Exotic geometries from non-abelian GLSMs

[McGovern-JK soon]

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Outline

CY moduli spaces

- Context
- GLSMs
- Known examples

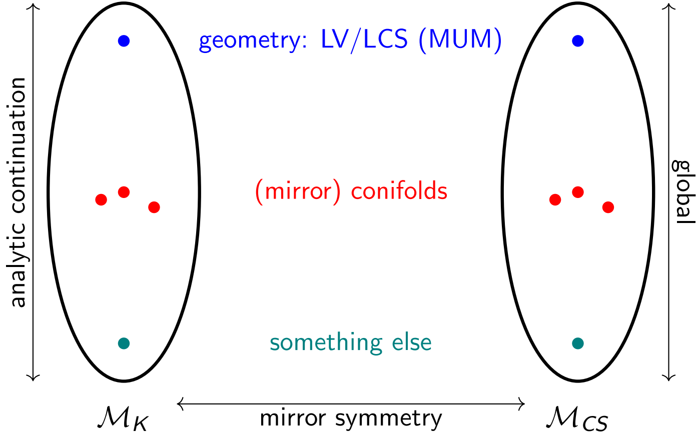
A new non-abelian model

- GLSM and phases
- Mirror symmetry and torsion-refined invariants

Outlook

Calabi-Yau moduli spaces

- Structure of moduli spaces of one-parameter models:



Motivation/Statement of the Problem

- What is "something else"?
  - Is it another geometry/non-linear sigma model? (sometimes)
  - If not, what is the correct mathematical and physical description? (usually hard)
  - Do new things happen when we consider CYs which are not complete intersections in toric spaces? (yes)
  - Can we do calculations? (often yes, but how to interpret?)
- Tools
  - 2D $(2,2)$ Gauged linear sigma models (GLSMs) [Witten 93]
  - Topological strings and mirror symmetry

GLSM data

- Symmetries
  - $(2,2)$-SUSY (including $U(1)_{V,A}$ R-symmetry)
  - Gauge symmetry: gauge group G (not necessarily abelian)
- Field content
  - Chiral multiplets: $\Phi^i = (\phi^i, \ldots)$
    - transform in some representation of G
  - Vector multiplets: $\Sigma^a = (\sigma^a, \ldots)$
- Parameters
  - FI-theta parameters (ζ^a, θ^a)
    - not renormalised in CY case
    - $t^a = \zeta^a - i\theta^a$: coordinates on $\mathcal{M}_K$
    - this talk: (ζ, θ) – one-parameter case
  - gauge couplings
- Superpotential $W(\Phi)$

Phases

- Phases of the GLSM are low-energy effective theories depending on the value of the FI parameter(s).
- Higgs vacua (all σ_i zero) are determined by the solutions of the D-term and F-term equations:

$$\mu(\phi) = \zeta \qquad dW(\phi) = 0$$

  - Gauge symmetry broken to a, potentially continuous, subgroup.
  - Parameter space is divided into chambers corresponding to different solutions.
  - Obtain low-energy effective theory by expanding about the vacuum.

Coulomb branch

- At phase boundaries, a Coulomb branch can emerge.
- Located in FI-theta-space determined by the critical loci of

$$\widetilde{W}_{eff} = -\langle t,\sigma\rangle - \sum_{i=1}^{\dim V} \langle Q_i,\sigma\rangle \left(\log\langle Q_i,\sigma\rangle - 1\right) + i\pi \sum_{\alpha>0}\langle \alpha,\sigma\rangle$$

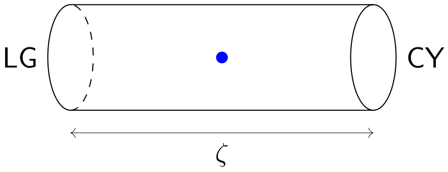
  - Q_i ... gauge charges of the chirals $\phi^i \in V$
  - $\alpha > 0$ ... positive roots of G
  - $\langle\cdot,\cdot\rangle$ ... $\mathfrak{t}_{\mathbb{C}}^* \times \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$
- This determines the singular points in $\mathcal{M}_K$

Quintic

- The quintic is a $G = U(1)$ GLSM with scalars $\phi = (p, x_1, \ldots, x_5)$ with gauge charges $(-5, 1, \ldots, 1)$, σ and FI-theta parameter $\zeta - i\theta$, and superpotential $W = pG_5(x)$.

