BPS quivers, Galois covers and dualities in supersymmetric quantum mechanics

Cyril Closset

University of Birmingham

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based on work in progress:
(i) with Johannes Aspman, Elias Furrer, Horia Magureanu, Jan Manschot
(ii) with James Wynne

Introduction: from singularities to quiver quantum mechanics

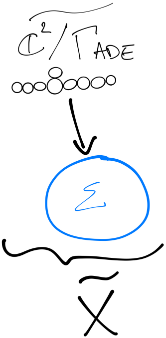
From singularities to 5d BPS quivers

Consider a local Calabi-Yau threefold $\tilde{\mathbf{X}}$ in Type IIA.

More precisely, consider the crepant resolution of a canonical singularity:

$$\pi \;:\; \tilde{\mathbf{X}} \longrightarrow \mathbf{X}$$

This is the traditional framework of geometric engineering. [Katz, Klemm, Vafa, 1996]



It gives us 4d $\mathcal{N}=2$ supersymmetric QFT/ gauge theory:

- Ruling $\leftrightarrow$ non-abelian gauge groups.
- D2-branes on fiber curves $\leftrightarrow$ W-bosons
- Gauge-theory instantons = worldsheet instantons.
  $\mathcal{F} = \mathcal{F}_{\text{pert}} + \sum_n c_n e^{-nt_f}$.
- Decouple the D2-branes on Σ, of size t_b.
  'Geometric engineering limit' (4d gauge theory limit):

$$t_f = \epsilon\;, \qquad \Lambda^{2N} = e^{-\frac{1}{g^2}} = \frac{e^{-t_b}}{\epsilon^{2N}} \text{ fixed}\;,\; \epsilon \to 0$$

(here for pure $SU(N)$)

From singularities to 5d BPS quivers

The '4d gauge theory limit' isn't very natural in IIA.

In recent years, we are more interested in understanding the map:

$$\mathrm{IIA}\ :\ \{\ \text{3-fold canonical singularities}\ \} \quad \longrightarrow \quad \{\ \mathcal{N}=2\ \text{SQFTs}\ \}$$

This map gives us Kaluza-Klein (KK) $\mathcal{N}=2$ theories, because:

$$\mathrm{M}\ :\ \{\ \text{3-fold canonical singularities}\ \} \quad \longrightarrow \quad \{\ \text{5d}\ \mathcal{N}=1\ \text{SCFTs}\ \}$$

since [Intriligator, Morrison, Seiberg, 1997] [Nekrasov, 1996; Lawrence, Nekrasov, 1997]

$$\begin{aligned} & \mathrm{M}[\mathbf{X}\times\mathbb{R}^{1,3}\times S^1] && = && \mathcal{T}_{\mathbf{X}}^{5\mathrm{d}}\ \text{SCFT on}\ \mathbb{R}^{1,3}\times S^1 \\ = \quad & \mathrm{IIA}[\mathbf{X}\times\mathbb{R}^{1,3}] && = && D_{S^1}\mathcal{T}_{\mathbf{X}}^{5\mathrm{d}}\ \mathcal{N}=2\ \text{KK theory on}\ \mathbb{R}^{1,3}. \end{aligned}$$

From singularities to 5d BPS quivers

Resolving $\mathbf{X}$ corresponds to going onto the Coulomb branch.

We are interested in the BPS states of the 4d $\mathcal{N} = 2$ KK theory:

$$\text{half-BPS excitations} \cong \text{D}p\text{-branes wrapping } \mathcal{C}_p \subset \tilde{\mathbf{X}}$$

They arise as (bound states of) 5d BPS states (wrapped M2-branes, electric), 5d BPS strings wrapped on S^1 (wrapped M5-branes, magnetic) and KK momentum ($D0$-branes).

- Stable D-brane states are not geometric except at large Kähler volume, due to worldsheet instanton corrections.
- We can first focus on understanding the **BPS category** of $D_{S^1}\mathcal{T}_{\mathbf{X}}^{5\text{d}}$. In the case at hand, it is identified with **the B-brane category**, which is independent on the Kähler parameters:

$$\mathscr{T}^{\text{BPS}}_{D_{S^1}\mathcal{T}_{\mathbf{X}}^{5\text{d}}} \cong \text{D}(\text{coh}\,\tilde{\mathbf{X}})$$

A B-brane is an object in the derived category of coherent sheaves of $\tilde{\mathbf{X}}$.

From singularities to 5d BPS quivers

In the small volume limit (near the 5d SCFT point), we often have a quiver description.

