# Topological Strings and Spectral Theory: a Non-Perturbative Approach

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# Introduction and Motivation

In physics, perturbation theory often leads to divergent series expansions, and string theory is no exception.

$$F(g) \sim \sum_{k \geq 0} g^k a_k, \qquad a_k \sim k!$$

This asymptotic behavior signals that perturbation theory alone does not fully capture the physics of the system.

# Introduction

Given an asymptotic series

$$\sum_{k\geq 0} g^k a_k, \qquad a_k \sim k! \tag{1}$$

We say that a function $F(g)$ is a non-perturbative completion of (1) if

1) $F(g)$ is a well defined function of $g$ at least in some domain

2) $F(g)$ reproduces (1) when expanded around $g = 0$

# Introduction

Given an asymptotic series, its non-perturbative completion is not unique.

If $F(g)$ is a non-pert completion, also $F(g) + e^{-1/g^2}$ is as well

To ensure uniqueness, one needs more information, such as a Hamiltonian or a differential equation with specified boundary conditions.

$$-\hbar\partial_x^2\varphi(x) + V(x)\varphi(x) = E\varphi(x)$$

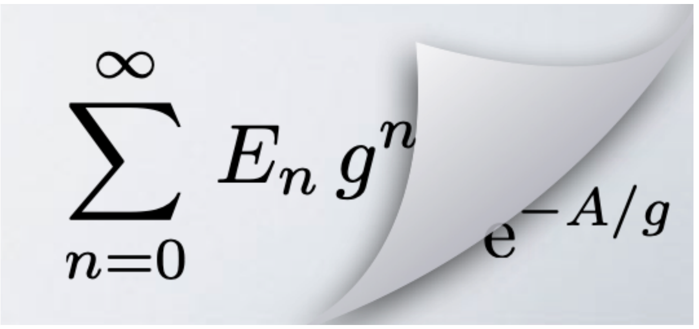
In string theory this uniqueness is not built in a priori.

# Introduction

One systematic way to approach the problem of finding a non-perturbative completion is via resurgence

→ Given a perturbative series, one can construct a non-perturbative completion using its (median) Borel summation

This approach is very general and concrete but often does not capture the full richness of the problem.

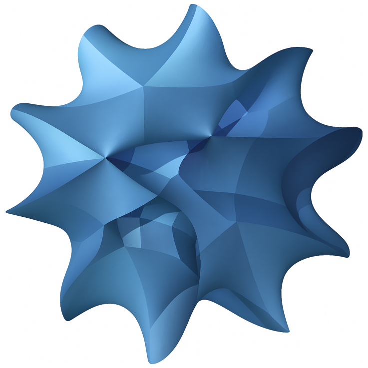


→ see Les Houches lectures Marcos Mariño

# Introduction

In string theory, however, we have another powerful tool: string dualities

→ Not only provide a concrete handle on non-perturbative effects but allow to uncover unexpected connections leading to beautiful new mathematics and new physics.



geometrical objects

algebraic objects

$$\int \mathrm{d}M \mathrm{e}^{-N \mathrm{Tr}(V(M))}$$

# Introduction

Today we focus on topological string on toric CY manifolds* and we study its non-perturbative structure via the topological string/spectral theory duality

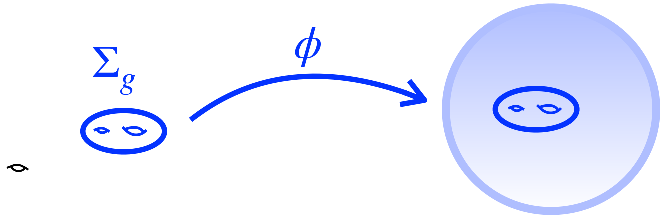
Topological string/
Enumerative geometry

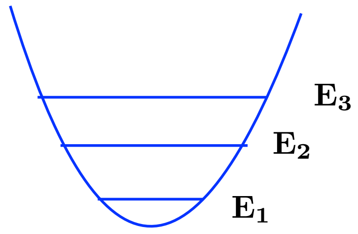
TS/ST

[AG-Hatsuda-Mariño]

Spectral theory of a class of
quantized mirror curves

$$O(\hat{x},\hat{p}), \quad [\hat{x},\hat{p}] = i\hbar$$





A-model objects

B-model objects

* + extension to elliptic cases [Hatsuda, Sciarappa, Zakany]

# Plan of the talk

1. Topological strings and special functions
2. Spectral theory of quantized mirror curves
3. TS/ST duality (closed sector)
4. Some applications of TS/ST
5. TS/ST duality (open sector)
6. Outlook

# Topological String Theory

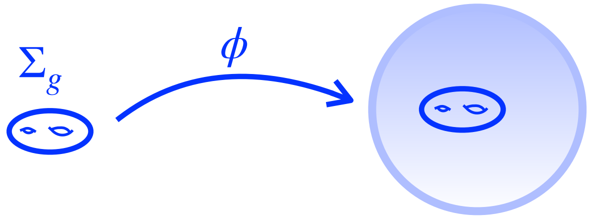
# Topological String Theory

In topological string theory the free energies encode in a precise way the enumerative geometry of the target space X

$$F_g(t) = \sum_{d \geq 1} N_g^d \, \mathrm{e}^{-dt}$$

$N_g^d$ are the Gromov-Witten (GW) invariants: "count" holomorphic maps

$$\phi : \quad \Sigma_g \quad \rightarrow \quad X$$

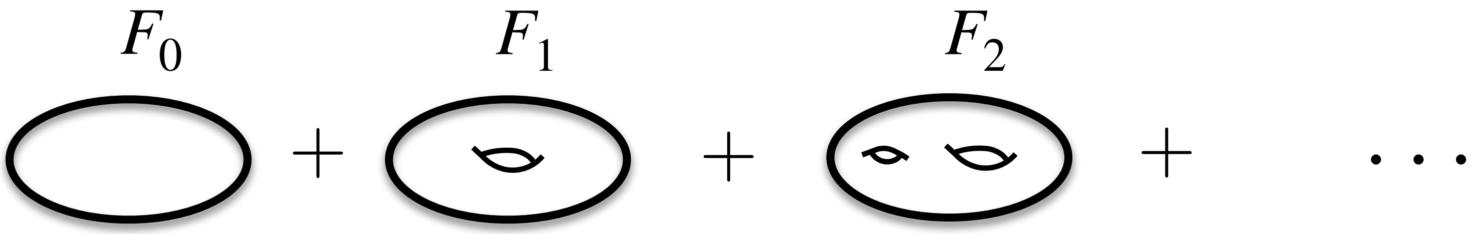


t: Kähler parameter of X

# Topological String Theory

The (formal) partition function Z is obtained by summing over all genera

$$\log Z = F = \sum_{g\geq 0} g_s^{2g-2} F_g(t)$$



Problem: $F_g \quad \sim (2g-2)! \quad g \gg 1$ → zero radius of convergence

[Gross- Periwal, Shenker, Drukker-Mariño-Putrov]

→ We are missing some non-perturbative physics

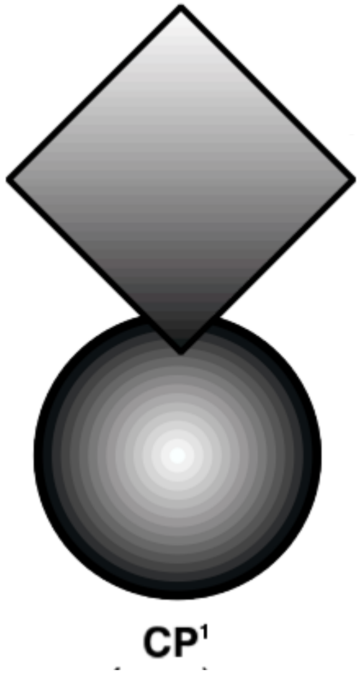
# Topological String Theory

One way to reorganize the information on the genus expansion is through the Gopakumar-Vafa function

## Example 1: resolved conifold

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$$

$$F(t, g_s) = \sum_{g \geq 0} g_s^{2g-2} F_g(t)$$



$$F_0(t) = -\,\mathrm{Li}_3(\exp(-t)),$$

$$F_1(t) = -\,\frac{1}{12}\mathrm{Li}_1(\exp(-t)),$$

$$F_g(t) = -\,\frac{(-1)^{g-1} B_{2g} \mathrm{Li}_{3-2g}(\exp(-t))}{2g(2g-2)!}, \quad g \geq 2 \, .$$

Figure from: 0410178

# Topological String Theory

## Example 1: resolved conifold

The corresponding Gopakumar-Vafa representation is

$$F^{\mathrm{GV}}(t, g_s) = \sum_{m \geq 1} \frac{e^{-mt}}{m} \left( 2 \sin \left( \frac{m g_s}{2} \right) \right)^{-2}$$

Two comments:

⟶ dense set of poles on the real axis at $g_s = \pi \mathbb{Q}$ (*)

⟶ away from the axis, i.e. $g_s \notin \mathbb{R}$, it is well defined because there are no poles and the sum over m converges