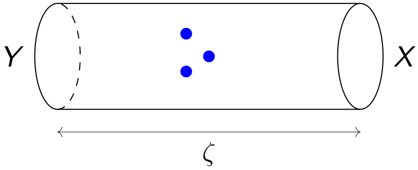
[Witten 93]

- Phases
  - $\zeta \gg 0$: $X_{\zeta \gg 0} = \{x \in \mathbb{P}^4 | G_5(x) = 0\}$ NLSM
  - $\zeta \ll 0$: $X_{\zeta \ll 0} = \{x \in \mathbb{C}^5/\mathbb{Z}_5, W^{LG} = \langle p \rangle G_5(x)\}$ Landau-Ginzburg orbifold
- Coulomb branch: $e^{-t} = -5^{-5}$
- Moduli space:



GLSMs with two geometric phases

- Sometimes, it can happen that one-parameter GLSMs have two geometric phases.



- X and Y are typically non-birational.
- The $\zeta \ll 0$-phase Y is, in some sense, unusual.
  - Strongly-coupled/non-perturbative physics (unbroken non-abelian G)
  - Non-complete intersections
  - Non-commutative resolutions of singular geometries
- Calabi-Yau differential operator has two MUM points.

[Almkvist-van Enckevort-van Straten-Zudilin 05]

Examples with two geometric phases

- Abelian model: $G = U(1)$ [Căldăraru-Distler-Hellerman-Pantev-Sharpe 07]
  - $\zeta \gg 0$: $\mathbb{P}^7[2,2,2,2]$
  - $\zeta \ll 0$: NC resolution of double cover of $\mathbb{P}^3$ branched over nodal octic
- Rødland model: $G = U(2)$ [Rødland 98][Hori-Tong 06]
  - $\zeta \gg 0$: codim 7 complete intersection in $G(2,7)$
  - $\zeta \ll 0$: Pfaffian CY in $\mathbb{P}^6$
- Hosono-Takagi model: $G = U(1)^2 \rtimes \mathbb{Z}_2$ [Hosono-Takagi 11][Hori 11]
  - $\zeta \gg 0$: codim 5 complete intersection in $(\mathbb{P}^4 \times \mathbb{P}^4)/\mathbb{Z}_2$
  - $\zeta \ll 0$: determinantal quintic, branched double cover
- Can we find new/different/more general ones?
  - So far all GLSM potentials were of the form $W = M_k^{ij} p^k x_i y_j$ all $\zeta \ll 0$-phases encoded in the mass matrix $M^{ij}(p)$.
  - Only $\mathbb{Z}_2$'s so far.

GLSM with cubics

- A generalisation of the Hosono-Takagi model.
- Consider a GLSM with gauge group $G = U(1)^3 \rtimes \mathbb{Z}_3$.
- Chiral matter

ϕ	p^1, p^2, p^3	x_1, x_2, x_3	y_1, y_2, y_3	z_1, z_2, z_3	FI
$U(1)_1$	-1	1	0	0	ζ
$U(1)_2$	-1	0	1	0	ζ
$U(1)_3$	-1	0	0	1	ζ

$$\mathbb{Z}_3: \qquad x_i \to y_i \to z_i \to x_i$$

- Superpotential

$$W = \sum_{ijkl=1}^{3} S_l^{ijk} p^l x_i y_j z_k = \sum_{ijk=1}^{3} S^{ijk}(p) x_i y_j z_k$$

$\zeta \gg 0$-phase: geometry

- Solving the D-term and F-term equations in the $\zeta \gg 0$-phase we get:

$$X_{\zeta\gg 0} = \{(x_i, y_i, z_i) \in (\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)/\mathbb{Z}_3 | S_l^{ijk} x_i y_j z_k = 0\}$$

  - This is a non-simply-connected Calabi-Yau threefold with

$$h^{11} = 1, \qquad H^3 = 30, \qquad c_2 \cdot H = 36, \qquad \chi = -30$$

[Candelas-Dale-Lutken-Schimmrigk 87][Candelas-Davies 08][Constantin-Gray-Lukas 16][...]

  - The associated Calabi-Yau operator is AESZ 17.
- Since AESZ 17 has two MUM points, folklore leads us to expect that the GLSM has a second geometric phase.
  - But this does not seem to happen in an obvious way.

$\zeta \ll 0$-phase - vacuum

- Vacuum
  - At $\sigma_i = 0$, the D-terms and F-terms are solved by setting $x_i = y_i = z_i = 0$.
  - The p-fields take values in a $p^i \in \mathbb{P}^2$.
  - The gauge symmetry is broken to

$$G_{\zeta \ll 0} = (U(1) \times U(1)) \rtimes \mathbb{Z}_3$$

    where "determinantal" $U(1) \subset G$ is broken.
  - We have a continuous unbroken non-abelian symmetry.
  - The phase is strongly coupled.