Definition [CC, Del Zotto, 2019]: a **5d BPS quiver** for a 5d SCFT $\mathcal{T}^{5d}_{\mathbf{X}}$ is any quiver with superpotential $(\mathcal{Q}_{\tilde{\mathbf{X}}}, W)$ such that:

$$\mathscr{T}^{\mathrm{BPS}}_{D_{S^1}\mathcal{T}^{5d}_{\mathbf{X}}} \cong \mathrm{D}(\mathrm{coh}\,\tilde{\mathbf{X}}) \cong \mathrm{D}(\mathcal{Q}_{\tilde{\mathbf{X}}}\mathrm{-rep})$$

- 5d BPS quivers are a natural generalisations of BPS quivers of ordinary 4d $\mathcal{N}=2$ SQFTs. [Fiol, Marino, 2002; Denef, 2002; Alim, Cecotti, Cordova, Espahbodi, Rastogi, Vafa, 2011]
- In geometric engineering in IIA, they are precisely the fractional brane quivers for branes probing the singularity $\mathbf{X}$.
- For our purpose today, the most important way to think of them is as **'meta' gauged $\mathcal{N}=4$ supersymmetric quantum mechanics.**

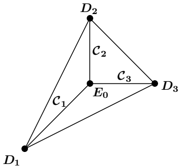
From singularities to 5d BPS quivers

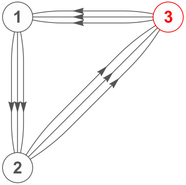
BPS quiver crash course (part 1):

- Quiver $\mathcal{Q} = (Q_0, Q_1)$: nodes $(i) \in Q_0$ and arrows $a \in Q_1$.
- simple BPS object $\longleftrightarrow$ fractional brane $\mathcal{E}_i$ $\longleftrightarrow$ quiver node (i);
- Dirac pairing $\longleftrightarrow$ open strings $\mathrm{Ext}^1(\mathcal{E}_i, \mathcal{E}_j)$ $\longleftrightarrow$ quiver arrows $a : (i) \to (j)$.

(Aslo a W, which we mostly ignore today.)

Example: $\mathbf{X} = \mathbb{C}^3/\mathbb{Z}_3$, $\tilde{\mathbf{X}} = \mathcal{O}_{\mathbb{P}^2}(-3)$ [Douglas, Moore, 1996]





This is a 5d BPS quiver for $\mathcal{T}^{5\mathrm{d}}_{\mathbf{X}} = E_0$, a rank-one 5d SCFT. [Morrison, Seiberg, 1997]

From singularities to 5d BPS quivers

BPS quiver crash course (part 2):

- A **quiver representation** $\mathfrak{R}(\mathcal{Q}) = (V, X)$ is the datum:

$$(i) \mapsto V_i \cong \mathbb{C}^{N_i} \ , \qquad [a : (i) \to (j)] \mapsto X_{ij} \in \mathrm{Hom}(V_i, V_j)$$

  The positive integers N_i are called the quiver ranks or dimension of $\mathfrak{R}$.

- Fixing some quiver ranks $\gamma = (N_i)$, we can consider the moduli space of (semi-stable) representations:

$$\mathcal{M}_\gamma = \left\{ X \middle| \partial_X W = 0 \right\}^{\mathrm{ss}} / \mathrm{GL}_\gamma$$

  Here we defined

$$\mathrm{GL}_\gamma \equiv \prod_i \mathrm{GL}(N_i)$$

- The stability condition is crucial in this definition. [King, 1994]

From singularities to 5d BPS quivers

4d $\mathcal{N}=2$ BPS states in terms of a **gauged** $\mathcal{N}=4$ **SUSY QM** on their worldline:

- Electro-magnetic charge $\gamma \in \Lambda \ \longleftrightarrow\ \gamma \in K(\mathrm{D}(\mathrm{coh}\, \tilde{\mathbf{X}})) \ \longleftrightarrow\$ quiver ranks.
- This defines a $\mathcal{N}=4$ SQM with gauge group

$$G = \prod_i U(N_i)$$

and the Higgs branch $\mathcal{M}_\gamma$ is the moduli space of representations:

$$\mathcal{M}_\gamma = \{X \,|\, \partial_X W = 0\ ,\ D(\xi) = 0\}/G\ .$$

The stability conditions arise from the Fayet-Iliopoulos (FI) terms in the D-terms.

- Quantising the Higgs branch gives us the BPS states. Roughly:

$$\mathscr{H} \cong H^{\bullet,\bullet}(\mathcal{M}_\gamma)\ .$$

- **Counting BPS states:**

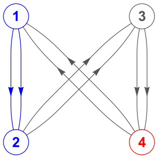
$$\Omega(\gamma) = \text{Witten index}[\mathrm{SQM}] = (-1)^{\mathrm{vdim}(\mathcal{M}_\gamma)}\chi(\mathcal{M}_\gamma) = \text{DT invariant of } \tilde{\mathbf{X}}$$

Towards BPS spectra from 5d BPS quivers?

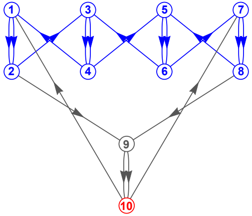
5d BPS quivers can be quite involved. Direct computation of spectra not practical.

