Complexity Bounds and Volume Growth in Effective Field Theories

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Based on:

2503.15601 with David Prieto, Mick van Vliet

+ work in progress

2410.23338 with Mick van Vliet

Motivation

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quantitive measure of information:
integers (F,D), sharp complexity

[Binyamini,Novikov '22][Binyamini,Novikov,Zak '23]

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- Interpret this as $V(\phi)$ is being too complex

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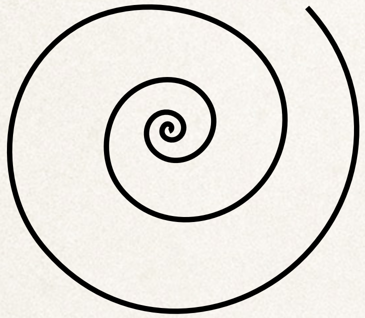

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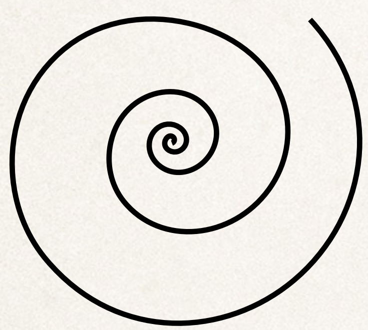

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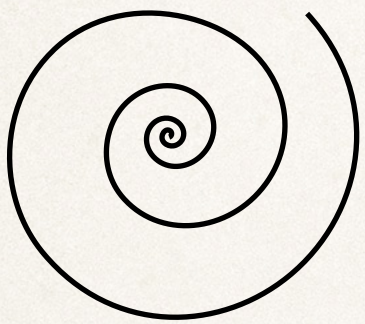

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- Interpret the resulting effective couplings as being too complex

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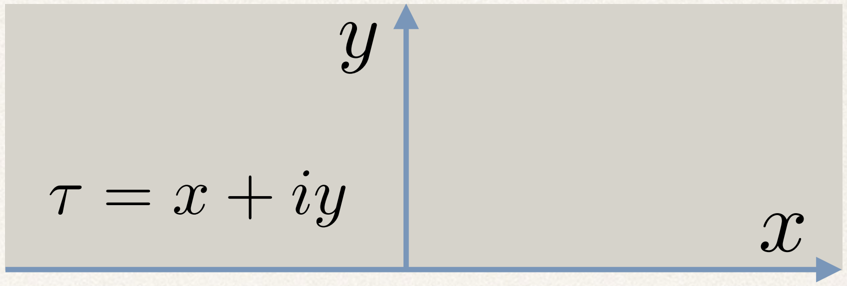
- Geometry of moduli spaces
  - upper half plane with hyperbolic metric

$$ds^2_{\mathbb{H}} = \frac{d\tau d\bar{\tau}}{(\mathrm{Im}\tau)^2}$$



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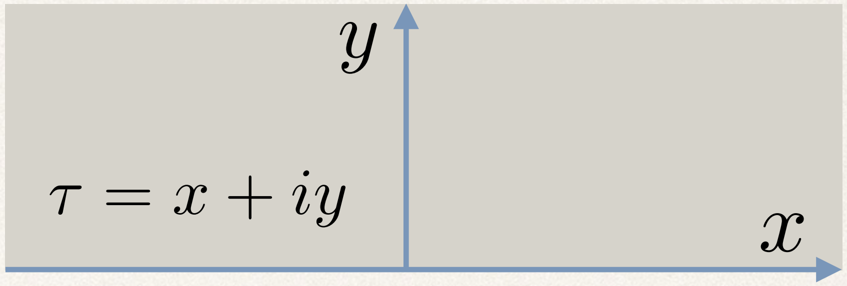

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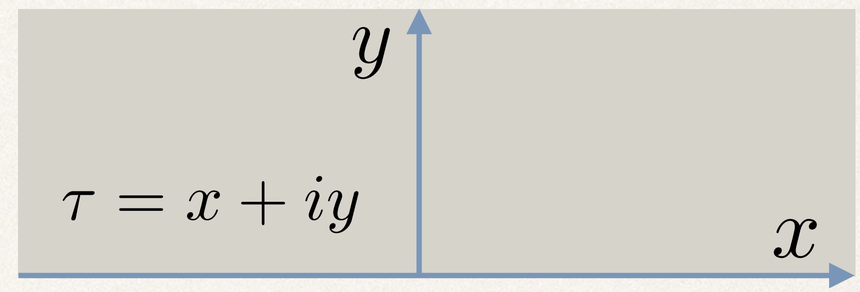

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$$\mathrm{vol}(\mathcal{M}_D) < D^{\dim(\mathcal{M})+\epsilon} \qquad D \to \infty$$

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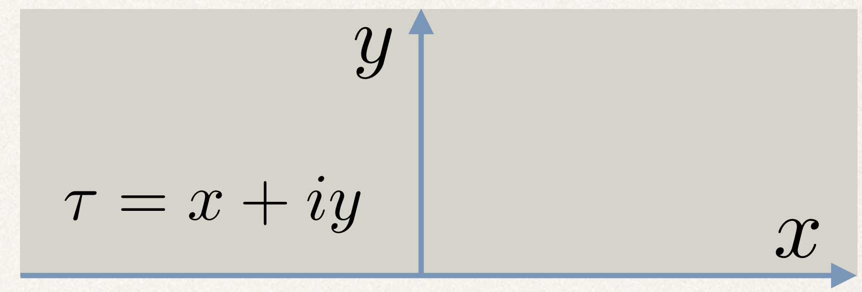

