal{X} \to \overline{\mathcal{X}}$ (with $\mathbb{Q}$-factorial terminal singularities).
  - $H^2(\mathcal{X}, \mathbb{R})$ = union of nef cones of crepant resolutions.
  - Prepotential on the Coulomb branch: triple intersection numbers of divisors in $\mathcal{X}$.
  - Web of 5-branes: separate the branes. Less clear with 7-branes.
- Higgs branch
  - Quaternionic enhancement of the moduli space of complex deformations of $\overline{\mathcal{X}}$: difficult question in algebraic geometry.
  - Magnetic quivers from web branes Ferlito–Hanany–Mekareeya–Zafrir 17, Cabrera–Hanany–Yagi 18, van Beest–Bourget–Eckhard–Sakura Schafer-Nameki 20, Bourget–Grimminger–Hanany–Sperling–Zhong 20, Akhond–Carta 21, ...

Questions

- Given a system of intersecting branes in IIB string theory engineering a 5d SCFT, can one realize the same 5d SCFT as M-theory on a canonical 3-fold singularity $\overline{\mathcal{X}}$?
- Can one give an algebro-geometric description of the class of canonical 3-fold singularities $\overline{\mathcal{X}}$ with a dual IIB string theory description?
- Webs of 5-branes $\rightsquigarrow$ M-theory dual is a toric Calabi–Yau 3-fold

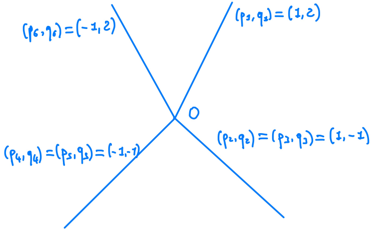
  Aharony–Hanany–Kol, Leung–Vafa 97
- Webs of 5-branes with 7-branes?
  - Webs of 5-branes with parallel configurations of 7-branes, using a IIA duality frame Bourget–Collinuci–Schafer-Nameki 23
  - Relation with polytope mutations and primitive T-cones

    Arias-Tamargo–Franco–Rodríguez-Gómez, 24, Bolla–Franco–Rodríguez-Gómez 24
  - General algebro-geometric construction based on mirror symmetry

    Alexeev–Argüz–B 24

Webs of 5-branes

- IIB string theory contains (p, q) 5-branes for coprime integers (p, q)
  - 5-brane: real 6 dimensional submanifold of $\mathbb{R}^{10} = \mathbb{R}^5 \times \mathbb{R}^2 \times \mathbb{R}^3$
- Web of 5-branes $\overline{W}$: Finite collection of (p_i, q_i) 5-branes living on $\mathbb{R}^5 \times \mathbb{R}_{\geq 0}(p_i, q_i)$, where $\sum_i (p_i, q_i) = 0$.



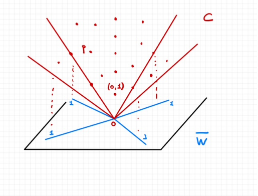
- Expectation: there exists a 5d SCFT on the common intersection $\mathbb{R}^5 \times \{0\}$ of the branes.

Affine toric Calabi–Yau 3-folds

- What is the M-theory dual canonical 3-fold singularity $\overline{\mathcal{X}}$ to a web of 5-brane $\overline{W}$?
- Answer: $\overline{\mathcal{X}}$ is an affine (Gorenstein) toric Calabi–Yau 3-fold.

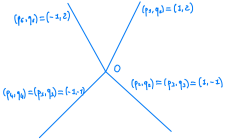
$$\overline{\mathcal{X}} = \mathrm{Spec} \bigoplus_{p \in C} \mathbb{C} z^p$$

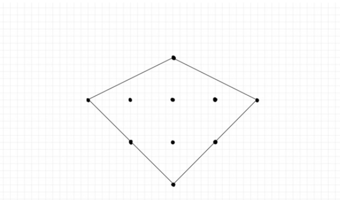
  - φ: continuous piecewise linear function on $\mathbb{R}^2$ with changes of slope given by numbers of 5-branes.
  - $C = \{(m,k) \in \mathbb{Z}^2 \times \mathbb{Z} \,|\, k \geq \varphi(m)\}$
  - $z^p \cdot z^{p'} = z^{p+p'}$



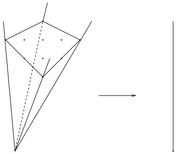
- Every affine (Gorenstein) toric Calabi–Yau 3-fold arises this way from a web of 5-branes.