For instance, the 5d SCFT that flows to 5d $SU(p)_q$ arises at the $Y^{p,q}$ singularity:

"5d $SU(2)_0$":



"5d $SU(5)_0$":



- For 4d $\mathcal{N} = 2$ SQFTs with finite chambers: mutation method. [Alim *et al.*, 2011]
- 5d BPS quivers never have finite chambers – infinite spectrum (KK towers).
- Much recent progress was made using **discrete symmetries** and collimation chambers. [Bonelli, Del Monte, Tanzini, 2020; Longhi, 2021; Del Monte, Longhi, 2021; Bridgeland, Del Monte, Giovenzana, 2024]
- Building on this body of work, we find new relations between 5d BPS quivers, and between their BPS spectra, using the concept of **Galois cover** first introduced in the physics literature (for 4d BPS quivers) in [Cecotti, Del Zotto, 2015].

Galois covers of 5d BPS quivers

Galois covers of BPS quivers

Consider a quiver $\mathcal{Q} = (Q_0, Q_1)$ with $|Q_0| = M$ nodes.

Let $\mathbb{G}$ be a finite abelian group. For definiteness, we can take $\mathbb{G} = \mathbb{Z}_n$.

Assign to every arrow $a \in Q_1$ a $\mathbb{G}$ grading:

$$\deg(a) = g_{(a)} \in \mathbb{G} \ .$$

[Cecotti, Del Zotto, 2015; cite mathematicians here...]

The **Galois covering quiver $\tilde{\mathcal{Q}} = (\tilde{Q}_0, \tilde{Q}_1)$ for this $\mathbb{G}$ grading** is obtained as follows:

- For every node $(i) \in Q_0$, we have $|\mathbb{G}|$ nodes (a 'block' of nodes)

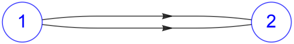
$$(i, g) \in \tilde{Q}_0 \ , \qquad \forall g \in \mathbb{G} \ .$$

- For every arrow $[a : (i) \to (j)]$, we have an arrow

$$[\tilde{a} : (i, g) \to (j, g')] \quad \text{if } g' - g = \deg(a) \ .$$

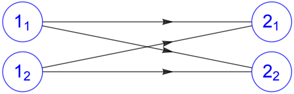
Galois covers of BPS quivers

Example 1: $\mathbb{G} = \mathbb{Z}_2$, 2-cover of the Kronecker quiver $\mathcal{Q} = K(2)$: [Gaiotto, Moore, Neitzke, 2008] [Cecotti, Del Zotto, 2015]



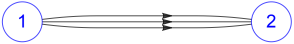
BPS quiver of pure $\mathcal{N} = 2$ SYM $SU(2)$.

$$\deg(a_0, a_1) = (0, 1)$$

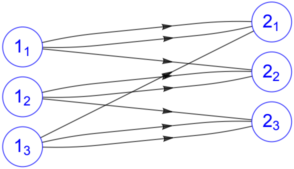


BPS quiver of $\mathcal{N} = 2$ SYM $SU(2)$, $N_f = 2$.

Example 2: $\mathbb{G} = \mathbb{Z}_3$, 3**-cover of** $\mathcal{Q} = K(3)$**.**

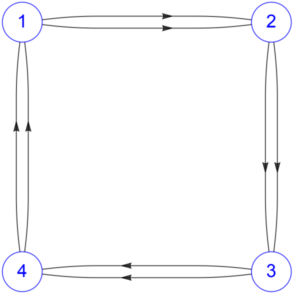


$$\deg(a_0, a_1, a_2) = (0, 0, 1)$$



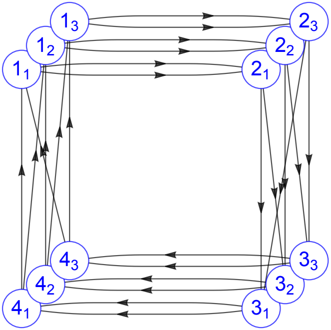
Galois covers of 5d BPS quivers

Example 3: $\mathbb{Z}_n$ covers of 5d $SU(2)_0$ gives the 5d $SU(2n)_0$ theory $(\mathbf{X} = C(Y^{2n,0}))$:



$$\deg(a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1)$$

$$= (0, 0, 0, 1, 0, 0, 0, 1)$$



3-cover: 5d $SU(6)_0$.

Note: the superpotential is lifted as well, so this is a genuine Galois covering.

Galois covering functors

The inverse of the Galois cover is the map:

$$F \ : \ \tilde{\mathcal{Q}} \longrightarrow \mathcal{Q} \cong \tilde{\mathcal{Q}}/\mathbb{G} \ .$$

Roughly, it is the gauging of a discrete symmetry $\mathbb{G}$ of the quiver $\tilde{\mathcal{Q}}$. This discrete symmetry permutes nodes and arrows and leaves the superpotential invariant.