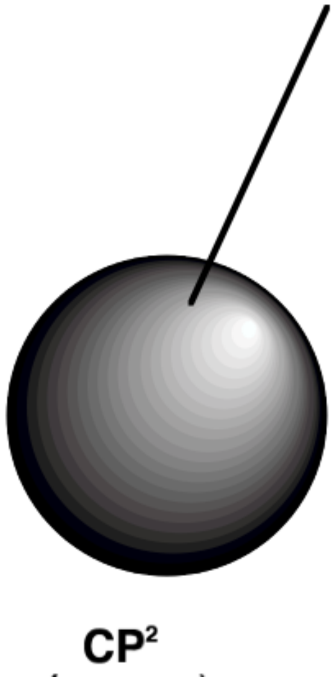
(*) this problematic point was first raised by [Hatsuda-Okuyama-Moryama]

# Topological String Theory

## Example 2: local $\mathbb{CP}^2$

$$F(t,g_s) = \sum_{g\geq 0} g_s^{2g-2}F_g(t)$$

$\mathcal{O}(-3) \to \mathbb{CP}^2$



One can compute $F_g(t)$ up to very high genus recursively (*) but no closed from expression expression for all genus exists

(*) up to $g \sim 200$ from [Haghighat-Klemm-Rauch]

Figure from: 0410178

# Topological String Theory

**Example 2: local $\mathbb{CP}^2$** . The Gopakumar-Vafa representation is

$$F^{\mathrm{GV}}(t,g_s)=\sum_{g\geq 0}\sum_{d}\sum_{w=1}^{\infty}\frac{1}{w}n_d^g\left(2\sin\frac{wg_s}{2}\right)^{2g-2}e^{-wdt}$$

$$=\sum_{m\geq 1}F_m(g_s)e^{-mt}$$

For the first few terms

$$F_1(g_s)=-\frac{3q^2}{(q^2-1)^2}$$

$$F_2(g_s)=\frac{3q^2\left(4q^4+7q^2+4\right)}{2(q^4-1)^2}$$

$$F_3(g_s)=\frac{-10q^{12}+27q^{10}+54q^8+62q^6+54q^4+27q^2+10}{(q^6-1)^2}$$

with $q=\mathrm{e}^{ig_s}$

# Topological String Theory

Example 2: local $\mathbb{CP}^2$ . The Gopakumar-Vafa representation is

$$F^{\mathrm{GV}}(t, g_s) = \sum_{m \geq 1} F_m(g_s) e^{-mt}$$

Two comments:

⟶ dense set of poles on the real axis at $g_s = \pi\mathbb{Q}$

⟶ away from the axis, $g_s \notin \mathbb{R}$, we do not have poles but the summation over m diverges $\log |F_m(g_s)| \sim m^2, \qquad m \gg 1$

[Hatsuda-Mariño-Moryama-Okuyama]

# Some points to keep in mind

→ Dense set of poles at $g_s \in \mathbb{R}$: $F^{\mathrm{GV}}$ alone can not provide a non-perturbative completion of the genus expansion

→ For the resolved conifold there have been many approaches to study non-perturbative effects, and at the end they are all give the same answer.

[Pasquetti - Schiappa, Krefl - Mkrtchyan, Hatsuda - Okuyama, Hatsuda, Bridgeland, Alexandrov-Pioline, Alim-Saha-Tulli, Alim-Teschner-Tulli, AG-Hao - Neitzke, Alim-Hollands-Tulli, Hattab-Palti …]

→ But not everything is like the resolved conifold

# Some points to keep in mind

- → The non-perturbative approach that I will use today is based on one particular duality: the TS/ST correspondence, and it applies to all local CY geometries (with mirror curve of genus $g \geq 1$)

- → There exists a special class of toric CYs ($A_n$ fibrations over $\mathbb{CP}^1$) for which it exists a non-perturbative duality with Chern–Simons matrix models on lens spaces [Aganagic-Klemm-Mariño-Vafa]

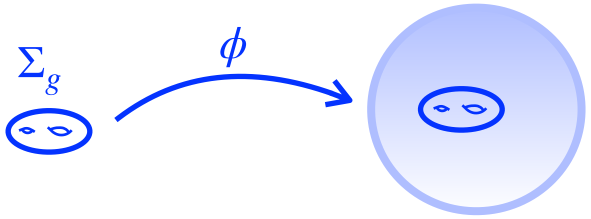
# Topological string and spectral theory

Topological string/
Enumerative geometry

TS/ST

[AG,Hatsuda,Mariño]

Spectral theory of a class of
quantized mirror curves



$$g_s = \frac{4\pi^2}{\hbar}$$

$$\mathrm{O}(\hat{x}, \hat{p}), \quad [\hat{x}, \hat{p}] = \mathrm{i}\hbar$$

Next: spectral theory of quantized mirror curves

# Spectral Theory of Quantized Mirror Curves

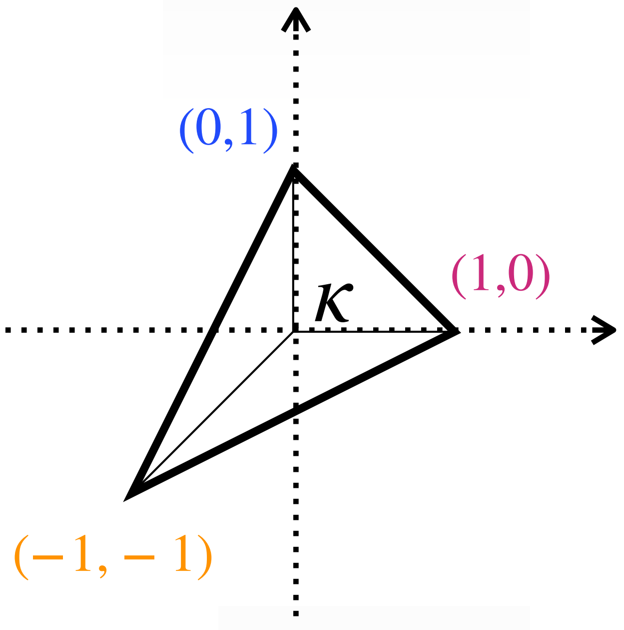
# Spectral Theory of Quantized Mirror Curves

Toric Calabi–Yau threefolds can be classified in terms of two-dimensional polytopes.

To each such polytope, we associate a curve known as the mirror curve.

[Batyrev , Chiang-Klemm-Yau-Zaslow, Hori-Vafa - …].

Example: local $\mathbb{P}_2$.



$$\xrightarrow{\text{mirror curve}} \qquad e^{p} + e^{x} + e^{-x-p} + \kappa = 0$$

$\kappa$: CY complex modulus

$(a, b)$ vertex $\longrightarrow$ $e^{ax+by}$

# Spectral Theory of Quantized Mirror Curves

One can quantize such mirror curves by using Weyl's prescription

$$e^{ax+by} \xrightarrow[\text{quantization}]{} e^{a\hat{x}+b\hat{p}}$$

where $\hat{x}$, $\hat{p}$ are momentum and position operators in one dimensional quantum mechanics: $\left[\hat{x}, \hat{p}\right] = \mathrm{i}\hbar$

$$e^{\hat{x}}\phi(x) = e^{x}\phi(x) \qquad\qquad e^{\hat{p}}\phi(x) = \phi(x - i\hbar)$$

[ Aganagic-Dijkgraaf-Klemm-Mariño-Vafa, Aganagic-Dijkgraaf-Cheng-Krefl-Vafa, Mironov-Morozov, Nekrasov-Shatashvili,…]

# Spectral Theory of Quantized Mirror Curves

Example: local $\mathbb{P}_2$

The quantization leads to $\mathcal{O} = \mathrm{e}^{\hat{p}} + \mathrm{e}^{\hat{x}} + \mathrm{e}^{-\hat{x}-\hat{p}}$ whose eigenvalue equation is

$$\phi(x - \mathrm{i}\hbar) + e^{x}\phi(x) + e^{-i\hbar/2}\mathrm{e}^{-x}\phi(x + i\hbar) + \kappa\phi(x) = 0$$

Example: local $\mathbb{P}_1 \times \mathbb{P}_1$

The quantization leads to $\mathcal{O} = \mathrm{e}^{\hat{p}} + \mathrm{e}^{-\hat{p}} + m\mathrm{e}^{-\hat{x}} + \mathrm{e}^{\hat{x}}$ whose eigenvalue equation is

$$\phi(x - \mathrm{i}\hbar) + \phi(x + \mathrm{i}\hbar) + (m\mathrm{e}^{-x} + \mathrm{e}^{x} + \kappa)\phi(x) = 0$$

# Spectral Theory of Quantized Mirror Curves

The **inverse** of the operators obtained from such quantization have a discrete spectrum and are of **trace class.**

The trace class property is very important, as it allows for the definition of **Fredholm determinants** and **Fermionic spectral traces**, which are key ingredients in the TS/ST correspondence.