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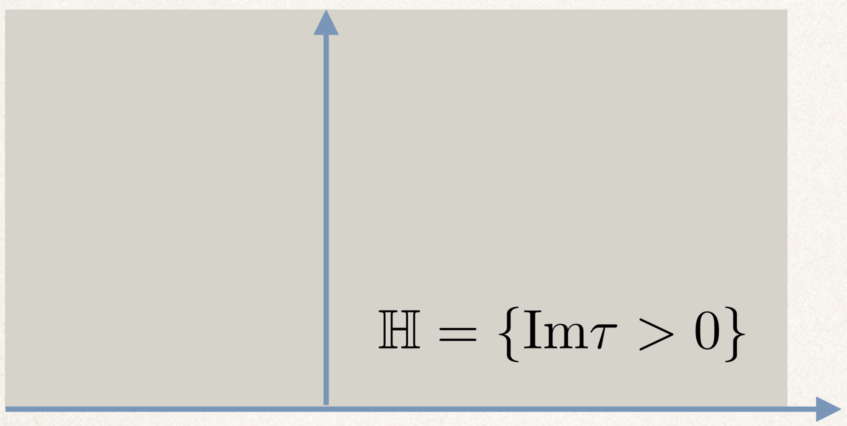
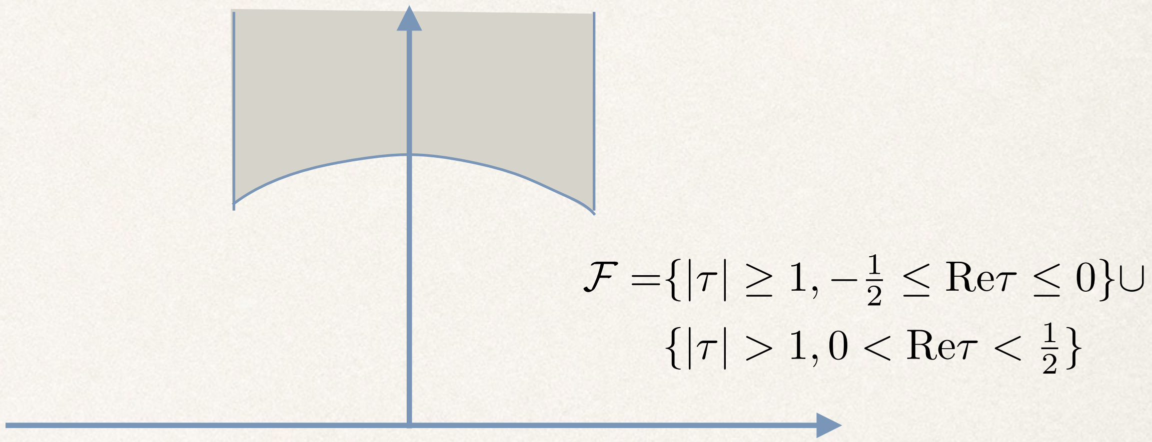
Should we find this puzzling?

swampy upper half plane



fundamental domain of $\mathrm{Sl}(2, \mathbb{Z})$



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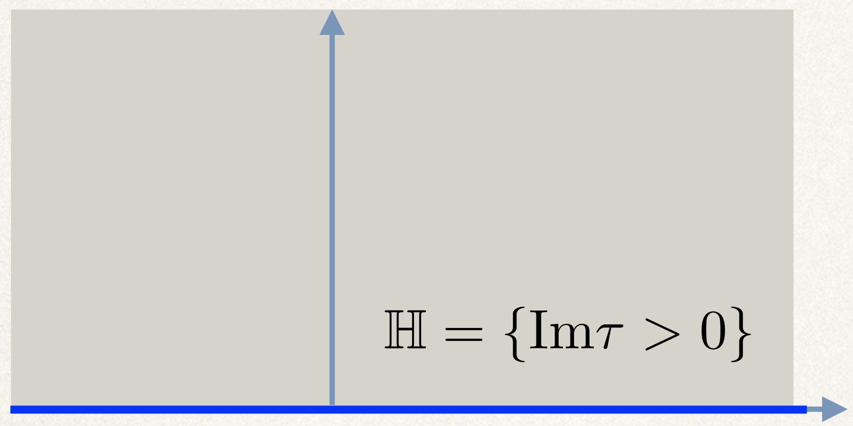

- exponentially growing volume
- dense set of infinite distance points on the real line + infinite distance point at $\mathrm{Im}\tau \to \infty$

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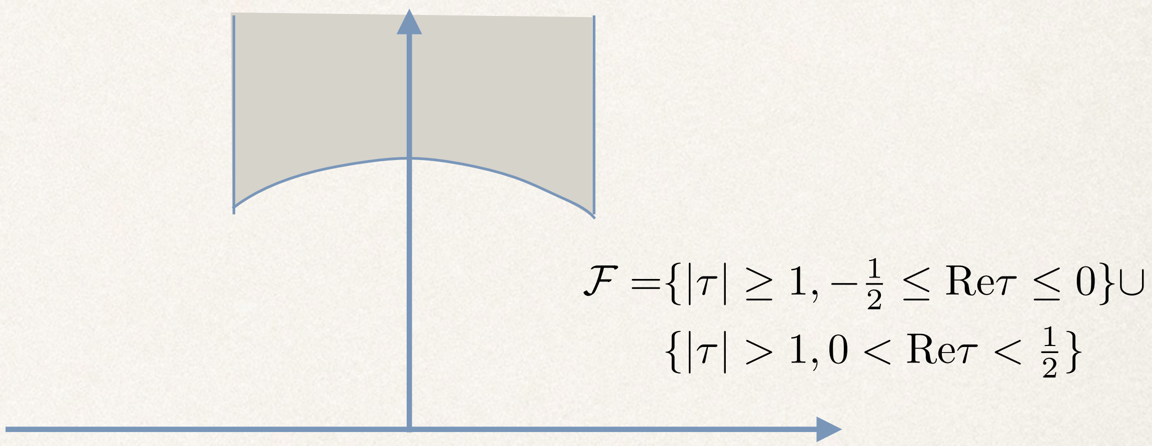

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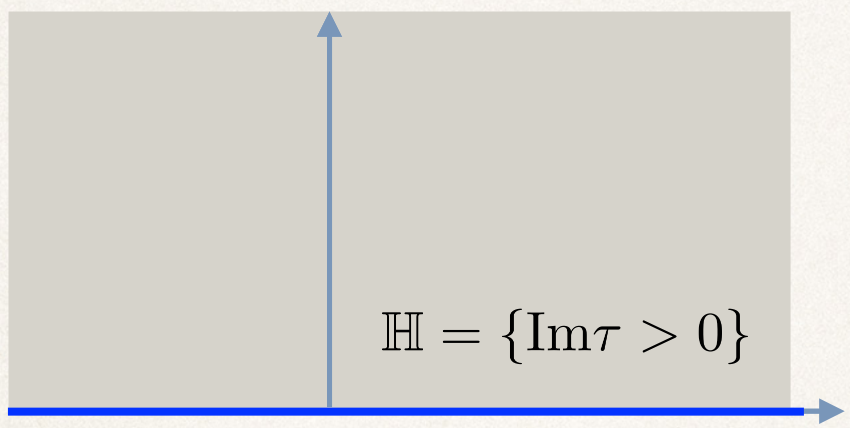