There exists induced actions on quiver representations:

$$F_\lambda : \tilde{\mathcal{Q}}-\text{rep} \longrightarrow \mathcal{Q}-\text{rep} \ , \qquad\qquad F^\lambda : \mathcal{Q}-\text{rep} \longrightarrow \tilde{\mathcal{Q}}-\text{rep} \ .$$

In particular, the push-down functor relates the quiver ranks

$$\gamma = \sum_i N_i \gamma_{(i)} \ , \qquad\qquad \tilde{\gamma} = \sum_i \sum_g N_{i,g} \tilde{\gamma}_{(i,g)} \ .$$

by summing the ranks of a 'block' in the $\tilde{\mathcal{Q}}$ quiver:

$$F_\lambda(\tilde{\gamma}) = \gamma \ , \qquad \text{with} \quad N_i = \sum_g N_{i,g} \ .$$

$\mathbb{G}$-action on the Higgs branch and fixed loci

Given the $\mathbb{G}$-grading of the quiver $\mathcal{Q}$, we can define a $T = U(1)$ action on the Higgs branch of the SQM at fixed quiver rank,

$$\mathcal{M}_\gamma^{\mathcal{Q}} \cong \{X \,|\partial_X W = 0\}^{ss}/\mathrm{GL}_\gamma \ , \qquad T \ : \ \mathcal{M}_\gamma \to \mathcal{M}_\gamma \ ,$$

Consider the isomorphism

$$\mathbb{G} \overset{\cong}{\to} \hat{\mathbb{G}} \equiv \mathrm{Hom}(\mathbb{G}, U(1)) \ : \ g \mapsto \hat{g} \ .$$

For definiteness, consider

$$\mathbb{G} = \mathbb{Z}_n = \{k \in \mathbb{Z} \mod n\} \ , \qquad g = k \mapsto \hat{g} = e^{\frac{2\pi i k}{n}} \ .$$

We then have the circle action on the moduli space of quiver representations

$$T \ : \ \mathcal{M}_\gamma^{\mathcal{Q}} \to \mathcal{M}_\gamma^{\mathcal{Q}}$$

induced by the grading of the arrows:

$$T \ : \ X \to \hat{g}_{(a)} X \ , \qquad \text{if } \ \mathfrak{R}[a] = X \text{ and } \deg(a) = g_{(a)} \ .$$

$\mathbb{G}$-action on the Higgs branch and fixed loci

The set of fixed loci of the T action is deeply related to the moduli spaces of the Galois covering quiver.

Consider all ranks $\tilde{\gamma} = (N_{i,g})$ of $\tilde{\mathcal{Q}}$ that are projected to the ranks $\gamma = (N_i)$ by the pull-down functor:

$$N_i = \sum_{g \in \mathbb{G}} N_{i,g}$$

Then the fixed locus are in 1-to-$|\mathbb{G}|$ correspondence with the moduli spaces

$$\mathcal{M}_{\tilde{\gamma}}^{\tilde{\mathcal{Q}}} \qquad \text{such that} \quad F_\lambda(\tilde{\gamma}) = \gamma \ .$$

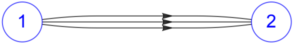
- This holds assuming some 'nice enough' properties – e.g. $\mathcal{M}_\gamma$ smooth, compact (?).
- The FI terms of $\tilde{\mathcal{Q}}$ are fined-tuned to preserve the $\mathbb{G}$ symmetry

$$\xi_{i,g} = \xi_i \quad \forall g \in \mathbb{G}$$

(Potential issues when we land on walls of marginal stability.)

$\mathbb{G}$-action on the Higgs branch and fixed loci

Example: 3**-cover of** $\mathcal{Q} = K(3)$**,** $\deg(a_0, a_1, a_2) = (0,0,1)$**.**



Take $\gamma = (N_i) = (1,1)$. Then

$$\mathcal{M}^{\mathcal{Q}}_{\gamma=(1,1)} = \mathbb{P}^2 \cong \{[X_0, X_1, X_2]\} \ .$$

We have

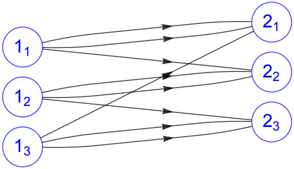
$$T \ : \ [X_0, X_1, X_2] \to [X_0, X_1, e^{\frac{2\pi i}{3}} X_2]$$

Fixed loci:

$$\{[X_0, X_1, 0]\} \cong \mathbb{P}^1$$

$$\{[0, 0, X_2]\} \cong \mathbb{P}^0 = \mathrm{pt}$$

Let $\tilde{\gamma} = (N_{1_1}, N_{1_2}, N_{1_3}; N_{2_1}, N_{2_2}, N_{2_3})$.



Then for $\tilde{\gamma} = (1,0,0;1,0,0)$ (and 3 permutations) we have

$$\mathcal{M}^{\tilde{\mathcal{Q}}}_{\tilde{\gamma}} \cong \mathbb{P}^1 \ .$$

For $\tilde{\gamma}' = (1,0,0;0,1,0)$ (and 3 permutations) we have

$$\mathcal{M}^{\tilde{\mathcal{Q}}}_{\tilde{\gamma}'} \cong \mathbb{P}^0 \ .$$

Relations between BPS indices

For any T-action on a variety $\mathcal{M}$ which has fixed loci $F_s \subset \mathcal{M}$, we have

$$\chi(\mathcal{M}) = \sum_s \chi(F_s) \ ,$$

e.g. by Atiyah-Bott localisation.