The quantization of a mirror curve of genus $g$ leads **g non-commuting trace class** quantum mechanical operators on the real line

[AG-Hatsuda-Mariño, Kashaev-Mariño, Laptev-Schwimmer-Takhtajan, Codesido-AG-Mariño]

# Spectral Theory of Quantized Mirror Curves

Example: local $\mathbb{P}_2$ (genus one mirror curve)

The quantization of the mirror curve gives

$$\phi(x - \mathrm{i}\hbar) + e^{x}\phi(x) + e^{-i\hbar/2}\mathrm{e}^{-x}\phi(x + i\hbar) + \kappa\phi(x) = 0$$

Spectral problem: look for $L^2(\mathbb{R})$ solutions which admit analytic continuation in the strip $\{x \in \mathbb{C} \mid |\mathrm{Im} x| < \hbar\}$

**Theorem:** The operator $\rho = \mathcal{O}^{-1}$ has a discrete spectrum $\{E_n^{-1}\}_{n\geq 0}$ and it is of trace class on $L^2(\mathbb{I}\mathbb{R})$

$$\mathrm{Tr}\rho^N = \sum_{n\geq 0} E_n^{-N} < \infty$$

[AG-Hatsuda-Mariño, Kashaev-Mariño, Laptev-Schwimmer-Takhtajan]

# Spectral Theory of Quantized Mirror Curves

Example: local $\mathbb{P}_2$

The **kernel** of the operator $\rho$ is [Kashaev-Mariño]

$$\rho(x,y) = \frac{\Phi_b\left(x + ib/3\right)}{\Phi_b\left(y - ib/3\right)} \frac{e^{\pi b(x+y)/3}}{2b\cosh\left[\pi\left(\frac{x-y}{b} + \frac{i}{6}\right)\right]}, \qquad b^2 = \frac{3\hbar}{2\pi}$$

where $\Phi_{\mathrm{b}}$ is the **Faddeev quantum dilogarithm**

If Im(b)>0 it reduces to

$$\Phi_b(x) = \frac{(\mathrm{e}^{2\pi \mathrm{b}(x+c_b)}, \mathrm{e}^{2i\pi b^2})_\infty}{(\mathrm{e}^{2\pi \mathrm{b}^{-1}(x+c_b)}, \mathrm{e}^{-2i\pi b^{-2}})_\infty}$$

$$2c_b = i(b + b^{-1})$$

# Spectral Theory of Quantized Mirror Curves

If $b^2 = \frac{n}{m} \in \mathbb{Q}$ , the Faddeev dilogarithm simplifies and we get

$$\Phi_b(z) = \frac{\exp\left[\frac{i}{2\pi nm}\,\mathrm{Li}_2\left(e^{\tilde{z}}\right) + \left(1 + \frac{i}{2\pi nm}\,\tilde{z}\right)\ln\left(1 - e^{\tilde{z}}\right)\right]}{D_m\left(e^{\frac{\tilde{z}}{m}}; e^{i2\pi\frac{n}{m}}\right) D_n\left(e^{\frac{\tilde{z}}{n}}; e^{i2\pi\frac{m}{n}}\right)}$$

[Garoufalidis,Kashaev]

where $D_k(X;q) = \prod_{\ell=1}^{k-1}\left(1 - q^{\ell}X\right)^{\ell/k}$ , $\tilde{z} = 2\pi\sqrt{nm}\,z + i\pi(n+m)$

As a consequence the kernel of our operators also simplifies at these values.

# Spectral Theory of Quantized Mirror Curves

For example for local $\mathbb{P}^2$ at $\hbar = 2\pi \leftrightarrow b^2 = 3$ we have

$$\rho(x, y) = \sqrt{\frac{\sinh(y)}{\sinh(3y)}} \frac{1}{2\pi \cosh\left(x - y + \frac{i\pi}{6}\right)} \sqrt{\frac{\sinh(x)}{\sinh(3x)}}$$

We will see that also on the string theory side when

$$g_s = \frac{4\pi^2}{\hbar} = 2\pi$$

many simplifications occurs (the theory is essentially one-loop exact).

[Codesido-AG-Mariño]

# Spectral Theory of Quantized Mirror Curves

The quantization of mirror curves of genus one leads to one trace class operator $\rho$, and a convenient way to encode information about its spectrum is by using the Fredholm determinant

$$\det\left(1+\kappa\rho\right)=\prod_{n\geq 0}\left(1+\frac{\kappa}{E_n}\right)$$

The Fredholm determinant of a trace class operator is an entire function of $\kappa$ whose zeros correspond to the spectrum

# Spectral Theory of Quantized Mirror Curves

Another key object in the TS/ST are the fermionic spectral traces $Z(N,\hbar)$ that appear in the small $\kappa$ expansion of the determinant

$$\det\left(1+\kappa\rho\right)=\sum_{N\geq 0}\kappa^N Z(N,\hbar)$$

where $Z(N,\hbar)=\frac{1}{N!}\sum_{\sigma\in S_N}(-1)^{\mathrm{sgn}(\sigma)}\int_{\mathbb{R}^N}\mathrm{d}x_1\cdots\mathrm{d}x_N\prod_{i=1}^{N}\rho(x_i,x_{\sigma(i)})$

partition function of an non-interacting Fermi gas in one dimension with N particles and density matrix $\rho$

Example: $Z(1,\hbar)=\mathrm{Tr}\rho$ or $Z(2,\hbar)=\frac{1}{2}\left((\mathrm{Tr}\rho)^2-\mathrm{Tr}\rho^2\right)$

# Spectral Theory of Quantized Mirror Curves

We can also express such spectral traces as matrix models. For the example of local $\mathbb{P}^2$ we have [Mariño-Zakany]

$$Z(N,\hbar) = \frac{1}{N!}\int_{\mathbb{R}^N} \frac{d^N u}{(2\pi)^N} \prod_{i=1}^{N} e^{-V(u_i,\hbar)} \frac{\prod_{i<j} 4\sinh^2\left(\frac{u_i-u_j}{2}\right)}{\prod_{i,j} 2\cosh\left(\frac{u_i-u_j}{2} + i\frac{\pi}{6}\right)}$$

where $e^{-V(u,\hbar)} = e^{\frac{2\pi b}{3}u} \dfrac{\Phi_b(u+i\frac{\pi}{3})}{\Phi_b(u-i\frac{\pi}{3})}$

# Spectral Theory of Quantized Mirror Curves

What about geometries with **higher genus mirror curves**?

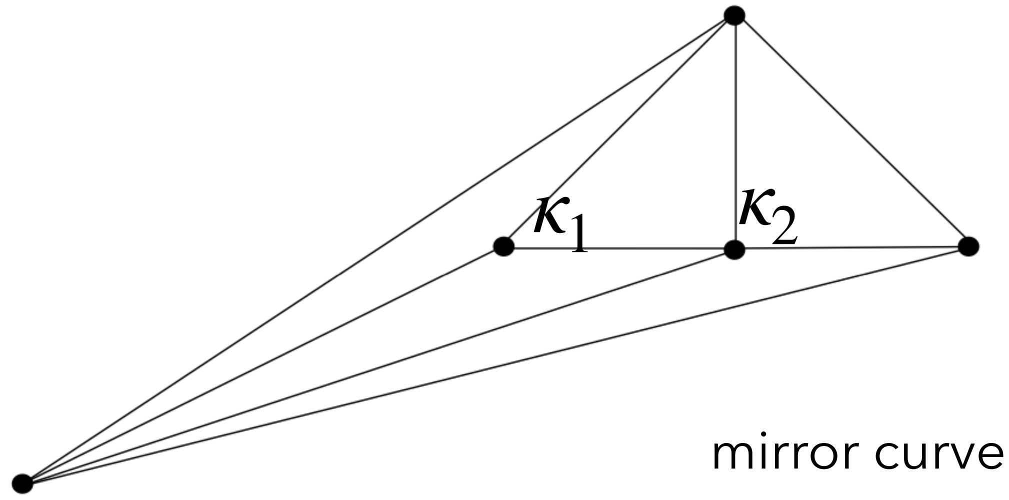
The quantization of mirror curves of genus g leads to **g non-commuting trace class quantum mechanical operators on the real line.**

We then need to define a **generalized notion of Fredholm determinant**, which is entire in all g spectral parameters and takes into account the spectral properties of all these operators simultaneously.

