S-folds and 4d $\mathcal{N} = 2$ SCFTs

Simone Giacomelli

University of Milano-Bicocca

Strings and Geometry 2025

April 8th 2025

Based on: SG, R. Savelli, G. Zoccarato arXiv:2405.00101[hep-th]; SG, W. Harding, N. Mekareeya, A. Mininno arXiv:2411.03425[hep-th] (and work in progress).

Field Theories with extended supersymmetry

Theories with eight supercharges constitute an important laboratory for the study of nonperturbative phenomena in QFT.

4d $\mathcal{N} = 2$ SCFTs have a $SU(2)_R \times U(1)_R$ symmetry and display an intricate moduli space of vacua:

- **Coulomb Branch (CB):** where vector multiplet scalars have nonzero vev ($SU(2)_R$ unbroken);
- **Higgs Branch (HB):** where hypermultiplet scalars have nonzero vev ($U(1)_R$ unbroken).

It can be exploited for classifying SCFTs at low rank (CB dimension). Argyres, Lotito, Lü, Martone '15-'16; Bourget, Grimminger, Martone, Zafrir '21.

Geometric realizations of $\mathcal{N} = 2$ theories

We have discovered many strongly-coupled SCFTs via stringy constructions:

- Type IIB compactifications (geometric engineering);
- Class $\mathcal{S}$.

These methods do not allow us to study RG flows triggered by relevant (mass) deformations.

We propose a geometric method which allows us to efficiently study such RG flows:

$D3$ branes probing a S-fold background in Type IIB.	$\longleftrightarrow$	6d orbi-instanton theories compactiified on T^2.

Geometric realizations of $\mathcal{N} = 2$ theories

We have discovered many strongly-coupled SCFTs via stringy constructions:

- Type IIB compactifications (geometric engineering);
- Class $\mathcal{S}$.

These methods do not allow us to study RG flows triggered by relevant (mass) deformations.

We propose a geometric method which allows us to efficiently study such RG flows:

$D3$ branes probing a S-fold background in Type IIB.	$\longleftrightarrow$	6d orbi-instanton theories compactiified on T^2.

$\mathcal{N}=2$ theories from $D3$ branes

We consider 7-branes with constant axio-dilaton. Mukhi, Dasgupta '96.

G	$\emptyset$	$SU(2)$	$SU(3)$	$SO(8)$	E_6	E_7	E_8
Δ_7	$6/5$	$4/3$	$3/2$	2	3	4	6

The angular variable around the 7-brane has periodicity $2\pi/\Delta_7$.

We probe the 7-brane with a stack of N $D3$ branes:

	0	1	2	3	4	5	6	7	8	9
7-brane	x	x	x	x	x	x	x	x		
D3 brane	x	x	x	x						

The gauge symmetry G on the 7-brane becomes the global symmetry in 4d.

$\mathcal{S}$-folds + 7-branes

We combine the 7-brane with a $\mathbb{Z}_\ell$ quotient of the 89-plane. To preserve $\mathcal{N} = 2$ supersymmetry this must be accompanied (for $\ell \neq 1$) by the action of $\mathbb{Z}_{\ell\Delta_7} \subset SL(2, \mathbb{Z})$. Apruzzi, SG, Schafer-Nameki '20.

We find the following possibilities:

- For $\ell = 1$ $\Delta_7 = 1, 6/5, 4/3, 3/2, 2, 3, 4$ and 6;
- For $\ell = 2$ $\Delta_7 = 1, 3/2, 2$ and 3;
- For $\ell = 3$ $\Delta_7 = 1, 4/3$ and 2;
- For $\ell = 4$ $\Delta_7 = 1$ and $3/2$;
- For $\ell = 5$ $\Delta_7 = 6/5$;
- For $\ell = 6$ $\Delta_7 = 1$.

Each possibility leads to an infinite family of 4d $\mathcal{N} = 2$ SCFTs which include all rank-1 theories!

$\mathcal{S}$-folds + 7-branes

We combine the 7-brane with a $\mathbb{Z}_\ell$ quotient of the 89-plane. To preserve $\mathcal{N}=2$ supersymmetry this must be accompanied (for $\ell \neq 1$) by the action of $\mathbb{Z}_{\ell\Delta_7} \subset SL(2,\mathbb{Z})$. Apruzzi, SG, Schafer-Nameki '20.