Given the relations just discussed, we therefore obtain relations between the BPS indices induced by the Galois covering functor:

$$\Omega_Q(\gamma) = \frac{1}{|\mathbb{G}|} \sum_{\tilde{\gamma} \,|\, F_\lambda(\tilde{\gamma}) = \gamma} (-1)^{\mathrm{vdim}(\mathcal{M}_{\tilde{\gamma}}) - \mathrm{vdim}(\mathcal{M}_\gamma)} \Omega_{\tilde{Q}}(\tilde{\gamma})$$

- These can be checked experimentally in a large number of examples.
- There are various subtleties whenever the moduli spaces are “too singular”. (Under investigation.)

Galois covers as orbifolds $\mathbf{X}/\mathbb{G}$

Consider 5d BPS quivers related to a singularity $\mathbf{X}$:

$$\mathbf{X} \quad \longleftrightarrow \quad D_{S^1}\mathcal{T}^{5\text{d}}_{\mathbf{X}} \quad \longleftrightarrow \quad \mathcal{Q}_{\mathbf{X}}$$

We make the following **conjecture:** the $\mathbb{G}$-Galois covering of $\mathcal{Q}_{\mathbf{X}}$ is always the fractional brane quiver for some $\mathbb{G}$-orbifold of $\mathbf{X}$:

$$\mathcal{Q}_{\mathbf{X}} = \tilde{\mathcal{Q}}_{\mathbf{Y}}/\mathbb{G} \qquad \Leftrightarrow \qquad \mathbf{Y} = \mathbf{X}/\mathbb{G}$$

Example: The 5d $SU(2n)_0$ BPS quiver is the n-cover of the 5d $SU(2)_0$ BPS quiver:

$$\mathbf{Y} = C(Y^{2n,0}) = \mathbf{X}/\mathbb{Z}_n = C(Y^{2,0})/\mathbb{Z}_n$$

- For toric singularities and their toric abelian orbifolds, one can prove this using brane tiling techniques. [Hanany, Herzog, Vegh, 2006]
- Not all orbifolds arise in this way. (We want a ‘block quiver structure’.)
- Proof of general conjecture an interesting challenge.

Dualities and trialities in $\mathcal{N} = 2$ SQM

Gauged $\mathcal{N} = 2$ SQM

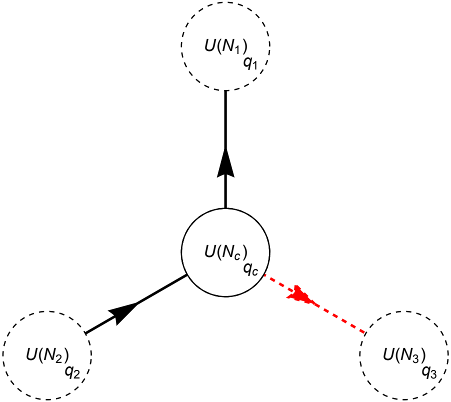
Consider now half of the supersymmetry. The **gauged $\mathcal{N} = 2$ SQM** consists of:

- Vector multiplet $\mathcal{V} = (A_t, \sigma, \lambda, \bar{\lambda}, D)$;
- Chiral multiplets $\Phi = (\phi, \psi)$;
- Fermi multiplets $\Lambda = (\Lambda, \mathcal{G})$
- 1d $\mathcal{N} = 2$ superpotentials $E(\Phi)$ and $J(\Phi)$ for each Λ.

Considering unitary gauge groups, this allows us to construct (representations of) **generalised quivers.**

Note that, for each $U(N)$ gauge group, we have the freedom to turn on some "1d Chern–Simons term" – a background gauge charge:

$$L = q \int dt \, \mathrm{Tr}(A_t) \ .$$



Gauged $\mathcal{N}=2$ SQM: Γ-SQCD

We consider the following system, which we call Γ-SQCD:

$$1\text{d } \mathcal{N}=2\ U(N_c)_q \quad \text{with} \quad N_1\ \Phi \oplus N_2\ \tilde{\Phi} \oplus N_3\ \Lambda \oplus N_1 N_2\ \Gamma$$

Here we have

$$Q_c \equiv q_c + \frac{1}{2}(N_1 - N_2 - N_3) \in \mathbb{Z}$$

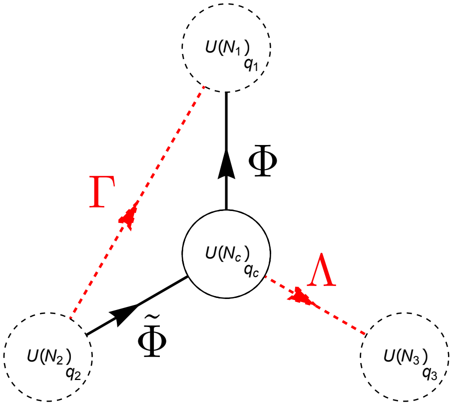
and we introduced the Γ fields, which are gauge-singlets "fermionic mesons" with a E-term:

$$E_\Gamma = \tilde{\Phi}\Phi$$

We also have the FI term:

$$L_{\mathrm{FI}} = \zeta \int dt D$$

For $\zeta \neq 0$, we expect **a finite number** of supersymmetric ground states.