[Codesido-AG-Mariño]

# Spectral Theory of Quantized Mirror Curves

Example: crepant resolution of $\mathbb{C}^3/\mathbb{Z}^5$ geometry (genus two mirror curve)



mirror curve

$$\longrightarrow$$

Representation 1

$$e^x + e^y + e^{-3x-y} + \kappa_1 e^{-x} + \kappa_2 = 0$$

Representation 2

$$e^{2x} + e^{p+x} + e^{-2x-p} + \kappa_2 e^x + \kappa_1 = 0$$

# Spectral Theory of Quantized Mirror Curves

We can construct $g = 2$ non-commuting trace class operators, one for each $\kappa_i$

$$\underbrace{e^x + e^p + e^{-3x-p} + \kappa_1 e^{-x}} + \kappa_2 = 0$$

quantization gives $\mathrm{O}_2$ with trace class inverse $\rho_2$

$$\underbrace{e^{2x} + e^{p+x} + e^{-2x-p} + \kappa_2 e^x} + \kappa_1 = 0$$

quantization gives $\mathrm{O}_1$ with trace class inverse $\rho_1$

# Spectral Theory of Quantized Mirror Curves

We can construct $g = 2$ non-commuting trace class operators, one for each $\kappa_i$

$$\underbrace{e^{x} + e^{p} + e^{-3x-p}}_{\mathrm{O}_0} + \kappa_1 e^{-x} + \kappa_2 = 0$$

Two trace class operators $A_1 = \mathrm{O}_0^{-1}\mathrm{e}^{-x}$ and $A_2 = \mathrm{O}_0{}^{-1}$

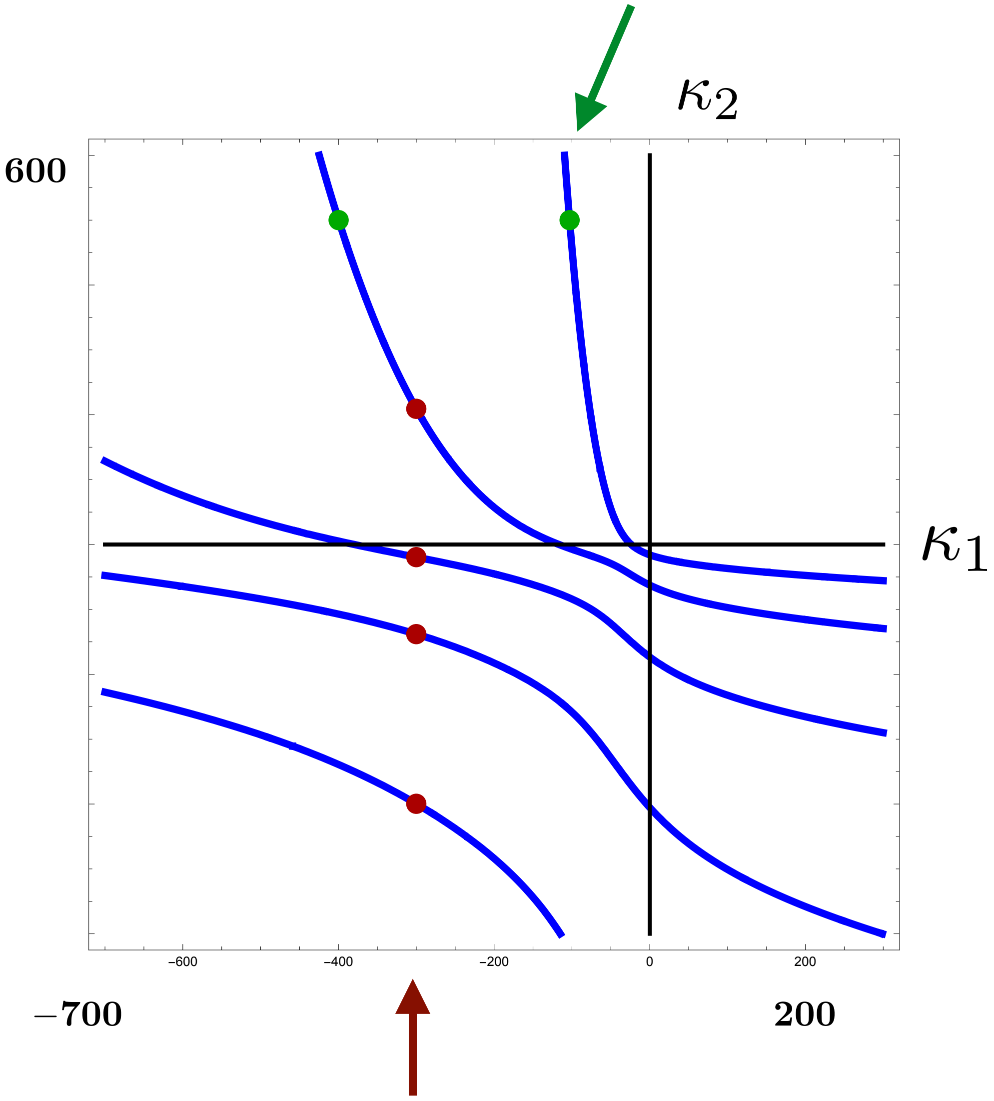
The generalized Fredholm determinant is entire in $\kappa_i$

$$\det\left(1 + \kappa_1 A_1 + \kappa_2 A_2\right) = \det\left(1 + \kappa_1 \rho_1\right) \det\left( 1 + \kappa_2 \rho_2 \Big|_{\kappa_1 = 0} \right)$$

$$= \det\left(1 + \kappa_2 \rho_2\right) \det\left( 1 + \kappa_1 \rho_1 \Big|_{\kappa_2 = 0} \right)$$

# Spectral Theory of Quantized Mirror Curves

$\Xi(\kappa_1^{(n)}, 500) = 0$ : spectrum of $\rho_1$



$\Xi(-300, \kappa_2^{(n)}) = 0$ spectrum of $\rho_2$

Blue line: vanishing of the determinant

$$\Xi(\kappa_1, \kappa_2) = \det\left(1 + \kappa_1 A_1 + \kappa_2 A_2\right)$$
$$= \det\left(1 + \kappa_1 \rho_1\right) \det\left( 1 + \kappa_2 \rho_2 \Big|_{\kappa_1=0} \right)$$
$$= \det\left(1 + \kappa_2 \rho_2\right) \det\left( 1 + \kappa_1 \rho_1 \Big|_{\kappa_2=0} \right)$$

$$\rho_2 = \left(e^x + e^p + e^{-3x-p} + \kappa_1 e^{-x}\right)^{-1}$$

$$\rho_1 = \left(e^{2x} + e^{p+x} + e^{-2x-p} + \kappa_2 e^x\right)^{-1}$$

# Spectral Theory of Quantized Mirror Curves

The fermionic spectral traces decomposition gives

$$\det\left(1+\kappa_1 \mathrm{A}_1+\kappa_2 \mathrm{A}_2\right)=\sum_{N_1,N_2,\geq 0} Z(N_1,N_2,\hbar)\kappa_1^{N_1}\kappa_2^{N_2}$$

where

$$Z(N_1,N_2,\hbar)=\frac{1}{N_1!N_2!}\sum_{\sigma\in S_N}(-1)^{\sigma}\int \mathrm{d}^N x\left(\prod_{i=1}^{N_1}A_1(x_{\sigma(i)},x_i)\right)\left(\prod_{j=1+N_1}^{N_1+N_2}A_2(x_{\sigma(j)},x_j)\right)$$

Example $Z(1,1,\hbar)=\mathrm{Tr}A_1\mathrm{Tr}A_2-\mathrm{Tr}(A_1A_2)$

$$Z(2,1,\hbar)=\mathrm{Tr}\left(A_1^2A_2\right)-\frac{1}{2}\mathrm{Tr}\left(A_1^2\right)\mathrm{Tr}A_2+\frac{1}{2}\left(\mathrm{Tr}A_1\right)^2\mathrm{Tr}A_2-\mathrm{Tr}A_1\,\mathrm{Tr}\left(A_1A_2\right)$$

$Z(N_1,N_2,\hbar)$ also has a matrix model representation which corresponds to a two-cut model.

# Spectral Theory of Quantized Mirror Curves

**Question:** can we compute spectral quantities (spectrum, determinant, eigenfunctions , … ) explicitly in an analytic manner?

A perturbative WKB analysis of these problems reveals an intriguing connection with the refined topological string special functions in the Nekrasov-Shatashvili (NS) limit [Aganagic-Dijgraf-Cheng-Krefl-Vafa, Nekrasov-Shatashvili, Mironov-Morozov, … ].

Can we go beyond WKB and obtain an exact solution?

# Spectral Theory of Quantized Mirror Curves

Can we go beyond WKB and obtain an exact solution?

**Answer:** Yes. Non-perturbative effects to the NS approach are encoded in the usual unrefined Gopakumar-Vafa phase of topological string theory.

$\hbar \to 0$

spectral theory of quantized mirror curves

$\hbar \to \infty$

- NS limit of refined top string
- WKB approximation

- top string theory in the GV phase with $g_s = \dfrac{4\pi^2}{\hbar}$
- enumerative geometry

# Topological String and Spectral Theory

Non-perturbative topological string

TS/ST ⟷

[AG-Hatsuda-Mariño]

Spectral theory of a class of quantized mirror curves

$$g_s = \frac{4\pi^2}{\hbar}$$

→ bridges perturbative expansions in one theory with non-perturbative phenomena in its dual counterpart → derivation of exact, closed-form expressions for many quantities on both sides of the correspondence.