We find the following possibilities:

- For $\ell = 1$ $\Delta_7 = 1, 6/5, 4/3, 3/2, 2, 3, 4$ and 6;
- For $\ell = 2$ $\Delta_7 = 1, 3/2, 2$ and 3;
- For $\ell = 3$ $\Delta_7 = 1, 4/3$ and 2;
- For $\ell = 4$ $\Delta_7 = 1$ and $3/2$;
- For $\ell = 5$ $\Delta_7 = 6/5$;
- For $\ell = 6$ $\Delta_7 = 1$.

Each possibility leads to an infinite family of 4d $\mathcal{N}=2$ SCFTs which include all rank-1 theories!

Orbi-instanton theories in 6d

Realized in M-theory by probing with M5 branes a ADE singularity $\mathbb{C}^2/\Gamma_{ADE}$ inside a M9 wall: Del Zotto, Heckman, Tomasiello, Vafa '14.

	0	1	2	3	4	5	6	7	8	9	10
$M9$	x	x	x	x	x	x	x	x	x	x	
$M5$	x	x	x	x	x	x					
Γ_{ADE}							x	x	x	x	

For $\Gamma_{ADE}=\mathbb{Z}_M$ they are the UV completion of:

$$G - SU(m_1) - \cdots - SU(m_n) - \boxed{M}$$

- G is of SU or USp type;
- For given M the 6d theories are labelled by $Hom(\mathbb{Z}_M, E_8)$ (**holonomy at infinity** for the gauge field on the $M9$).

Orbi-instanton theories in 6d

Realized in M-theory by probing with M5 branes a ADE singularity $\mathbb{C}^2/\Gamma_{ADE}$ inside a M9 wall: Del Zotto, Heckman, Tomasiello, Vafa '14.

	0	1	2	3	4	5	6	7	8	9	10
$M9$	x	x	x	x	x	x	x	x	x	x	
$M5$	x	x	x	x	x	x					
Γ_{ADE}							x	x	x	x	

For $\Gamma_{ADE} = \mathbb{Z}_M$ they are the UV completion of:

$$G - SU(m_1) - \cdots - SU(m_n) - \boxed{M}$$

- G is of SU or USp type;
- For given M the 6d theories are labelled by $Hom(\mathbb{Z}_M, E_8)$ (**holonomy at infinity** for the gauge field on the $M9$).

4d $\mathcal{N}=2$ theories from orbi-instantons

Compactifying the 6d theory on T^2 we find a 4d SCFT.

If the 6d theory has global symmetry $G_F=\hat{G}/\mathbb{Z}_\ell$ where

$$\mathbb{Z}_\ell \subset \mathcal{Z}(\hat{G}) \quad (\hat{G} \text{ is simply-connected})$$

we can turn on **almost-commuting** holonomies P and Q on T^2

$$PQ=\omega QP; \quad \omega \in \mathbb{Z}_\ell$$

while preserving $\mathcal{N}=2$ susy (**SW twist**). Ohmori, Tachikawa, Zafir '18.

The 6d theory admits a $\mathbb{Z}_\ell$ SW twist only for a subset of possible holonomies and if M is a multiple of ℓ. Heckman, Lawrie, Lin, Zhang, Zoccarato '22.

4d $\mathcal{N}=2$ theories from orbi-instantons

Compactifying the 6d theory on T^2 we find a 4d SCFT.

If the 6d theory has global symmetry $G_F = \hat{G}/\mathbb{Z}_\ell$ where

$$\mathbb{Z}_\ell \subset \mathcal{Z}(\hat{G}) \quad (\hat{G} \text{ is simply-connected})$$

we can turn on **almost-commuting** holonomies P and Q on T^2

$$PQ = \omega QP; \quad \omega \in \mathbb{Z}_\ell$$

while preserving $\mathcal{N}=2$ susy (**SW twist**). Ohmori, Tachikawa, Zafrir '18.

The 6d theory admits a $\mathbb{Z}_\ell$ SW twist only for a subset of possible holonomies and if M is a multiple of ℓ. Heckman, Lawrie, Lin, Zhang, Zoccarato '22.

Combining orbifolds and S-folds

We probe with N $D3$ branes the following background:

	z_1	z_2	z_3	T^2
7-brane	x	x		
$\mathbb{Z}_\ell$	ω_ℓ	ω_ℓ^{-1}	ω_ℓ	ω_ℓ^{-1}
$\mathbb{Z}_k$	ω_k	ω_k^{-1}		

$$\omega_\ell = e^{2\pi i/\ell} \qquad \omega_k = e^{2\pi i/k}$$

For each value of N, k we find several SCFT's (of rank $\sim Nk$):

- $O(k^4)$ for $\ell = 2$;
- $O(k^2)$ for $\ell = 3$;
- $k + 1$ for $\ell = 4$ and 1 for $\ell > 4$.