The Witten index of Γ-SQCD

This theory has a flavour symmetry $G_F = U(N_1) \times U(N_2) \times U(N_3)/U(1)$.

The flavoured Witten index

$$\mathrm{I}_W(y;\zeta) = \mathrm{Tr}((-1)^{\mathrm{F}} y^{Q_F}) \in \mathbb{Z}(y)$$

can be computed by supersymmetric localisation on the 1d Coulomb branch:

$$\begin{aligned}\mathrm{I}_W(y;\zeta) \quad &= \frac{1}{N_c!} \oint_{\mathrm{JK}(\zeta)} \prod_{a=1}^{N_c} \left[-\frac{dx_a}{2\pi i x_a}\, x_a^{Q_c} \frac{\prod_{k=1}^{N_3}(1-\frac{x_a}{y_{3,k}})}{\prod_{i=1}^{N_1}(1-\frac{x_a}{y_{1,i}}) \prod_{j=1}^{N_2}(1-\frac{y_{2,j}}{x_a})} \right] \\ &\times \prod_{a \neq b}^{N_c} (1-\frac{x_a}{x_b}) \times \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} (1-\frac{y_{2,j}}{y_{1,i}})\end{aligned}$$

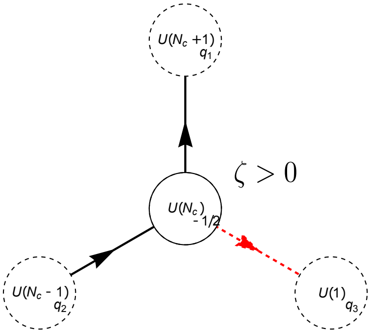
Here the JK residue depends on the sign of the FI parameter ζ. [Kim, Yi, Hori, 2014]
We can have **non-trivial wall-crossing:**

$$\Delta \mathrm{I}_W(y) \equiv \mathrm{I}_W(y;\zeta>0) - \mathrm{I}_W(y;\zeta<0) \neq 0$$

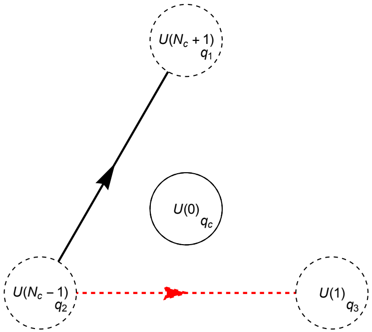
Confinement of gauged $\mathcal{N} = 2$ SQM

Definition: A gauged SQM "confines" if the ground state can be described by free fields.

Example: We claim that the following two descriptions are dual:



$U(N_c)_{-1/2}$, $(N_1, N_2, N_3) = (N_c{+}1, N_c{-}1, 1)$.



Free chirals and fermis.

This was found "experimentally" looking at 1d-3d coupled systems. [CC, Khlaif, 2023]

Question: How can we determine whether a 1d theory confines (ideally without doing any computation)?

Seiberg-like dualities of $\mathcal{N} = 2$ gauged SQM

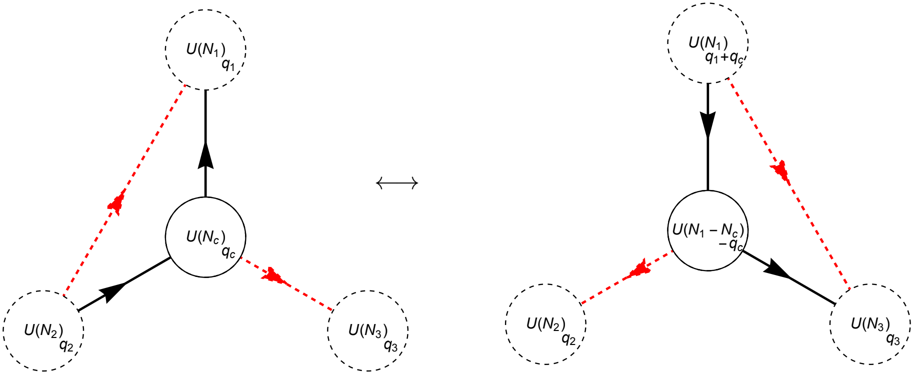
[CC, Wynne, to appear]

To answer this question, we derived new Seiberg-like dualities of $U(N_c)$ $\mathcal{N} = 2$ SQM.

- These are **infrared dualities:** in quantum mechanics, this simply means that the supersymmetric ground states have are isomorphic.
- The dualities, like the ground states, depend on the sign of the FI term.
- There are two basic dualities: the one for $\zeta > 0$ and for $\zeta < 0$, respectively, which we may call right-mutation and left-mutation.
- We can easily prove that the Witten indices matches across the dualities.
- In many examples, we can also explicitly show that the ground states are isomorphic.
- This is an interesting toy model of an infrared duality in QFT, where everything is (in principle) under control.
- These dualities are similar but logically distinct to the 1d $\mathcal{N} = 4$ Seiberg dualities/mutations, which have been better studied (sort of...).