# Topological String and Spectral Theory

→ Let us look at some concrete statements

# Topological String and Spectral Theory

Non-perturbative topological string

TS/SST

$\longleftrightarrow$

[AG-Hatsuda-Mariño]

Spectral theory of a class of quantized mirror curves

$$g_s = \frac{4\pi^2}{\hbar}$$

string theory special functions are naturally expressed by using Kahler parameters

spectral theory objects are naturally expressed by complex moduli

$t_I$

quantum mirror map

$\kappa_I$

# Topological String and Spectral Theory

Let us first focus on geometries with genus one mirror curves. The quantization leads to one trace class operator $\rho$. A key identity in the TS/ST correspondence is [AG-Hatsuda-Mariño]

$$\boxed{\det\left(1+\kappa\rho\right) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{J}(\mu+\mathrm{i}2\pi n,\hbar)}} \qquad \kappa = \mathrm{e}^{\mu}$$

spectral theory

topological string

# Topological String and Spectral Theory

Where grand potential (*)

$$\mathrm{J}(\mu, \hbar) \sim F^{\mathrm{GV}}\left(\frac{2\pi}{\hbar}t, \frac{4\pi^2}{\hbar}\right) + \left(\frac{\hbar}{2\pi}\partial_\hbar + \frac{t}{2\pi}\partial_t\right) F^{\mathrm{NS}}\left(t, \hbar\right)$$

NS limit of refined topological
string partition function

The grand potential $\mathrm{J}$ is a well defined non-perturbative function of all its parameter.

(*) extension of the ABJM grand potential [Hatsuda- Mariño-Moryama-Okuyama]

# Topological String and Spectral Theory

$$\mathrm{J}(\mu,\hbar) \sim F^{\mathrm{GV}}\left(\frac{2\pi}{\hbar}t, \frac{4\pi^2}{\hbar}\right) + \left(\frac{\hbar}{2\pi}\partial_\hbar + \frac{t}{2\pi}\partial_t\right) F^{\mathrm{NS}}\left(t,\hbar\right)$$

Recall: $g_s = \dfrac{4\pi^2}{\hbar}$

$\longrightarrow$ $F^{\mathrm{GV}}$: non-perturbative in $\hbar$ at $t$ fixed

$\longrightarrow$ $F^{\mathrm{NS}}$: non-perturbative in $g_s$ at $t^D = \dfrac{2\pi}{\hbar}t$ fixed

# Topological String and Spectral Theory

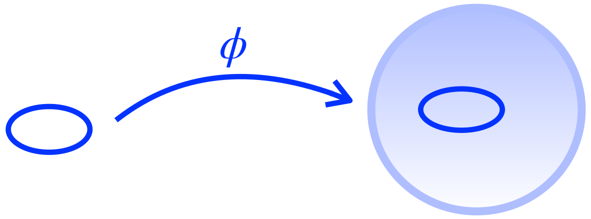
By using string theory special functions, we obtain exact and explicit expressions for the Fredholm determinants associated with quantum mirror curves

**Example:** local $\mathbb{P}^2$, set $\hbar = 2\pi$. Then we have

$$\det\left(1+\kappa\rho\right) \sim \theta_3\left(\xi - \frac{3}{8}, \frac{9}{4}\tau\right)$$

where $\xi = \frac{1}{2\pi^2}\left(t\partial_t^2 F_0 - \partial_t F_0\right)$ and $\tau = \frac{2\mathrm{i}}{\pi}\partial_t^2 F_0$ with $t = t(\kappa) = \text{(quantum) mirror map}$

$F_0$ : genus zero GW invariants of local $\mathbb{P}^2$



# Topological String and Spectral Theory

How do we think about the determinant from the topological strings point of view?

$$\det\left(1+\kappa\rho\right) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{J}(\mu+\mathrm{i}2\pi n,\hbar)}$$

⟶ The particular representation of the determinant on the rhs makes contact with the large radius frame

⟶ $J(\mu,\hbar)$ gives a non-perturbative completion for topological string in the large radius frame

⟶ the sum over $n$ allow us to move away from large radius and obtain an

entire object which is well defined over the full moduli space, parametrized by $\kappa$

Let's explore other regions

# Topological String and Spectral Theory

Let us study the representation of the determinant in terms of fermionic spectral traces

$$\det\left(1+\kappa\rho\right)=\sum_{N\geq 0} Z(N,\hbar)\kappa^N$$

where $Z(N,\hbar)=\frac{1}{N!}\sum_{\sigma\in S_N}(-1)^{\mathrm{sgn}(\sigma)}\int_{\mathbb{R}^N}\mathrm{d}x_1\cdots\mathrm{d}x_N\prod_{i=1}^{N}\rho(x_i,x_{\sigma(i)})$ are the

Fermionic spectral traces.

# Topological String and Spectral Theory

We find that

$$\log Z(N, \hbar) \xrightarrow{\hbar\ ,\ N \to \infty} \sum_{g \geq 0} F_g(\lambda)\ g_s^{2g-2}$$

$$\lambda = \frac{N}{\hbar} \qquad \text{fixed}$$

$$g_s = \frac{4\pi^2}{\hbar}$$

free energy of top. string in the conifold frame (equivalent of magnetic frame in Seiberg-Witten theory)

and $\lambda$ is the vanishing period at the conifold point.

# Topological String and Spectral Theory

Hence we found that

$$\det\left(1+\kappa\rho\right)=\sum_{N\geq 0} Z(N,\hbar)\kappa^{N}$$

① $Z(N,\hbar)$ is a well defined non-perturbative object since $\rho$ is of trace class

② $Z(N,\hbar)$ produce the genus expansion of top string in the conifold frame in the 't Hooft limit

→ $Z(N,\hbar)$: non-perturbative completion of top string in the conifold frame

# Topological String and Spectral Theory

How does the two expansions of the determinant talks to each other?

$$\det\left(1+\kappa\rho\right) = \sum_{N\geq 0} Z(N,\hbar)\kappa^N$$

non-perturbative partition function

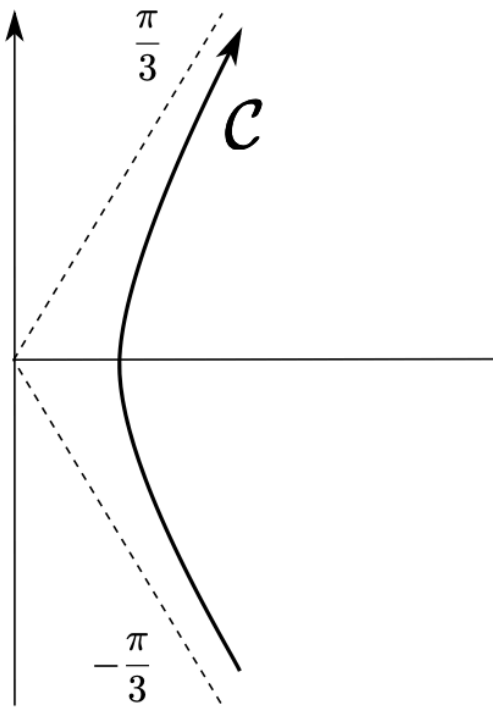
in the conifold frame

$$\det\left(1+\kappa\rho\right) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{J}(\mu+\mathrm{i}2\pi n,\hbar)}$$

non-perturbative partition

function in large radius frame

We have the relation $Z(N,\hbar) = \dfrac{1}{2\pi i}\displaystyle\int_C e^{J(\mu,\hbar)-N\mu}\,\mathrm{d}\mu$



→ Integral transformation as a change of frame: non-perturbative analogous of [Aganagic,Bouchard,Klemm]

# Topological String and Spectral Theory

What about higher genus mirror curve? Very similar but with more operators.

Example: crepant resolution of $\mathbb{C}^3/\mathbb{Z}^5$ geometry (genus two mirror curve)

$$\det\left(1+\kappa_1 \mathrm{A}_1+\kappa_2 \mathrm{A}_2\right)=\sum_{n_1, n_2 \in \mathbb{Z}} \mathrm{e}^{\mathrm{J}(\mu_1+\mathrm{i}2\pi n_1, \mu_2+\mathrm{i}2\pi n_2, \hbar)} \qquad \kappa_i=\mathrm{e}^{\mu_i}$$

where J has always the same structure

$$\mathrm{J}(\mu_1, \mu_2, \hbar) \sim F^{\mathrm{GV}}\left(\frac{2\pi}{\hbar} t_1, \frac{2\pi}{\hbar} t_2, \frac{4\pi^2}{\hbar}\right)+\left(\frac{\hbar}{2\pi} \partial_\hbar+\frac{1}{2\pi} \frac{t_i}{2\pi} \partial_{t_i}\right) F^{\mathrm{NS}}\left(t_1, t_2, \hbar\right)$$

# Topological String and Spectral Theory

This is an example of how the interplay between spectral theory and topological string is  powerful:

→ gives new results in spectral theory

→ provides a concrete handle on topological string theory at the non-perturbative level

# Topological String and Spectral Theory

Many more applications

→ new result in spectral theory and relativistic integrable systems [Many] NEXT

→ new integer invariants from spectral traces at finite N [Gu-Mariño, …]

→ quantum modularity structure in spectral traces [Fantini-Rella]

→ number theoretic identities NEXT

→ relation to **q**-isomonodromic tau functions [Bonelli-AG-Tanzini, ….]