Dual fan description



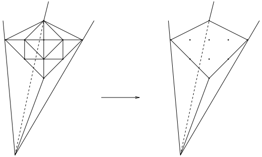


- $\overline{W}$: web of 5-branes $\rightsquigarrow$ dual polytope P to $\overline{W}$:
- $\overline{\mathcal{X}}$: dual toric Calabi–Yau 3-fold whose fan is the cone $C^\vee$ over P.
  - Toric morphism $\pi = z^{(0,1)} : \overline{\mathcal{X}} \to \mathbb{C}$, with $\pi^{-1}(t) = (\mathbb{C}^\star)^2$ for $t \neq 0$.
  - $\overline{\mathcal{X}}_0$: union of affine toric varieties.

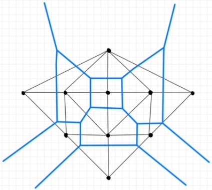


Crepant resolutions of $\overline{\mathcal{X}}$

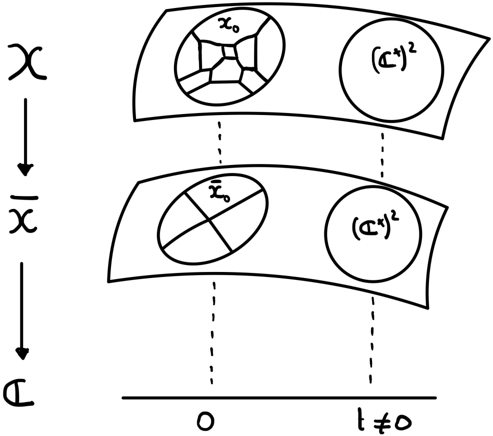
- Crepant resolutions $\mathcal{X} \to \overline{\mathcal{X}}$ are in 1:1 correspondence with regular triangulations of P.



- Maximal triangulations of P are dual to perturbations W of $\overline{W}$ – Coulomb branch of the 5d SCFT (GKZ secondary fan)



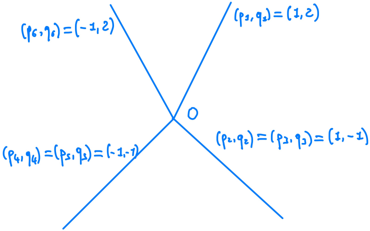
- $\mathcal{X}_0$: toric irreducible components with momentum polytopes faces of the perturbed web W.

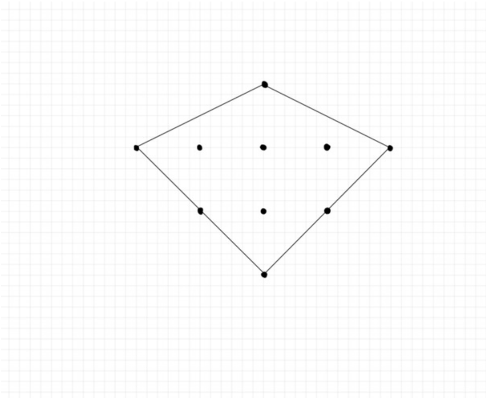


$\mathcal{X}$
$\overline{\mathcal{X}}$
$\mathbb{C}$
x_0
$\bar{x}_0$
$(\mathbb{C}^*)^2$
$(\mathbb{C}^*)^2$
0
$t \neq 0$

Webs of 5-branes and toric local mirror symmetry

- $\overline{W}$: web of 5-branes $\rightsquigarrow$ dual polytope P to $\overline{W}$:



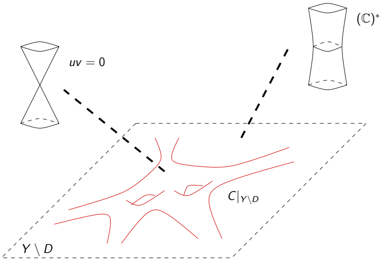


- (Y, D, L): polarised toric log Calabi–Yau pair associated to P
  - Y: toric surface, D: toric boundary divisor
  - L: ample line bundle on Y

$$Y = \mathrm{Proj} \bigoplus_{k \geq 0} H^0(Y, L^{\otimes k}) = \mathrm{Proj} \bigoplus_{p \in C^\vee} \mathbb{C} z^p$$