Seiberg-like dualities of $\mathcal{N} = 2$ gauged SQM

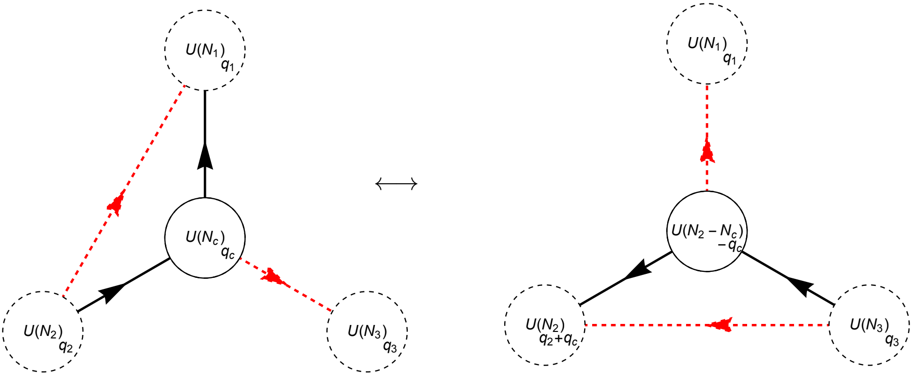
Right-mutation: duality for $U(N_c)_{q_c}$ with $\zeta > 0$.



$U(N_c)_{q_c}$, $\zeta > 0$ $\longleftrightarrow$ $U(N_1 - N_c)_{-q_c}$, $\zeta^D < 0$

Seiberg-like dualities of $\mathcal{N} = 2$ gauged SQM

Left-mutation: duality for $U(N_c)_{q_c}$ with $\zeta < 0$.



$U(N_c)_{q_c}$, $\zeta < 0$ $\longleftrightarrow$ $U(N_2 - N_c)_{-q_c}$, $\zeta^D > 0$

Note the shift of the 1d CS term of the adjacent 'flavour' node, which becomes relevant when we embed this into a larger 1d $\mathcal{N} = 2$ quiver.

Duality + trivial wall-crossing (TWC) = triality

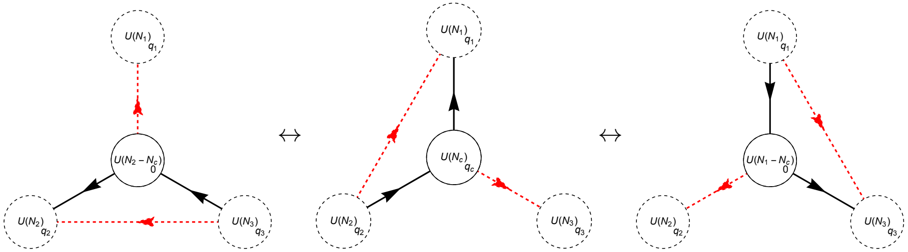
Sometimes wall-crossing is trivial (TWC). We can consider this as an infrared duality as well:

$$U(N_c)_{q_c} \ , \ \zeta > 0 \qquad \longleftrightarrow \qquad U(N_c)_{q_1} \ , \ \zeta < 0$$

A sufficient set of conditions to have trivial wall-crossing is

$$-\mathcal{A} - 1 < q_c < \mathcal{A} + 1 \ , \qquad \mathcal{A} \equiv \frac{1}{2}(N_1 + N_2 - N_3) - N_c \ .$$

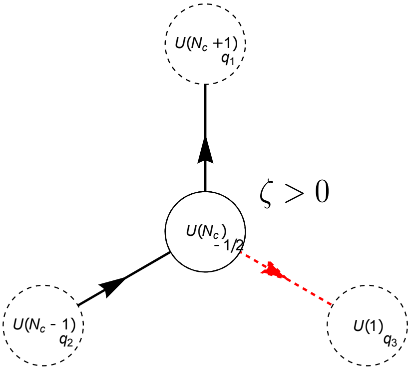
Combining TWC with the mutation dualities, we obtain various types of trialities.
A very special case is $\mathcal{A} = 0$ and $q_c = 0$:



This is the 1d reduction of the 2d $\mathcal{N} = (0, 2)$ triality of [Gadded, Gukov, Putrov, 2013].

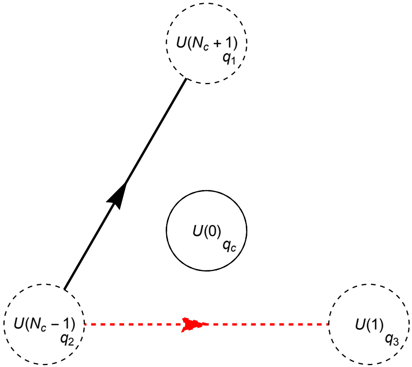
Duality + trivial wall-crossing (TWC) = triality

This explains our earlier example of confinement, which is part of a triality of this type:



$U(N_c)_{-1/2}$, $(N_1, N_2, N_3) = (N_c{+}1, N_c{-}1, 1)$.