→ connection with Hofstadter butterfly [Hatsuda-Katsura-Tachikawa, ….]

→ application to 3d susy gauge theories matrix models [Moryama-Nosaka, ….]

→ new results for $\mathcal{N} = 2\ SU(N)$ 4dim Seiberg-Witten theory [Bonelli-AG-Tanzini, ….]

→ extension of K-theoretic blowup equations [Gu-Haghighat-Sun-Wang, …]

• • •

# Spectral Theory of Quantized Mirror Curves

Example: local $\mathbb{P}_1 \times \mathbb{P}_1$

$$\phi(x - \mathrm{i}\hbar) + \phi(x + \mathrm{i}\hbar) + (m\mathrm{e}^{-x} + \mathrm{e}^{x} + \kappa)\phi(x) = 0$$

Spectral problem: look for $L^2(\mathbb{R})$ solutions which admit analytic continuation in the strip $\{x \in \mathbb{C} \mid |\mathrm{Im}x| < \hbar\}$

This is also the Baxter equation of two-particle relativistic Toda lattice

The spectrum is then determined by the vanishing of the Fredholm determinant

$$\det\left(1 + \kappa\rho\right) = \prod_{n\geq 0}\left(1 + \frac{\kappa}{E_n}\right)$$

# Spectral Theory of Quantized Mirror Curves

$$\det\left(1+\kappa\rho\right)=0 \quad \xleftrightarrow{\text{TS/ST}} \quad \partial_t F^{\mathrm{NS}}(t,\hbar)\underbrace{+\frac{\hbar}{2\pi}\partial_t F^{\mathrm{NS}}\left(\frac{2\pi}{\hbar}t,\frac{4\pi^2}{\hbar}\right)}=2\pi(n+1/2)$$

[AG,Hatsuda,Mariño -
Wang, Zhang,Huang - …]

non-perturbative corrections to the
NS quantization condition

together with the quantum mirror map $t \equiv t(E,\hbar)$ this gives the energy levels $E_n$ of the system.

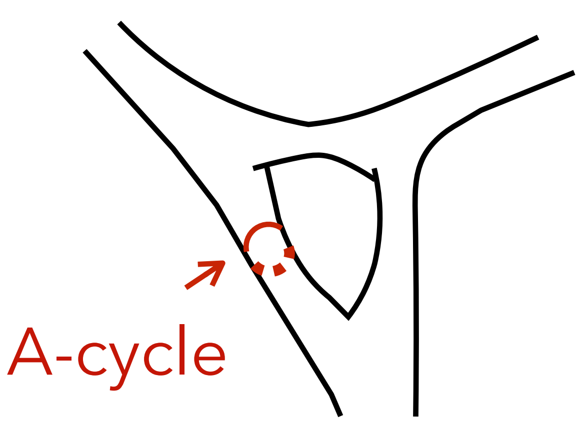
Important point to keep in mind: in the context of quantum mirror curves, 5d gauge theories, and relativistic integrable systems, the naive uplift of the Bethe/Gauge correspondence from 4d to 5d fails. A whole new tower of non-perturbative effects must be taken into account both at the level of spectrum and eigenfunctions.

# Some number theoretic identities

# Some number theoretic identities

Comparing the two sides of the duality yields some interesting number-theoretic identities, for example identities for the periods at special points in the moduli space

Example local $\mathbb{P}^2$:



$$t(\kappa) = \oint_A \mathrm{d}x \; p(x, \kappa)$$

At the conifold point $\kappa_c = -\frac{1}{27}$ we get [Mariño-Zakany]

$$t(\kappa_c) = \frac{9}{2\pi} \operatorname{Im} \operatorname{Li}_2 \left(e^{i\pi/3}\right)$$

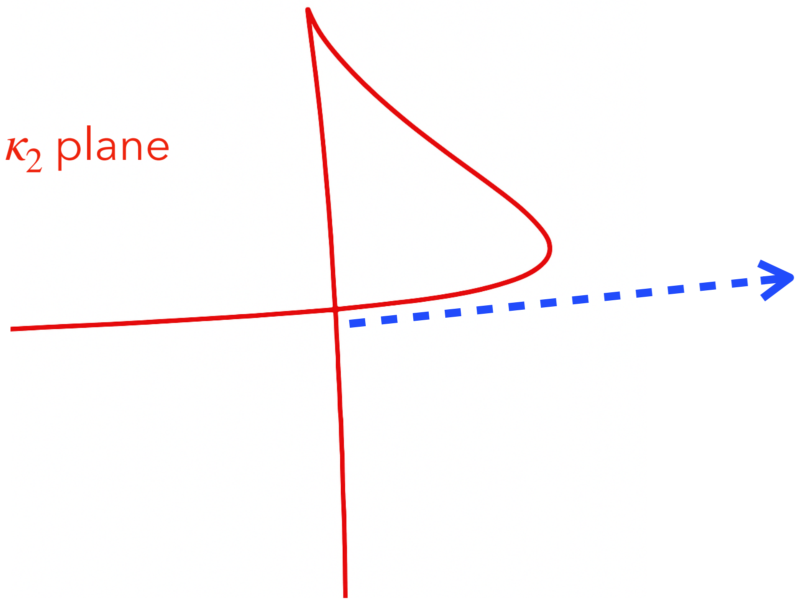
This was in fact a known identity [Rodriguez-Villegas].

In addition we also have $$F_0(t(\kappa_c)) = -3\zeta(3) + \frac{3\pi}{4} \operatorname{Im} \operatorname{Li}_2 \left(e^{i\pi/3}\right)$$

# Some number theoretic identities

Example: crepant resolution of $\mathbb{C}^3/\mathbb{Z}^5$:

In this case we get some number theoretic prediction for the periods at the maximal conifold point [Codesido-AG-Mariño]



$\kappa_1, \kappa_2$ plane

$\kappa_1 = 1/5, \quad \kappa_2 = -1/25$

maximal conifold point

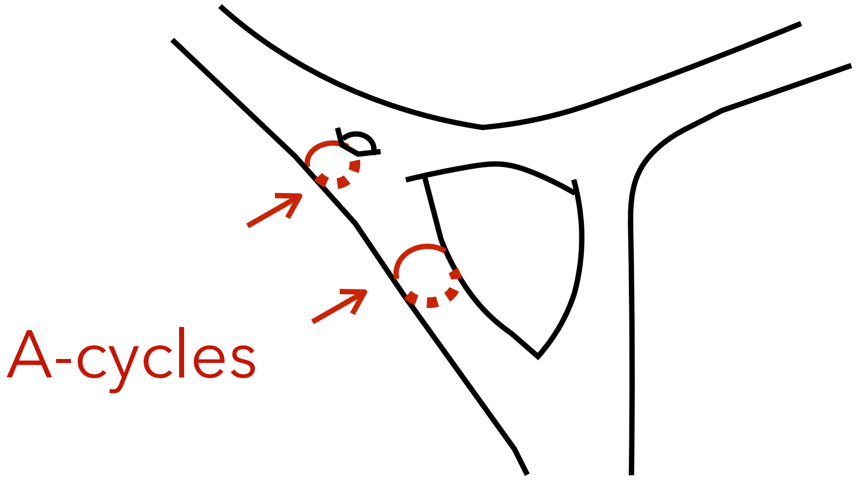
The conifold locus $\Delta(\kappa_1, \kappa_2) = 0$

# Some number theoretic identities

Example: crepant resolution of $\mathbb{C}^3/\mathbb{Z}^5$:

For example we find

$$\log 3125 - t_1\left(-\frac{1}{25},\frac{1}{5}\right) - 3t_2\left(-\frac{1}{25},\frac{1}{5}\right) = \frac{25}{\pi}D_2\left(e^{\frac{\pi i}{5}}\cdot\frac{1+\sqrt{5}}{2}\right)$$



Bloch-Wigner function

$$D_2(z) = \mathrm{Im}\left(\mathrm{Li}_2(z)\right) + \log\mid z\mid \arg(1-z)$$

This and others identities at the maximal conifold point have by now been proven by [Doran, Kerr, Sinha Babu] something about open strings

# Topological String and Spectral Theory

So far we focused on closed topological string. What about **open topological string?**

Rough idea

closed string ∼ Fredholm determinant

open string ∼ eigenfunctions (off-shell)

$g_s \to \infty$
$\hbar \to 0$

spectral theory of quantized mirror curves

$\longleftrightarrow$

$g_s \to 0$
$\hbar \to \infty$

- NS limit of refined top string
- WKB approximation

- top string pert theory
- enumerative geometry

# Topological String and Spectral Theory

The starting point is the GV free energy for the **open string wave function,** counting holomorphic maps from a Riemann surface of genus $g$ with $h$ boundaries into the target space

$$F_{\rm GV}^{\rm open}(x, \mathbf{t}, g_s) = \sum_{\mathbf{d}} \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \sum_{\ell} \sum_{w=1}^{\infty} \frac{i^h}{h!} n_{g,\mathbf{d},\ell} \frac{1}{w} \left( 2 \sin \frac{w g_s}{2} \right)^{2g-2}$$