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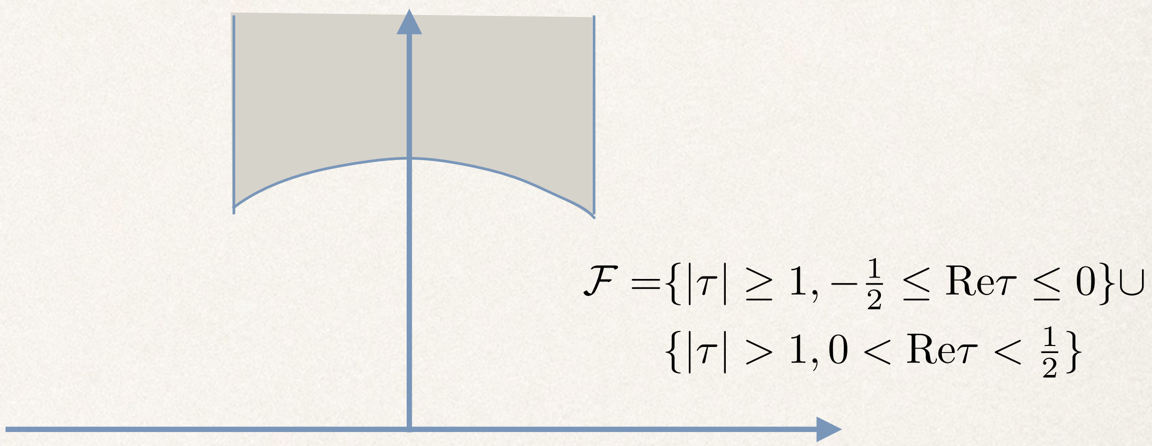

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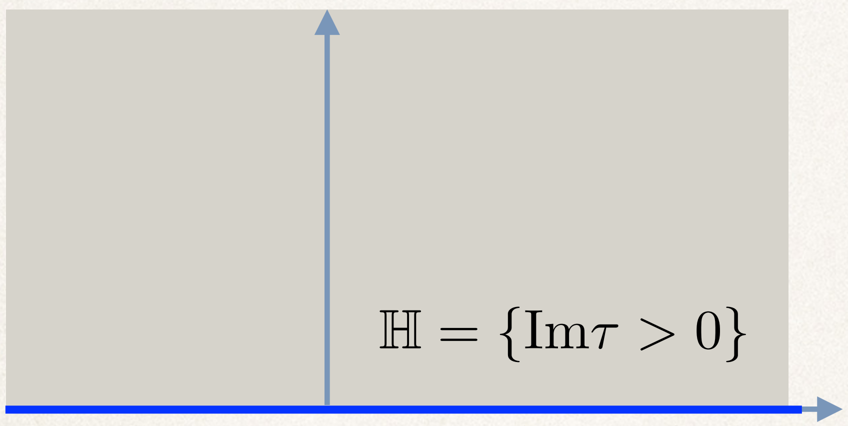

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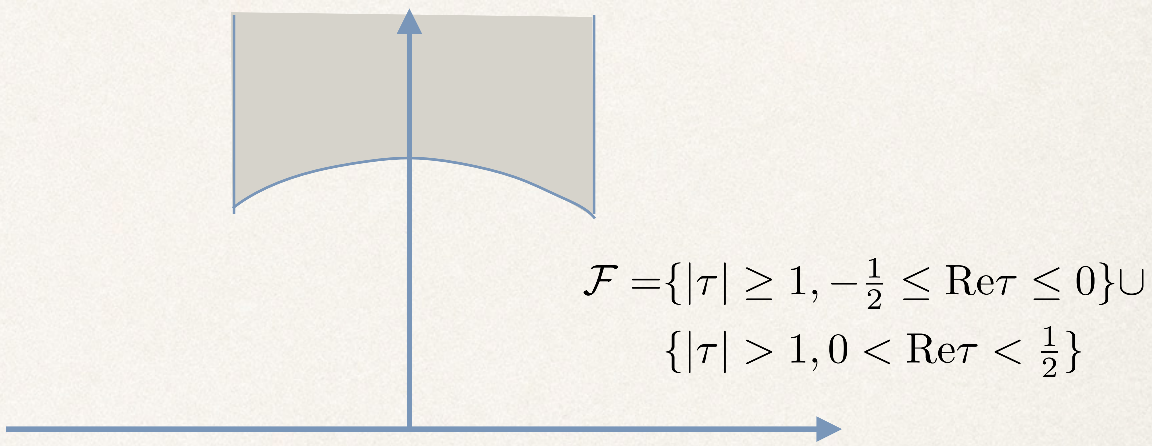

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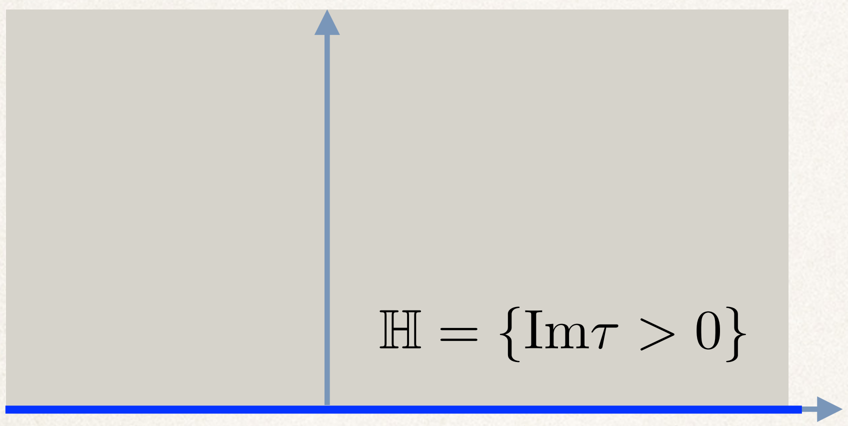

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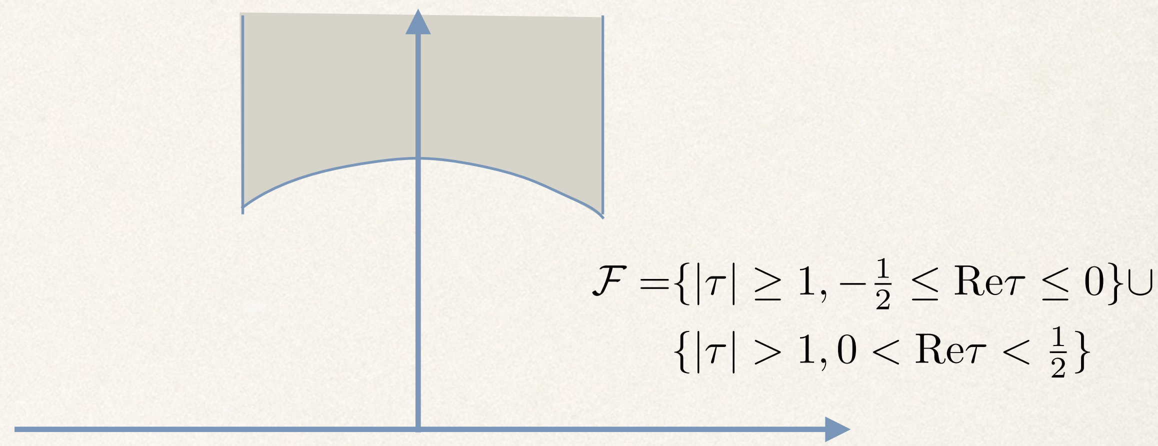