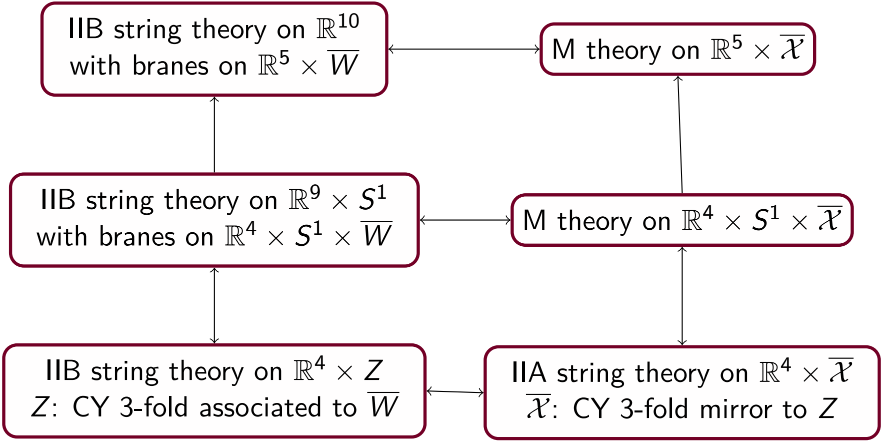
Polarised toric surfaces to 3-folds

- Mirror curve: $C \in |L| \rightsquigarrow C = \{s = 0\}$ where $s \in H^0(Y, L)$.
  - $s|_{(\mathbb{C}^\star)^2} = f(x, y)$: Laurent polynomial with Newton polygon P
- Z: CY 3-fold $\{uv = f(x, y)\} \subset \mathbb{C}^2 \times (\mathbb{C}^\star)^2$



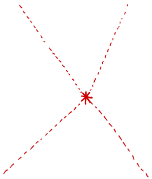
- $\mathcal{X} \to \overline{\mathcal{X}}$: M-theory dual 3-fold to $\overline{W}$ = mirror to maximal degenerations of Z.

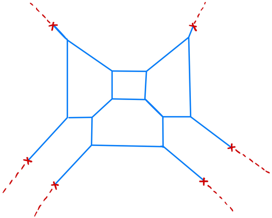
M-theory dual 3-fold $\overline{\mathcal{X}}$ to $\overline{W}$ via toric mirror symmetry



Webs of 5-branes and 7-branes

- IIB string theory on $\mathbb{R}^5 \times \mathbb{R}^2 \times \mathbb{R}^3$
  - (p,q) 5-branes on $\mathbb{R}^5 \times \overline{W}$
  - (p,q) 7-branes on $\mathbb{R}^5 \times \{point\} \times \mathbb{R}^3$



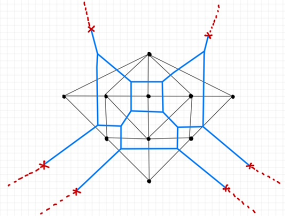


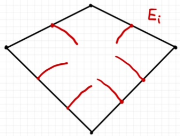
- (p,q) 5-branes ending on (p,q) 7-branes: DeWolfe–Hanany–Iqbal–Katz 99, Benini–Benvenuti–Tachikawa 09
- Problem: not all configurations are "consistent/supersymmetric": applying Hanany–Witten moves can lead to negative numbers of 5-branes.
  - s-rule/consistency condition: after Hanany–Witten moves, always get non-negative numbers of 5-branes (and the web cannot be decomposed into unlinked disconnected parts).

- Algebro-geometric proposal: Alexeev–Argüz–B, 24
  - $(\overline{Y}, \overline{D}, \overline{L})$: polarized toric surface, defined by polytope $\overline{P}$ corresponding to the original web of 5-branes.
  - (Y, D): log Calabi–Yau surface obtained by a non-toric blow-up on $\overline{D}$ for each 7-brane.
  - Exceptional curve E_i: if a_i 5-branes end on the corresponding 7-brane,

$$L := \overline{L} - \sum_i a_i E_i$$

  - (Y, D, L) invariant under Hanany–Witten moves (geometric cluster mutations).