$\mathcal{A} = -\frac{1}{2}$ and $q_c = -\frac{1}{2}$ and non-trivial WC.



Free chirals and fermis?

Duality + trivial wall-crossing (TWC) = triality

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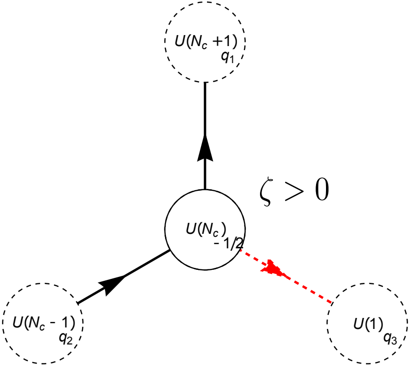

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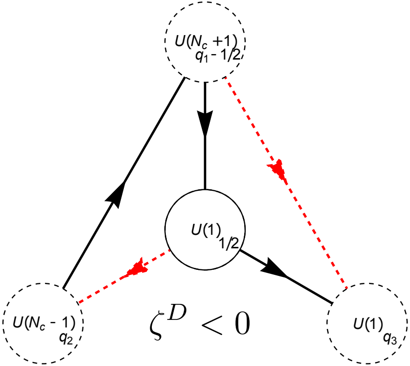

Abelian dual with $\zeta^D < 0$.

$\mathcal{A}^D = \frac{1}{2}$ and $q_c^D = -\frac{1}{2}$: TWC!

Duality + trivial wall-crossing (TWC) = triality

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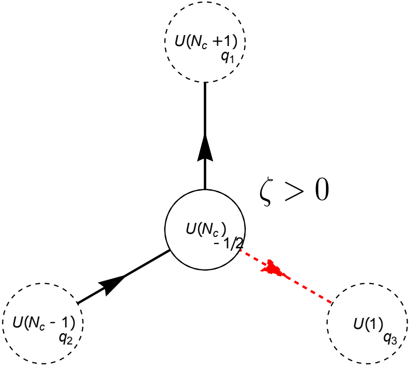

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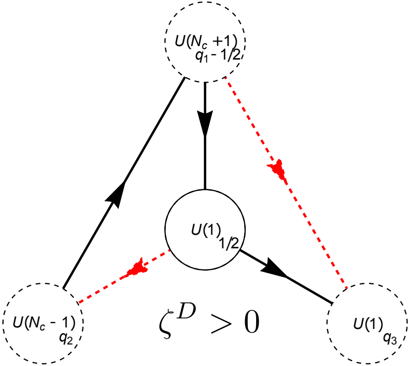

Abelian dual with $\zeta^D > 0$.

$\mathcal{A}^D = \frac{1}{2}$ and $q_c^D = -\frac{1}{2}$: TWC!

Duality + trivial wall-crossing (TWC) = triality

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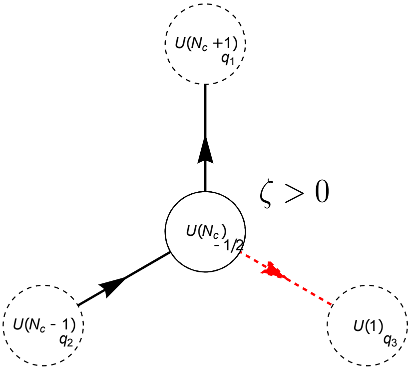

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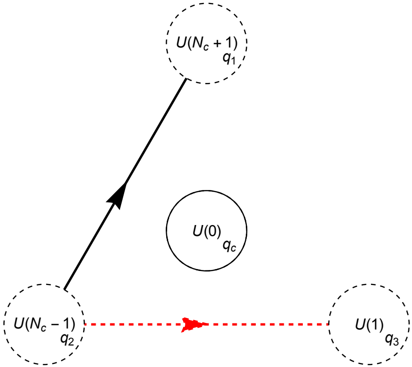

Free chirals and fermis!

Summary and outlook

Summary:

- We are having fun with 1d supersymmetric models.
- For $\mathcal{N} = 4$ SQM quivers:
  - We further explored the notion of Galois cover.
  - We obtained new relations between BPS spectra of distinct 5d SCFTs on a circle.

  For $\mathcal{N} = 2$ SQM quivers:
  - We discovered new Seiberg-like (mutation) dualities.
  - This gives us new simple criteria (sufficient conditions) for "1d confinement".

Outlook:

- Can we "bootstrap" the BPS spectra of complicated 5d BPS quivers starting from simpler ones?
- Systematic relations between KS algebras and WC invariants through $\mathbb{G}$ gauging?
- Full proof of 1d $\mathcal{N} = 2$ duality at the level of the ground states?
- Other 1d Seiberg dualities (e.g. for other gauge groups)?