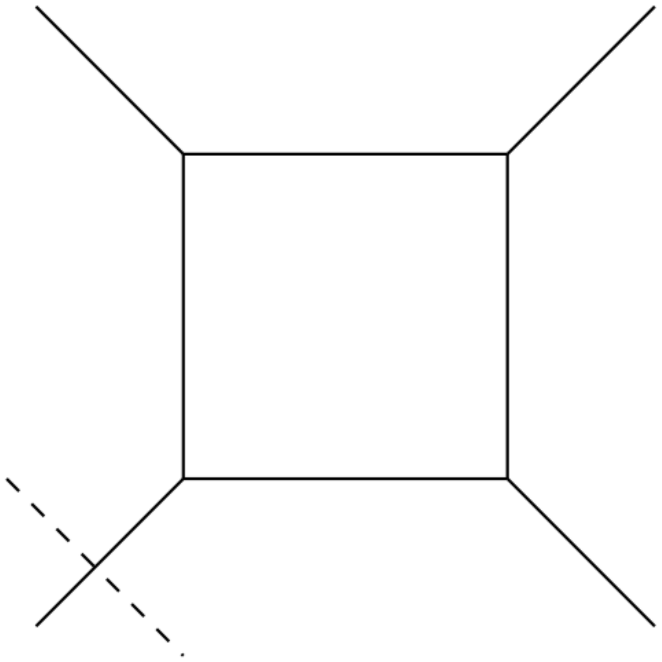
$$\times \prod_{i=1}^{h} \left( 2 \sin \frac{w \ell_i g_s}{2} \right) \frac{1}{\ell_1 \cdots \ell_h} X^{-w(\ell_1 + \cdots + \ell_h)} e^{-w \mathbf{d} \cdot \mathbf{t}}, \qquad X = e^x$$

with the requirement that the boundaries of the Riemann surface end on a Lagrangian manifold in the target space.

As with the closed string, we also have a dense set of poles on the axis $g_s \in \mathbb{R}$

# Topological String and Spectral Theory

We will focus on local $\mathbb{P}_1 \times \mathbb{P}_1$ and we take a Lagrangian brane with topology of $\mathbb{R}^2 \times S^1$ which lies on the external leg of the toric diagram



[Aganagic-Vafa]

# Topological String and Spectral Theory

Example: local $\mathbb{P}_1 \times \mathbb{P}_1$

The open topological string wavefunction corresponding to a brane inserted in the outer leg of toric diagram, can be computed via the refined topological vertex.

$$F^{\mathit{open}}_{\mathit{GV}}\left(x, t_F, t_B, g_s\right) = \frac{e^{i\frac{g_s}{2}} e^{\frac{t_F}{2}-x}\left(2e^{i\frac{g_s}{2}} e^{-\frac{t_F}{2}-x} - 1 - e^{-t_F}\right)}{\left(1-e^{ig_s}\right)\left(1-e^{-t_F}\right)^2\left(1-e^{i\frac{g_s}{2}} e^{\frac{t_F}{2}-x}\right)\left(1-e^{i\frac{g_s}{2}} e^{-\frac{t_F}{2}-x}\right)} e^{-t_B} + \mathcal{O}\left(e^{-2\,t_B}\right)$$

where $t_{B,F}$ are the Kähler parameters (fiber and base)

Note: this functions has poles at $x = \pm\frac{1}{2}t_F + ig_s\left(\frac{1}{2}+n\right) \ , \quad n \in \mathbb{N}$

# Topological String and Spectral Theory

Example: local $\mathbb{P}_1 \times \mathbb{P}_1$

The quantization leads to $\mathcal{O} = \mathrm{e}^{\hat{p}} + \mathrm{e}^{-\hat{p}} + m\mathrm{e}^{-\hat{x}} + \mathrm{e}^{\hat{x}}$ whose eigenvalue equation is

$$\phi(x - \mathrm{i}\hbar, \kappa) + \phi(x + \mathrm{i}\hbar, \kappa) + (m\mathrm{e}^{-x} + \mathrm{e}^{x} + \kappa)\phi(x, \kappa) = 0$$

when we think from the point of view of topological string theory, we should identify x is the open string modulus.

[Aganagic-Vafa, Aganagic-Dijkgraaf-Klemm-Mariño-Vafa, Aganagic - Dijkgraaf -Cheng - Krefl - Vafa, …]

# Topological String and Spectral Theory

Example: local $\mathbb{P}_1 \times \mathbb{P}_1$

$$\phi(x - \mathrm{i}\hbar, \kappa) + \phi(x + \mathrm{i}\hbar, \kappa) + (m\mathrm{e}^{-x} + \mathrm{e}^{x} + \kappa)\phi(x, \kappa) = 0$$

Spectral problem: look for $L^2(\mathbb{R})$ solutions which admit an analytic continuation in the strip $\{x \in \mathbb{C} \mid |\mathrm{Im} x| < \hbar\}$

→ unique family of on-shell eigenfunctions $\phi_n(x) = \phi(x, \kappa = -E_n)$

However, from a stringy perspective, we don't need to be on-shell. Ideally we look for solutions which are entire in x at generic values of $\kappa$ [Maldacena-Moore-Seiberg-Shih]

# Topological String and Spectral Theory

We find that it is indeed possible to construct such entire solutions using special combinations of topological string special functions in the GV and NS phase

[Mariño-Zakany, AG-François]

$$\phi(x) = \sum_{k\in\mathbb{Z}} \left( \mathrm{e}^{\mathrm{J}(x,\mu+\mathrm{i}2\pi k,\xi,\hbar)} + \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2}+\frac{\pi x}{\hbar}+\mathrm{J}(-x-\mathrm{i}\pi,\mu+\mathrm{i}\pi+\mathrm{i}2\pi k,\xi,\hbar)} \right)$$

where

$$\mathrm{J}(x,\mu,\xi,\hbar) = \mathrm{J}^{\mathrm{closed}}(\mu,\xi,\hbar) + \mathrm{J}^{\mathrm{open}}(x,\mu,\xi,\hbar)$$

grand potential for closed strings discussed previously

open string grand potential

# Topological String and Spectral Theory

$$J^{open}\left(x,\mu,\xi,\hbar\right) = F^{open}_{NS}\left(x,t_F,t_B,\hbar\right) + F^{open}_{GV}\left(\frac{2\pi}{\hbar}x,\frac{2\pi}{\hbar}t_F,\frac{2\pi}{\hbar}t_B,\frac{4\pi^2}{\hbar}\right)$$

↓ ↓

(refined) open topological string wavefunction corresponding to a brane inserted in the outer leg of toric diagram, can be computed via the refined topological vertex.

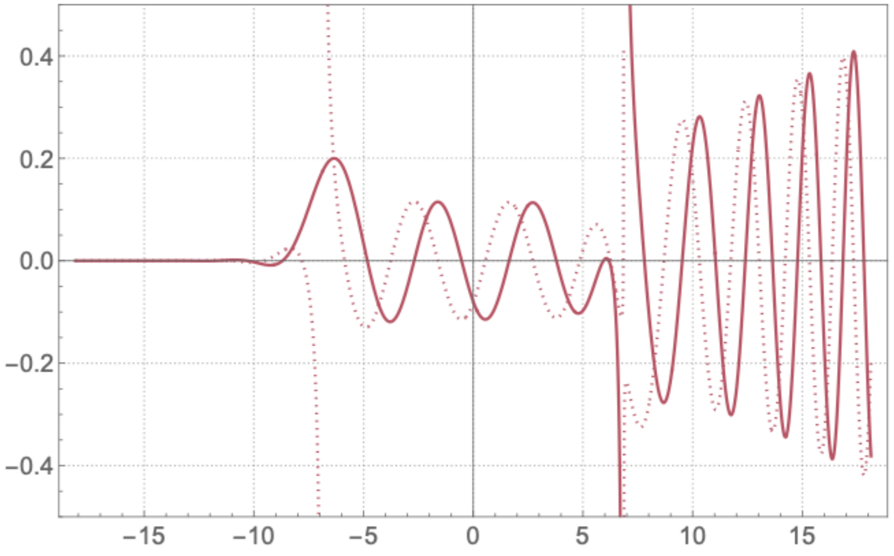
$$F^{open}_{GV}\left(x,t_F,t_B,g_s\right) = \frac{e^{i\frac{g_s}{2}}e^{\frac{t_F}{2}-x}\left(2e^{i\frac{g_s}{2}}e^{-\frac{t_F}{2}-x}-1-e^{-t_F}\right)}{\left(1-e^{ig_s}\right)\left(1-e^{-t_F}\right)^2\left(1-e^{i\frac{g_s}{2}}e^{\frac{t_F}{2}-x}\right)\left(1-e^{i\frac{g_s}{2}}e^{-\frac{t_F}{2}-x}\right)}e^{-t_B}+\mathcal{O}\left(e^{-2\,t_B}\right)$$

$$F^{open}_{NS}\left(x,t_F,t_B,\hbar\right) = \frac{\mathrm{e}^{\mathrm{i}\hbar}\mathrm{e}^{\frac{t_F}{2}-x}\left(1+\mathrm{e}^{-t_F}+\mathrm{e}^{\mathrm{i}\hbar}\left(1+\mathrm{e}^{\mathrm{i}\hbar}\right)\mathrm{e}^{-\frac{t_F}{2}-x}\right)}{\left(1-\mathrm{e}^{\mathrm{i}\hbar}\right)\left(1-\mathrm{e}^{\mathrm{i}\hbar}\mathrm{e}^{-t_F}\right)\left(1-\mathrm{e}^{-\mathrm{i}\hbar}\mathrm{e}^{-t_F}\right)\left(1+\mathrm{e}^{\mathrm{i}\hbar}\mathrm{e}^{\frac{t_F}{2}-x}\right)\left(1+\mathrm{e}^{\mathrm{i}\hbar}\mathrm{e}^{-\frac{t_F}{2}-x}\right)}\mathrm{e}^{-t_B}+\mathcal{O}\left(e^{-2\,t_B}\right)$$