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- actual moduli space of elliptic curve

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↓ period integrals of elliptic curve

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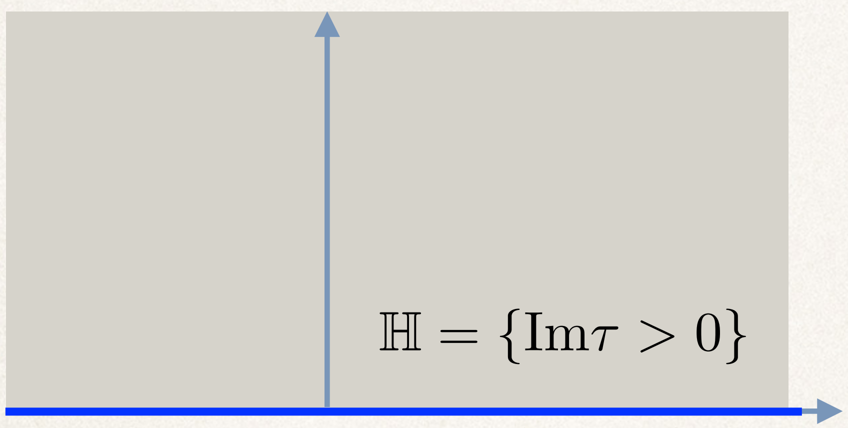

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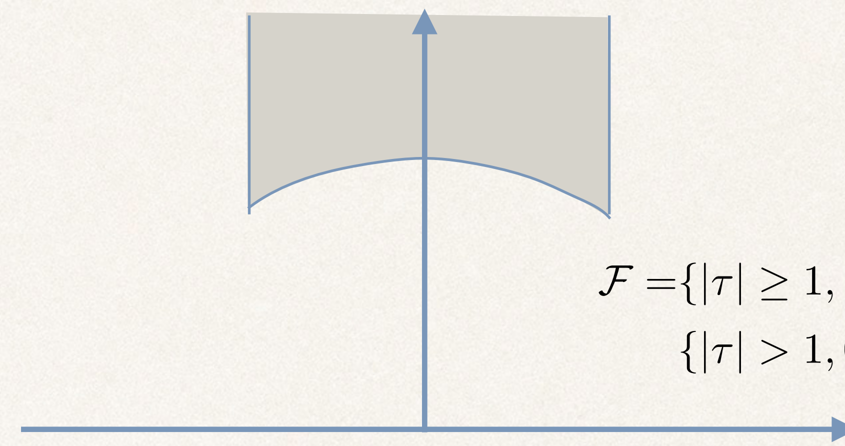

$$\mathcal{F} = \{|\tau| \geq 1, -\tfrac{1}{2} \leq \mathrm{Re}\tau \leq 0\} \cup \{|\tau| > 1, 0 < \mathrm{Re}\tau < \tfrac{1}{2}\}$$

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Geometric complexity in Tame Geometry

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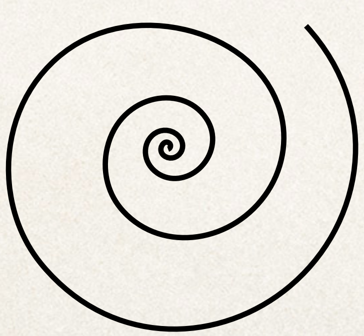

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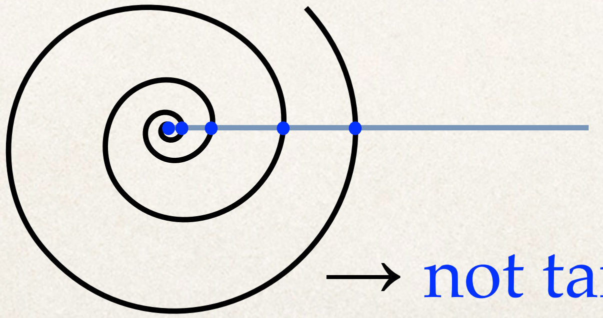

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$\longrightarrow$ not tame: infinite intersections with real line

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- Last years: sharp o-minimality [Binyamini,Novikov '22][Binyamini,Novikov,Zack]

tame principle that allows notion of complexity (keep track of finite information)

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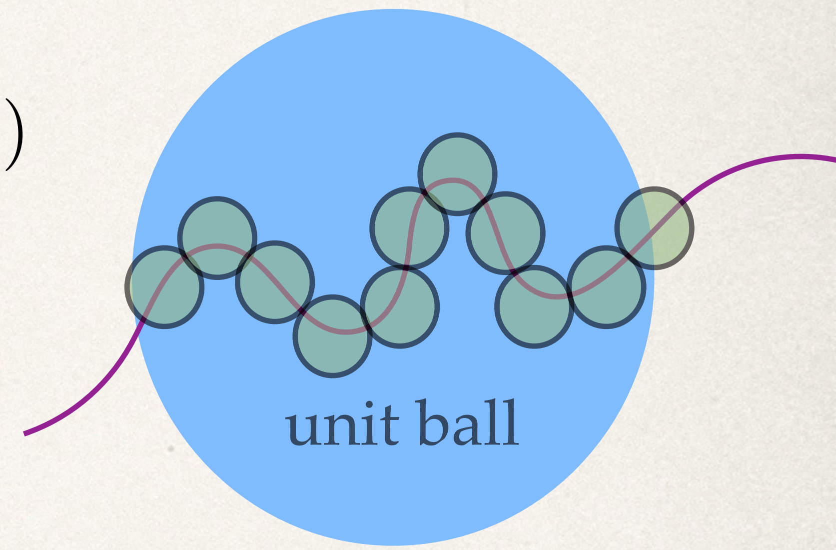