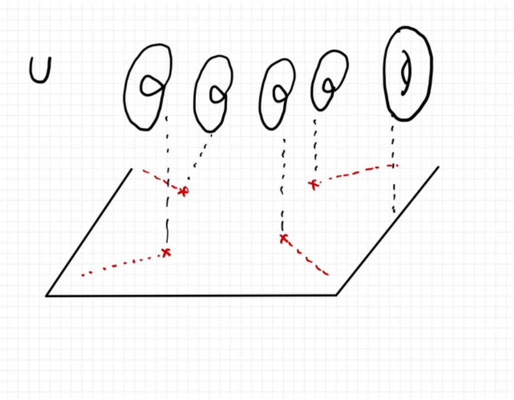




Algebraic cluster mutations: Arias-Tamargo–Franco–Rodríguez-Gómez, 24, Bolla–Franco–Rodríguez-Gómez 24

Consistent webs of 5-branes with 7-branes

- Non-compact surface $U = Y \setminus D$
- Toric case: $U = (\mathbb{C}^\star)^2 = T^2 \times \mathbb{R}^2$
- In general: T^2-fibration over $\mathbb{R}^2$ with nodal fibers over the 7-branes. Topologically, F-theory lift of the system of 7-branes.



Consistent webs of 5-branes with 7-branes

Theorem (AAB): A web of 5-branes with 7-branes is consistent $\iff$ L is nef ($L \cdot S \geq 0$ for every effective curve S in Y) and either

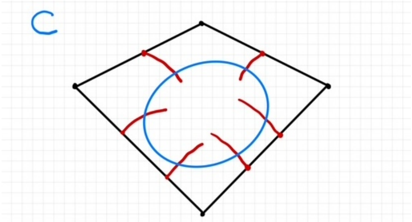
- $L^2 > 0$, or
- $L^2 = 0$ and $L = kE$, $k \geq 1$, E smooth elliptic curve such that $E \cdot D = 0$.

- Proof uses a result of Jeremy Blanc on the factorization of volume preserving birational transformations of $(\mathbb{C}^*)^2$, in terms of elementary cluster transformations.

Consistent webs to smooth curves $C \in |L|$

Lemma (AAB)

If a web of 5-branes with 7-branes is consistent, then there exists a smooth curve $C \in |L|$ in the associated polarized log Calabi–Yau surface.



- Existence of $C \in |L| \rightsquigarrow$ Calabi–Yau 3-fold Z as before.
- M-theory dual Calabi–Yau 3-fold $\mathcal{X} \to \overline{\mathcal{X}}$? Mirror to Z!
  - Problem: non-toric mirror symmetry.
  - Z birational to a toric mirror, so $\mathcal{X}$ should be a non-toric complex structure deformation of a toric Calabi–Yau (move on the Higgs branch).

Mirror symmetry

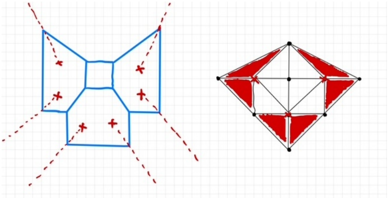
- How to construct the mirror to Z? Two approaches:
- Construct the smooth Calabi–Yau 3-fold $\mathcal{X}$ as deformation of a (generally singular) toric Calabi–Yau 3-fold. Then, get the canonical 3-fold singularity $\overline{\mathcal{X}}$ by contracting compact cyles.
  - Uses deformation theory
  - Explicit description of the crepant resolution, good to study the Coulomb branch and the prepotential.
- Directly construct $\overline{\mathcal{X}}$ running the Gross–Sibert intrinsic mirror symmetry construction.
  - Uses enumerative geometry (open A-model worldsheet instantons)
  - Explicit description of $\overline{\mathcal{X}}$ by equations or algebra of functions.

Deformation theory construction: pushing 7-branes

Lemma (AAB)

If a web of 5-branes with 7-branes is consistent, then there is no obstruction to "push in" the 7-branes along their monodromy invariant directions until they are no longer attached to any 5-branes.

- Pushing in 7-branes: flops in a degeneration of $(Y, D, L) \rightsquigarrow$ reformulate results in birational geometry (the existence of a minimal model after a sequence of flops for a maximal degeneration of (Y, D, L)).