For $\ell\Delta_7 = 6$ these SCFT's coincide with SW theories. The underlying orbi-instantons have $M = \ell k$. SG, Savelli, Zoccarato '24.

Combining orbifolds and S-folds

We probe with N $D3$ branes the following background:

	z_1	z_2	z_3	T^2
7-brane	x	x		
$\mathbb{Z}_\ell$	ω_ℓ	ω_ℓ^{-1}	ω_ℓ	ω_ℓ^{-1}
$\mathbb{Z}_k$	ω_k	ω_k^{-1}		

$$\omega_\ell = e^{2\pi i/\ell}$$
$$\omega_k = e^{2\pi i/k}$$

For each value of N, k we find several SCFT's (of rank $\sim Nk$):

- $O(k^4)$ for $\ell = 2$;
- $O(k^2)$ for $\ell = 3$;
- $k + 1$ for $\ell = 4$ and 1 for $\ell > 4$.

For $\ell\Delta_7 = 6$ these SCFT's coincide with SW theories. The underlying orbi-instantons have $M = \ell k$. SG, Savelli, Zoccarato '24.

SW theories and their relevant deformations

SW theories (with $\ell\Delta_7 = 6$) have global symmetry

$$G_F \supseteq \mathcal{H} \times SU(k); \quad \mathcal{H} \subset G$$

We can find other SCFTs via mass deformations:

1. Those preserving $SU(k)$ are implemented by changing Δ_7.
2. Those preserving $\mathcal{H}$ correspond to activating the B-field.

For both types of mass deformations we can determine:

- The Coulomb branch spectrum;
- The magnetic quiver describing the Higgs branch;
- 't Hooft anomalies for the global symmetries.

Theories with $\ell = 1$ and Class $\mathcal{S}$

Compactifying orbi-instanton theories on T^2 without SW twist ($\ell = 1$) we get A-type Class $\mathcal{S}$ theories. Mekareeya, Ohmori, Tachikawa, Zafrir '17.

We find a subset of Class $\mathcal{S}$ trinions (spheres with 3 punctures).

For A-type Class $\mathcal{S}$ theories punctures are classified by partitions of N, the number of $M5$ branes.

Compactification of orbi-instantons leads to trinions such that:

- One puncture labelled by a partition with 2 elements
- One labelled by a partition with 3 elements.

All such trinions arise in this way! SG, Harding, Mekareeya, Mininno '24.

Theories with $\ell = 1$ and Class $\mathcal{S}$

Compactifying orbi-instanton theories on T^2 without SW twist ($\ell = 1$) we get A-type Class $\mathcal{S}$ theories. Mekareeya, Ohmori, Tachikawa, Zafrir '17.

We find a subset of Class $\mathcal{S}$ trinions (spheres with 3 punctures).

For A-type Class $\mathcal{S}$ theories punctures are classified by partitions of N, the number of $M5$ branes.

Compactification of orbi-instantons leads to trinions such that:

- One puncture labelled by a partition with 2 elements
- One labelled by a partition with 3 elements.

All such trinions arise in this way! SG, Harding, Mekareeya, Mininno '24.

All Class $\mathcal{S}$ theories from orbi-instantons

We study mass deformations of the 4d theory by analyzing the FI deformations of the 3d mirror gauge theory. Benini, Tachikawa, Xie '10.

We find that: SG, Harding, Mekareeya, Mininno '24.

- With a mass deformation we can generate all trinions with one puncture labelled by a partition with 2 elements;
- With a further deformation we find all other trinions (including T_N theory);
- We can then generate all theories with $n \geq 4$ punctures on the sphere by activating $n - 2$ mass deformations.

With further mass deformations we get either higher genus theories or Argyres-Douglas type models.

Higgs branch and magnetic quivers

The HB of our theories is described by a unitary magnetic quiver, which is derived from the known quiver of the underlying 6d theory.

For the theories with e.g. $\Delta_7 = 3$ and $\ell = 2$ we find

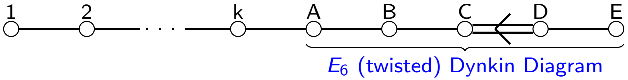
If we change Δ_7 we just need to replace the Dynkin diagram.

All theories for given k and Δ_7 are related by higgsing (proved via **quiver subtraction**).

Higgs branch and magnetic quivers

The HB of our theories is described by a unitary magnetic quiver, which is derived from the known quiver of the underlying 6d theory.