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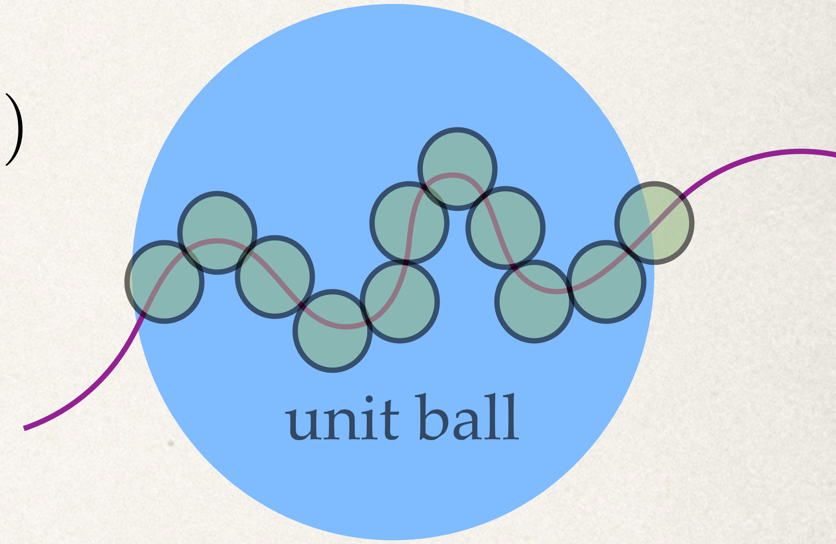
  - Upper: Volume growth → later



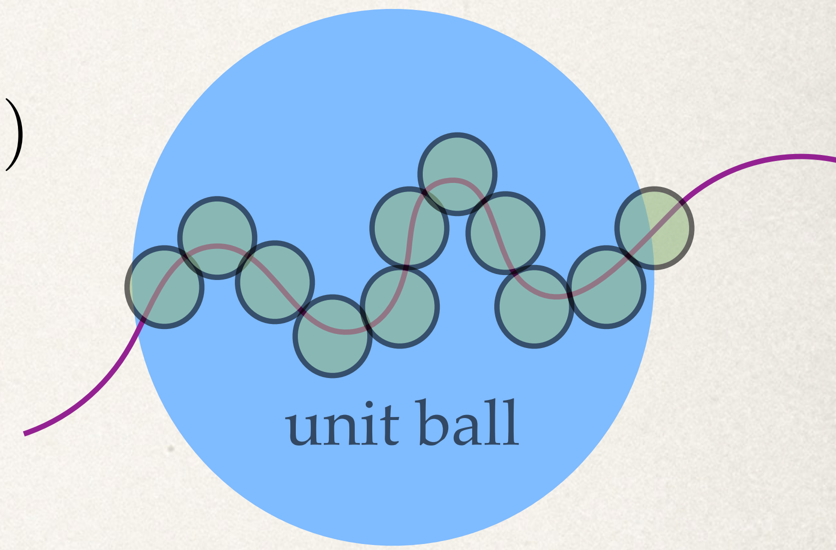
Intuition from polynomials

- Complexity for polynomials → D - degree of polynomial; F - number of variables

  $$P(x) = a_1 x^2 + a_2 x + a_3$$

  → amount of information needed to specify polynomial (real coefficients)

- Bounds from complexity:
  - Upper: Number of zeros, minima, and maxima of $P(x)$
  - Lower: Number of data points to approximate $P(x)$ up to error ϵ: $N(\epsilon, F, D)$
  - Upper: Volume growth $\longrightarrow$ later



- Complexity of (semi-) algebraic sets $A \subset \mathbb{R}^n$: $A = \{P_i(x) = 0,\ Q_i(x) > 0\}$

  $\mathbb{R}_{\mathrm{alg}}$ → is o-minimal structure with notion of complexity

Beyond polynomials - Pfaffian functions

- How to deal with exponential function?

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  degree: $D = \deg(P) + \sum_{ij} \deg(P_{i,j})$

  format: $F = n + r$ (number of variables + number of non-trivial functions)

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up-shot: sharply o-minimal structures have clean notion of complexity
→ relevant examples are under construction

Tameness, volume growth, and embeddings

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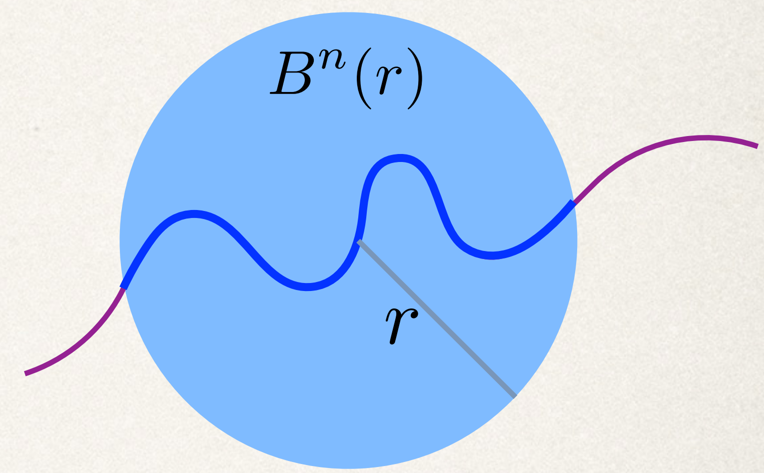

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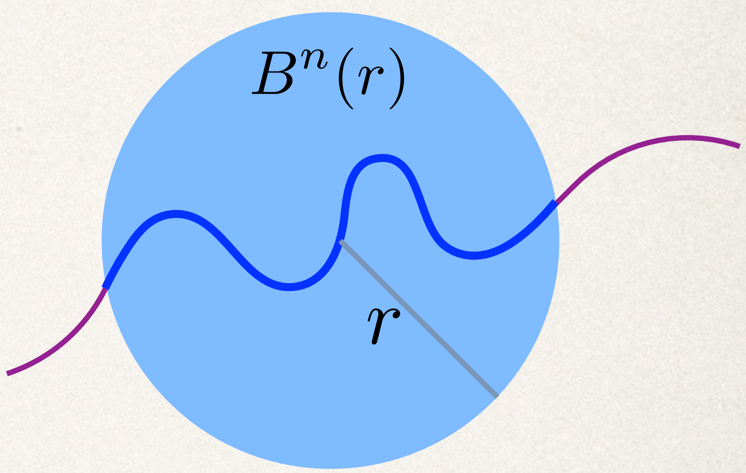

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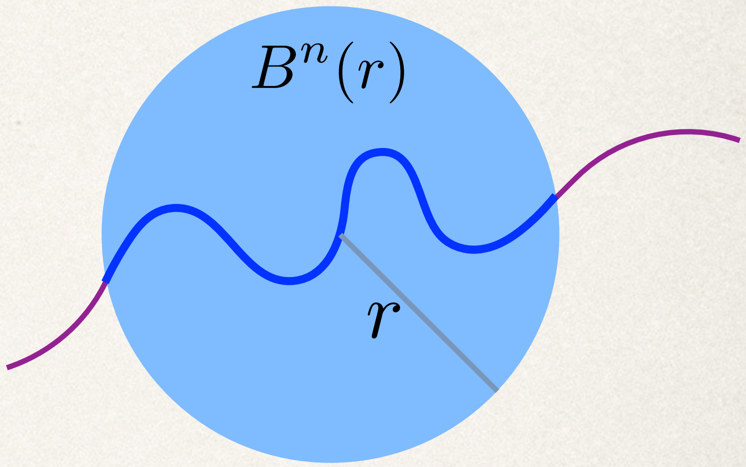