- Web of 5-branes with 7-branes $\rightsquigarrow$ Symington polygon for (Y, D, L)

Deformation theory construction

- P: Symington polygon for (Y, D, L) dual to a web of 5-branes and 7-branes

Theorem (AAB)

There is a regular triangulation T of P and a maximal degeneration of (Y, D, L) with special fiber Y_0 whose intersection complex is (P, T).

- $\mathcal{X}_0$: normal crossing surface with "dual intersection complex" (P, T).
  - irreducible components X_v of $\mathcal{X}_0$: vertices $v \in P$
  - v: non-singular point $\implies$ X_v is smooth toric (non-compact if $v \in \partial P$)
  - v: singular point $\implies$ X_v is a smooth log CY surface, which we obtain by a $\mathbb{Q}$-Gorenstein smoothing of a toric surface with quotient Wahl singularities locally of the form $\mathbb{C}^2/(\mathbb{Z}/n^2\mathbb{Z})_{1,an-1}$.
    Alternative description:
    $$\{xy = z^n\}/(\mathbb{Z}/n\mathbb{Z})_{1,-1,a}$$
    Deformation $\{xy = z^n + t\}/(\mathbb{Z}/n\mathbb{Z})_{1,-1,a}$.

Deformation theory construction

Theorem (AAB)

There exists a d-semistable gluing of X_v's to $\mathcal{X}_0$ and a smoothing $\mathcal{X} \to \Delta = \operatorname{Spec} \mathbb{C}[[t]]$ of it such that

- *The total space $\mathcal{X}$ is smooth and $\mathcal{X}_0 \subset \mathcal{X}$ is a reduced normal crossing divisor (i.e. $\mathcal{X} \to \Delta$ is a semistable degeneration).*
- *$\mathcal{X}$ is quasi-projective and $K_{\mathcal{X}} = 0$.*
- *There exist a contraction $\mathcal{X} \to \overline{\mathcal{X}}$, such that $\overline{\mathcal{X}}$ is affine with canonical singularities and $\mathcal{X} \to \overline{\mathcal{X}}$ is a projective crepant resolution*

- Proof uses deformation theory

Deformation theory construction

- The central fiber $\overline{\mathcal{X}}_0$ is:
  - a simple elliptic singularity if $L^2 = 0$.
  - a degenerate cusp singularity if $L^2 > 0$ and $\deg L|_D > 0$
  - a cusp singularity if $L^2 > 0$ and $\deg L|_D = 0$.

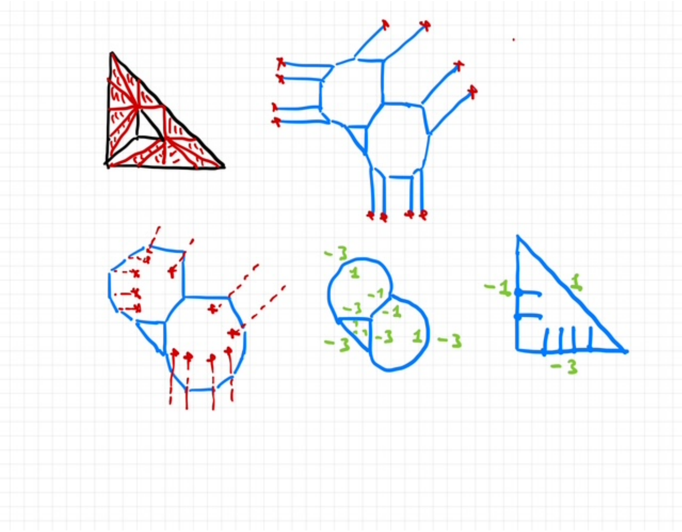
Conversely, we have the following:

Theorem (AAB)