For the theories with e.g. $\Delta_7 = 3$ and $\ell = 2$ we find



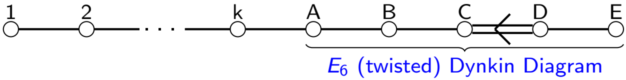
If we change Δ_7 we just need to replace the Dynkin diagram.

All theories for given k and Δ_7 are related by higgsing (proved via **quiver subtraction**).

Higgs branch and magnetic quivers

The HB of our theories is described by a unitary magnetic quiver, which is derived from the known quiver of the underlying 6d theory.

For the theories with e.g. $\Delta_7 = 3$ and $\ell = 2$ we find



If we change Δ_7 we just need to replace the Dynkin diagram.

All theories for given k and Δ_7 are related by higgsing (proved via **quiver subtraction**).

Holographic description and 't Hooft anomalies

The Type IIB setup implies the holographic dual is of the form

$$AdS_5 \times S^5/\Gamma \quad (\Gamma = \mathbb{Z}_k \times \mathbb{Z}_\ell)$$

From it we deduce the formulas: Aharony, Tachikawa '07; SG, Savelli, Zoccarato '24.

$$8a - 4c = (M\Delta_7)N^2 + (2M\Delta_7\,\epsilon + \Delta_7 - 1)N + \alpha N^0$$
$$TrR = 48(a-c) = 12(1-\Delta_7)N + \beta N^0$$
$$Tr(R\mathcal{H}^2) = -\Delta_7 I_{\mathcal{H}\hookrightarrow G} N + \gamma N^0$$

The coefficients α, β and γ require a one-loop computation in supergravity.
We determine them from the known central charges for $N=1$.

Holographic description and 't Hooft anomalies

The Type IIB setup implies the holographic dual is of the form

$$AdS_5 \times S^5/\Gamma \quad (\Gamma = \mathbb{Z}_k \times \mathbb{Z}_\ell)$$

From it we deduce the formulas: Aharony, Tachikawa '07; SG, Savelli, Zoccarato '24.

$$8a - 4c = (M\Delta_7)N^2 + (2M\Delta_7\,\epsilon + \Delta_7 - 1)N + \alpha N^0$$
$$TrR = 48(a - c) = 12(1 - \Delta_7)N + \beta N^0$$
$$Tr(R\mathcal{H}^2) = -\Delta_7 I_{\mathcal{H} \hookrightarrow G} N + \gamma N^0$$

The coefficients α, β and γ require a one-loop computation in supergravity.
We determine them from the known central charges for $N = 1$.

Generalized S-folds and the 4d $\mathcal{N} = 2$ landscape

To identify all theories of a given rank we should:

- List all orbi-instanton theories whose SW compactification gives a theory of the required rank. These are at the top of the mass deformation trees.
- Using our rules derive for each SW theory its descendants upon relevant deformation.

This procedure is algorithmic and **does not require** a detailed knowledge of all theories of lower rank.

We recover all theories of rank 1 and $\sim 90\%$ of known rank 2 theories.
We find 115 rank 3 theories, including 17 out of the 31 lagrangian theories.

Generalized S-folds and the 4d $\mathcal{N} = 2$ landscape

To identify all theories of a given rank we should:

- List all orbi-instanton theories whose SW compactification gives a theory of the required rank. These are at the top of the mass deformation trees.
- Using our rules derive for each SW theory its descendants upon relevant deformation.

This procedure is algorithmic and **does not require** a detailed knowledge of all theories of lower rank.

We recover all theories of rank 1 and $\sim 90\%$ of known rank 2 theories.
We find 115 rank 3 theories, including 17 out of the 31 lagrangian theories.

Concluding remarks

We have developed a general geometric framework which allows us to study a vast class of 4d $\mathcal{N} = 2$ theories and understand RG flows between them.

Possible generalizations:

- Understand more general mass deformations;
- Consider orbi-instantons of type D or E;
- Consider other twists of the 6d theories, not of SW type;
- 4d theories with minimal supersymmetry.

Thank You!

Concluding remarks

We have developed a general geometric framework which allows us to study a vast class of 4d $\mathcal{N} = 2$ theories and understand RG flows between them.

Possible generalizations:

- Understand more general mass deformations;
- Consider orbi-instantons of type D or E;
- Consider other twists of the 6d theories, not of SW type;
- 4d theories with minimal supersymmetry.

Thank You!

Concluding remarks

We have developed a general geometric framework which allows us to study a vast class of 4d $\mathcal{N} = 2$ theories and understand RG flows between them.

Possible generalizations:

- Understand more general mass deformations;
- Consider orbi-instantons of type D or E;
- Consider other twists of the 6d theories, not of SW type;
- 4d theories with minimal supersymmetry.

Thank You!