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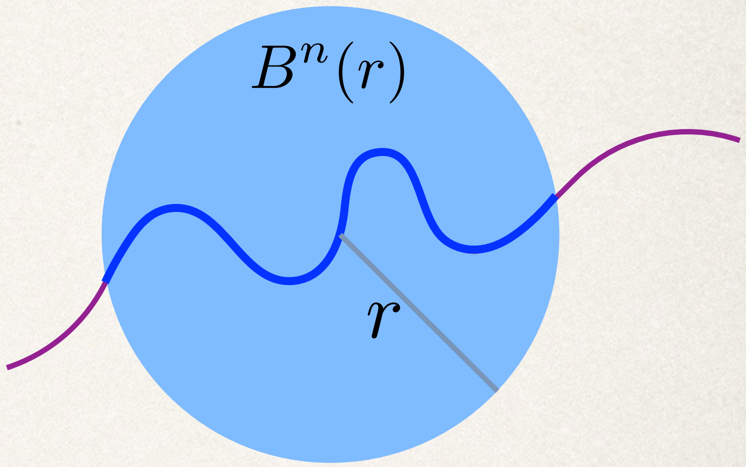

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- Sharply o-minimal set: $\mathcal{C}(n, A) = \mathcal{C}(F, D, n)$

explicitly computable function depending on the complexity (F, D) of the set [Binyamini,Novikov,Zack '23]

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- Key questions: Tame version of Nash's embedding theorem?

  $\rightarrow$ Conditions for the existence of tame π?

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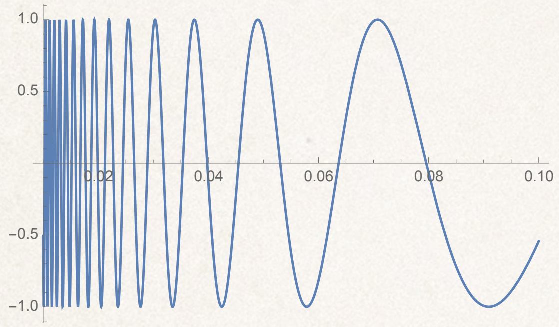

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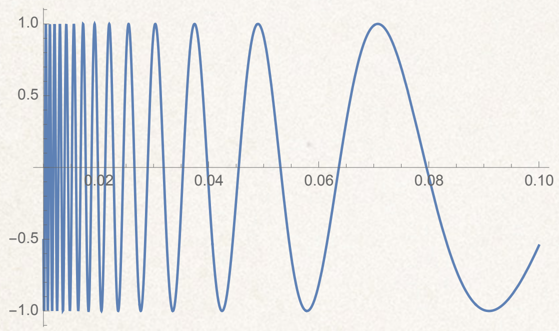

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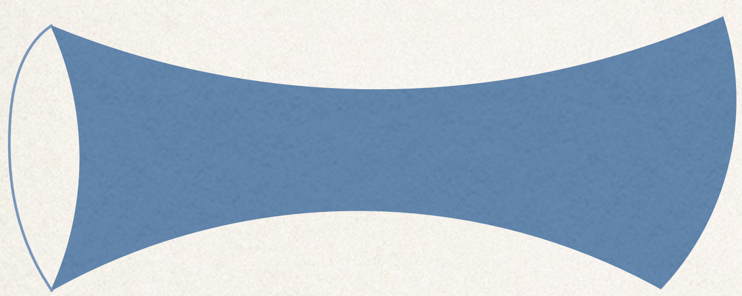

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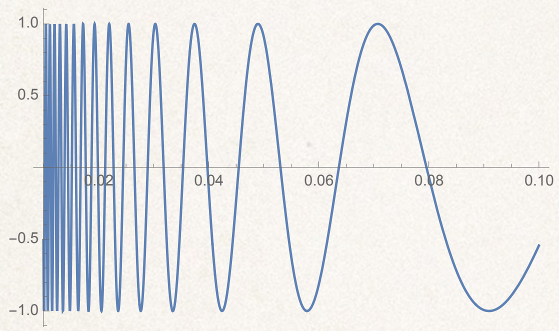

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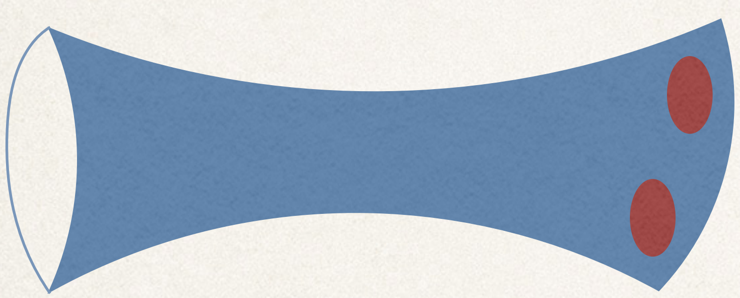

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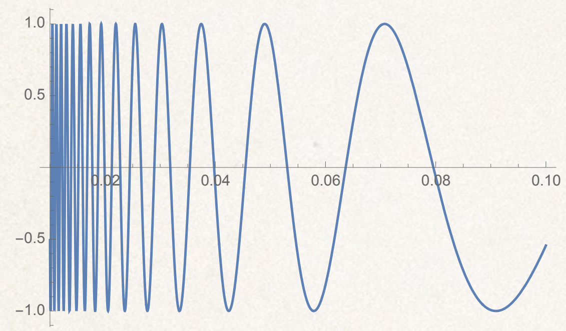

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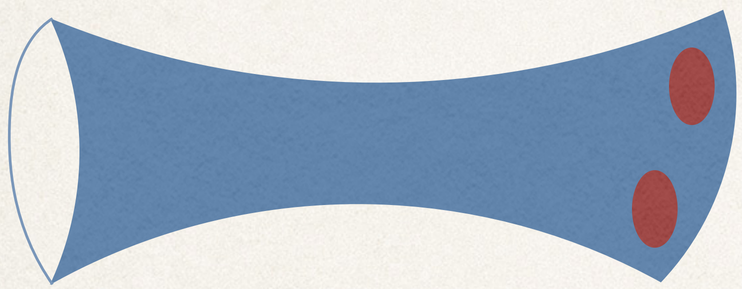

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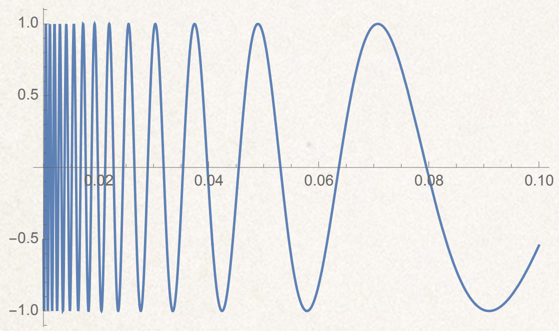