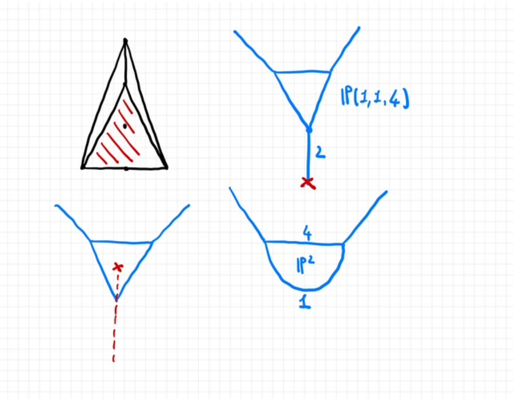
Let $\overline{\mathcal{X}}_0$ be either a simple elliptic, cusp, or degenerate cusp surface singularity. Let $\overline{\mathcal{X}} \to \Delta$ be a smoothing of $\overline{\mathcal{X}}_0$ admitting a semistable crepant resolution $\mathcal{X} \to \overline{\mathcal{X}}$. Then, the Calabi–Yau 3-fold $\overline{\mathcal{X}}$ is M-theory dual to a web of 5-branes with 7-branes.

Much more difficult to prove mathematically (uses intrinsic mirror symmetry). Need to establish that mirror symmetry is an involution.

T_4-theory: from a non-isolated toric singularity $xyz = u^4$ to an isolated non-toric singularity.



Non-toric deformation of local $\mathbb{P}(1,1,4)$ to $\mathbb{P}^2$



Remarks: degeneration of $\mathbb{P}^2$ to $\mathbb{P}(a^2, b^2, c^2)$ if and only if $a^2 + b^2 + c^2 = 3abc$ (Markov equation) Hacking–Prokhorov 05

Direct construction of $\overline{\mathcal{X}}$

- Gross–Siebert intrinsic mirror construction: very general.
  - Algebro-geometric realization of the expectation that the mirror is the moduli space of A-branes torus fibers of the SYZ fibration (including disk worldsheet instantons)

$$\overline{\mathcal{X}} = \mathrm{Spec} \bigoplus_{p \in C} \mathbb{C} \vartheta_p$$

$$\vartheta_p \cdot \vartheta_q = \sum_r N_{pqr} \vartheta_r$$

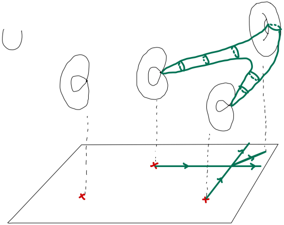
  - N_{pqr}: log Gromov–Witten invariants, algebro-geometric version of counts of holomorphic curves in $U = Y \setminus D$.

Direct construction of $\overline{\mathcal{X}}$

Theorem (AAB)

The equations of the mirror $\mathcal{X} \to \Delta$ can be obtained algorithmically from a scattering diagram with initial rays coming out of the 7-branes.

- Scattering diagram in mirror symmetry Kontsevich–Soibelman, Gross–Siebert, Gross–Hacking–Keel, ...
- If all 5-branes ending on 7-branes are parallel (no scattering), and we recover previous results of Bourget–Collinuci– Schafer-Nameki, 2023.



Worldsheet instantons as $\mathcal{N} = 2$ 4d BPS states.

- Consider the 4d $\mathcal{N} = 2$ theory on the worldvolume of a D3-brane probing the 7-branes.
  - Rank 1 theory (possibly not UV complete) with Coulomb branch the base B of the torus fibration on U.
  - Equivalently: worldvolume of an M5-brane wrapping around a torus fiber of $U \to \mathbb{R}^2$.
- The previous worldsheet instantons can be viewed as BPS states of this 4d $\mathcal{N} = 2$ theory:
  - String junctions between the D3-brane and the 7-branes.
  - M2-brane in U with boundary on a torus fiber of $U \to B$.
- Scattering diagram = Kontsevich–Soibelman wall-crossing formula for BPS states of this 4d $\mathcal{N} = 2$ theory. Cecotti–Vafa 09

Summary/future directions

- New class of non-toric non-compact Calabi–Yau 3-folds, obtained by geometric transition from toric Calabi–Yau 3-folds.
- All structures known for toric Calabi–Yau 3-folds should admit an interesting deformation:
  - Genus 0 mirror symmetry?
  - Homological mirror symmetry/dimer models/brane tilings/5d BPS quiver?
  - Topological vertex? Hayashi–Kim–Nishinaka 14,...
  - Topological recursion for $C^\circ \subset U$?
  - Topological string/Spectral theory correspondence?
- Main tool used in the toric case: localization, is missing.

Thank you for your attention !