# Topological String and Spectral Theory

It is indeed possible to construct such entire solutions as

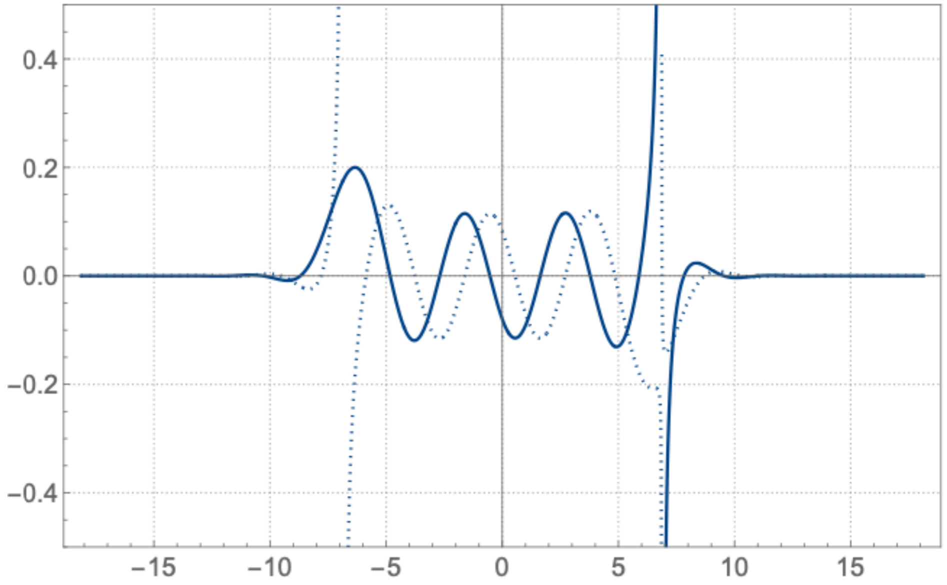
$$\phi(x) = \sum_{k\in\mathbb{Z}} \left( \mathrm{e}^{\mathrm{J}(x,\mu+\mathrm{i}2\pi k,\xi,\hbar)} + \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2}+\frac{\pi x}{\hbar}+\mathrm{J}(-x-\mathrm{i}\pi,\mu+\mathrm{i}\pi+\mathrm{i}2\pi k,\xi,\hbar)} \right)$$



$$\hbar = 4\sqrt{2}$$

$$\xi = \frac{2}{3}$$

$$\kappa = -27$$



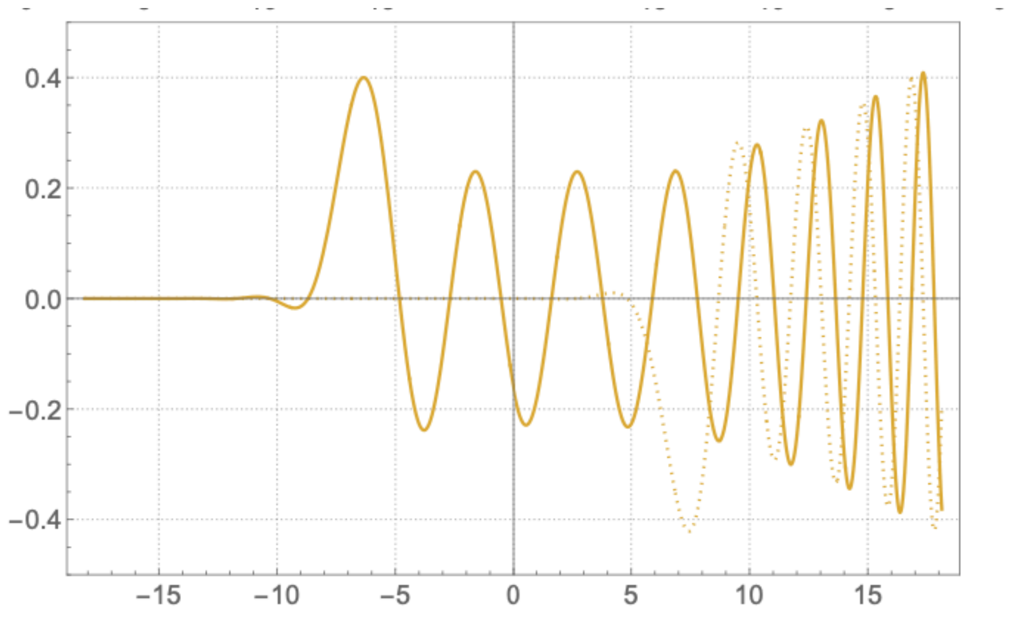
Each term individually is a formal solution, but they are not entire

# Topological String and Spectral Theory

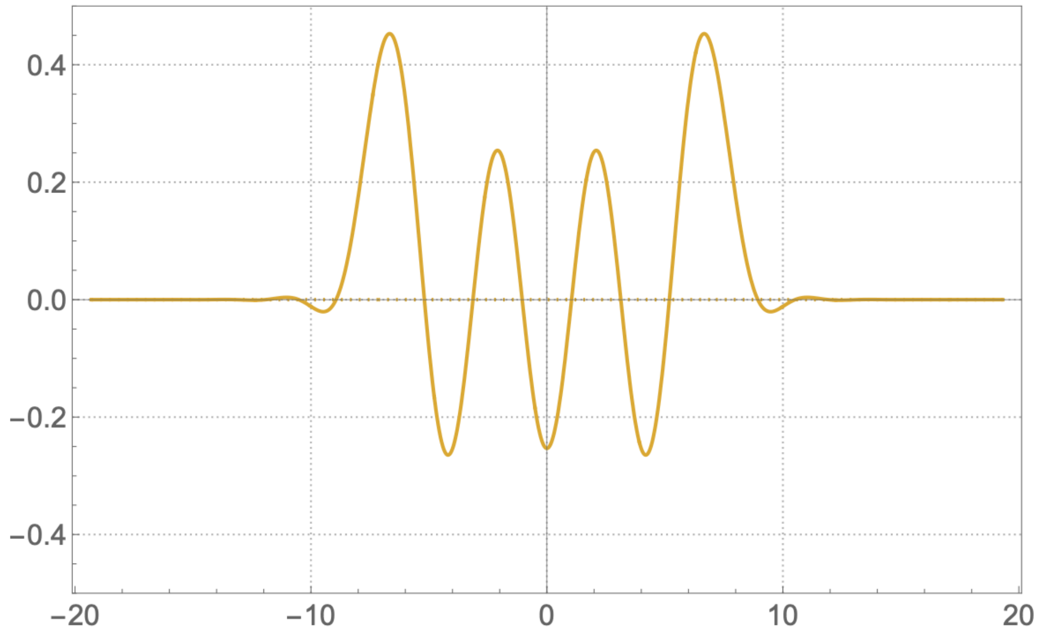
$$\phi(x) = \sum_{k\in\mathbb{Z}} \left( \mathrm{e}^{\mathrm{J}(x,\mu+\mathrm{i}2\pi k,\xi,\hbar)} + \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2}+\frac{\pi x}{\hbar}+\mathrm{J}(-x-\mathrm{i}\pi,\mu+\mathrm{i}\pi+\mathrm{i}2\pi k,\xi,\hbar)} \right)$$

For generic value of $\kappa = \mathrm{e}^{\mu}$ this

is entire in x but not $L^2(\mathbb{R})$



It is in $L^2(\mathbb{R})$ one at $\kappa = -E_n$



# Summary and Outlook

# Summary and Outlook

The topological string/spectral theory duality provides a precise non-perturbative relation between topological string theory on local Calabi–Yau threefolds and the spectral theory of quantized mirror curves.

⟶ many applications

Many open problems remain; from a physical perspective, perhaps the most interesting is understanding what is the physics of these non-perturbative effects and what is their geometric interpretation

Thank you!