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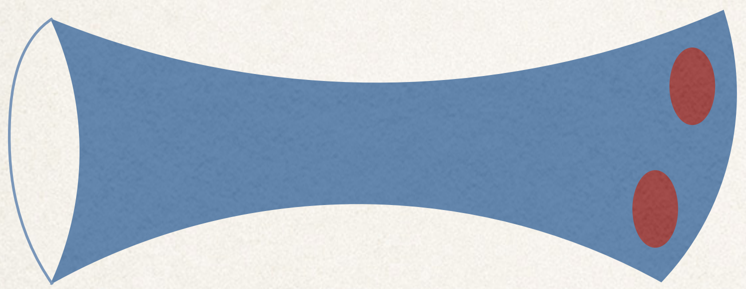

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(2) $B_1(r)$ and $B_2(r)$ are ϵ apart in $\mathbb{R}^n$ after embedding

[Fontenele,Xavier]

It's a tame world

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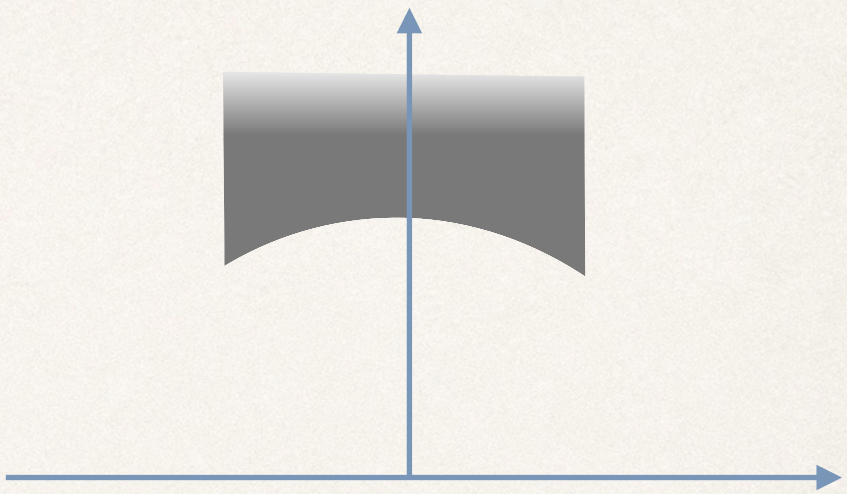

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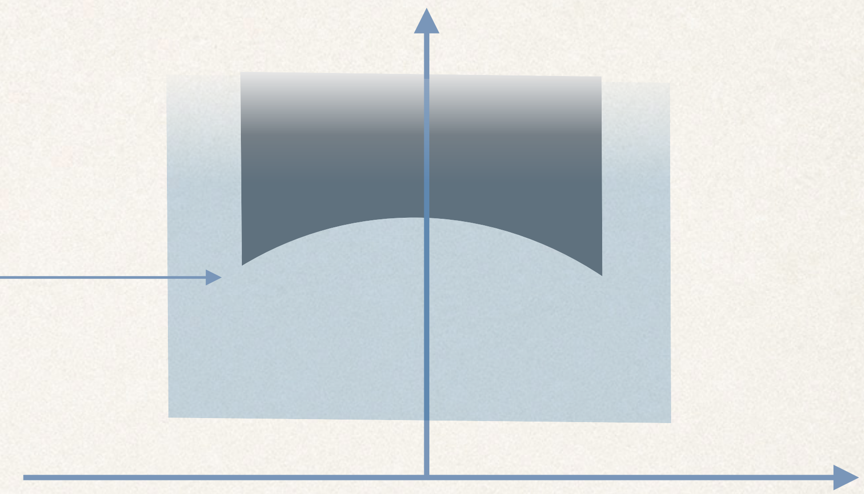

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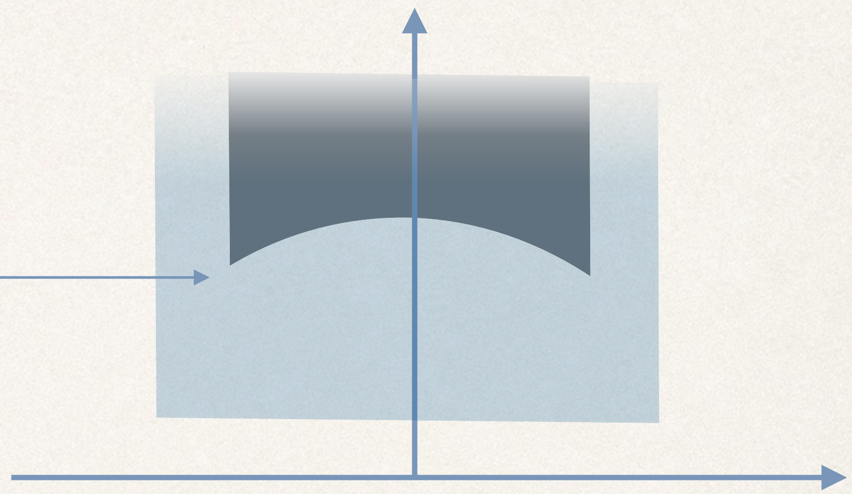

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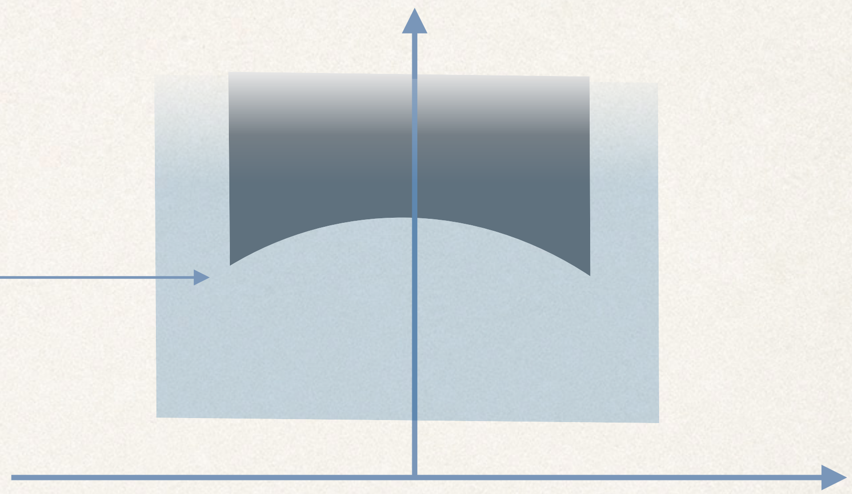

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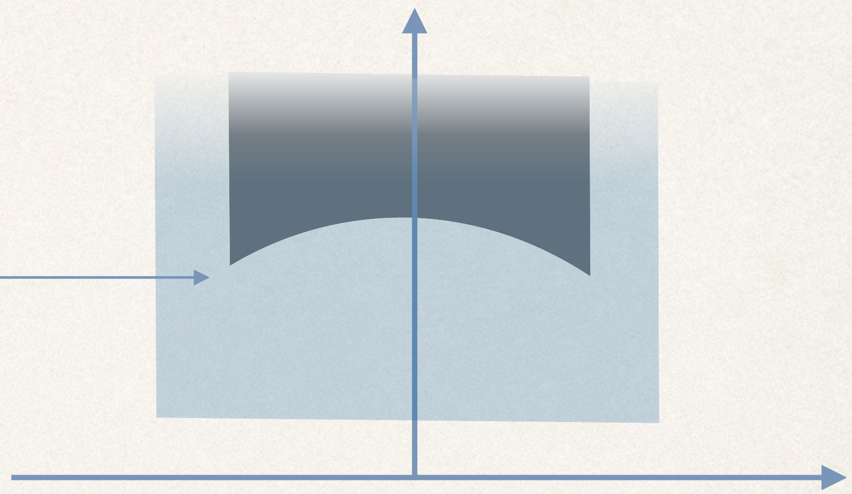

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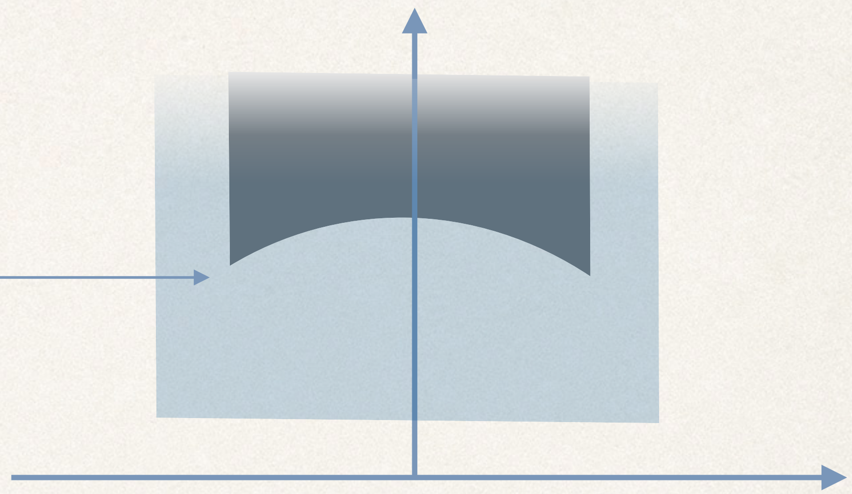

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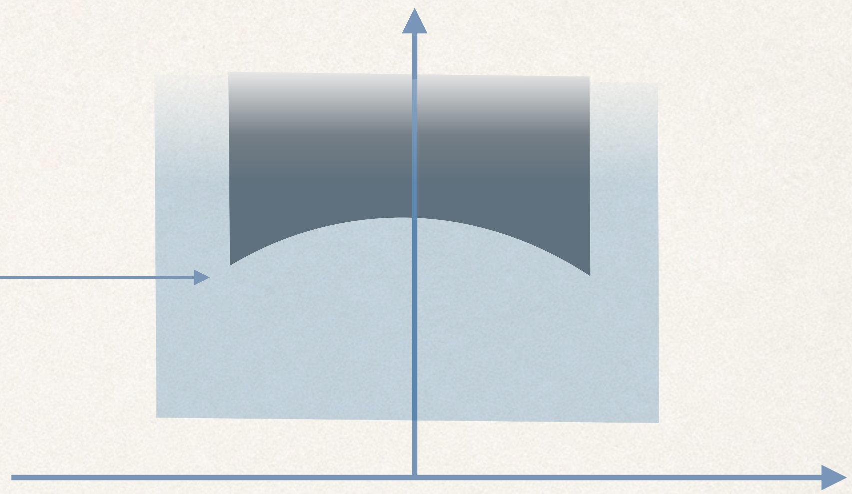

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$(\mathcal{M}, g_{\mathrm{cs}})$ have finite volume [Todorov '04]

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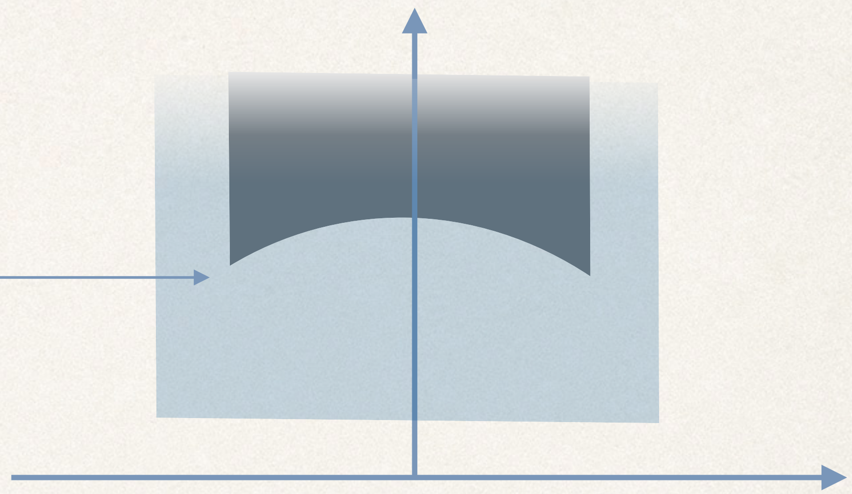

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$g_{\rm cs}$ metric on $\mathcal{M}$

$j(\tau)$ tame on $\mathcal{F}$ [Peterzil,Starchenko '04]

- General picture: $\mathcal{M}$ is complex structure moduli space of Calabi-Yau manifold

period map: $\Phi: \mathcal{M} \to \Gamma\backslash G/H$ is a tame map [Bakker,Klingler,Tsimerman '18]

$(\mathcal{M}, g_{\rm cs})$ have finite volume [Todorov '04]

$(\mathcal{M}, g_{\rm cs})$ have finite complexity (F, D) in progress

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  - vacuum locus always satisfied volume growth conjecture, fixed by (F, D)

    e.g. complexity behavior (F, D) of self-dual flux vacuum locus conjectured [TG,Monnee '23]

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- Future:
  - precise condition on existence of tame embedding (tame Nash)
  - complexity and the species scale
  - complexity to classify asymptotic limits

Geometry meets Physics:
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