Effective theory in the $\zeta \ll 0$-phase

- The low energy theory has a superpotential with a non-abelian "orbifold" action:

$$W_{\zeta \ll 0} = \sum_{ijk} S^{ijk}(\langle p \rangle) x_i y_j z_k$$

- This is an interacting theory.
- There is no mass term for the chirals.
  - The mechanism that leads to the geometry in other known examples does not work.
- This is a hybrid theory. It is not clear how a geometry is created. (hyperdeterminants?)
- But there is more.

Coulomb branch

- The Coulomb branch of the GLSM is at

$$e^{-t} = \frac{\sigma_i^3}{(\sigma_1+\sigma_2+\sigma_3)^3}, \qquad \frac{\sigma_i^3}{\sigma_j^3} = 1, \qquad i,j \in \{1,2,3\}$$

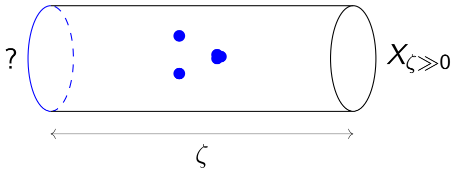
  - There are three points near the phase boundary (consistent with AESZ 17)

$$e^{-t} = -\frac{1}{27}, \pm\frac{i}{3\sqrt{3}}$$

- The strongly coupled $\zeta \ll 0$-phase also has a Coulomb branch.
  - Such a theory is non-regular. Coulomb and Higgs branch are not scale-separated. [Hori-Tong 06][Hori 11][Hori-JK 13]
  - Thanks to quantum effects due to the broken symmetry, the Coulomb branch is lifted except at $\zeta \to -\infty$. [Hori-Tong 06]

GLSM (in-)conclusion

- Moduli space



- We have found a Calabi-Yau GLSM with a phase that is hybrid-like with unbroken continuous symmetry, together with a Coulomb branch at infinity.
  - The phase looks singular and not at all like a non-linear sigma model with a smooth, compact target geometry.
  - However, monodromy indicates that there is no singularity.

Mirror picture

- Despite what the GLSM says, the Picard-Fuchs equation indicates that that the topological string should behave as if there were a geometry.
- Assume this is the case, and extract information from the mirror, monodromies, genus-g free energies, etc.

[BCOV 93][Huang-Klemm-Quackenbush 06]

- First attempt: assume the phase is a non-linear sigma model on a smooth geometry.
  - Monodromies, genus-0 free energy etc. would imply fractional triple intersection numbers: $H^3 = \frac{\chi}{13}$.
  - Something must be wrong.
- Second attempt: assume the phase is a non-commutative resolution of a singular geometry, leading to a theory with a B-field. [Vafa-Witten 94][Schimannek 21][Katz-Klemm-Schimannek-Sharpe 22]
  - This seems to work.

Geometry/Topology bootstrap I

- Consider a Calabi-Yau which has m terminal nodal singularities.
- Let $\hat{X}$ be a non-commutative (non-Kähler) resolution of the geometry with order N torsion: $\mathrm{Tors} H_2(\hat{X}, \mathbb{Z}) \simeq \mathbb{Z}_N$.
- This modifies the topological string partition function.

$$F^{(0)}|_{const} = \frac{\zeta(3)}{(2\pi i)^3}\left(\frac{\chi}{2} + mC_N\right)$$

  - Explicit formula for C_N from the Donaldson-Thomas partition function. [Katz-Klemm-Schimannek-Sharpe 22]
  - We expect $H^3 = \frac{2\left(\frac{\chi}{2}+mC_N\right)}{13}$.
  - Must determine m, N, H^3.

Geometry/Topology bootstrap II

- We get $c_2 = 32$ from the genus 1 free energy.

$$F^1(t) = -\frac{c_2}{24} t + O(e^{2\pi i t})$$

- Get $H^3 = 2$ from Yukawa couplings.
- Search databases for a suitable geometry and find N, m such that we get integer higher genus invariants (up to genus 4):

$$\mathbb{P}^5_{111223}[4,6], \qquad m = 63, \qquad N = 3$$

- Conjecture: This is the geometry we are looking for (assuming simply-connectedness).

Refined invariants

- Assuming the conjecture is true, we can compute $\mathbb{Z}_3$-torsion-refined GV-invariants.

[Schimannek 21][Katz-Klemm-Schimannek-Sharpe 22][see also Schimannek 25]

$$
\begin{aligned}
F_k &= \sum_{g=0}^{\infty} F_k^g(t)\lambda^{2g-2} \\
&= \sum_{g=0}^{\infty} \sum_{l=0}^{N-1} \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{m=1}^{\infty} \frac{n_{\beta,l}^{(g)}}{m} \left(2 \sin \frac{m\lambda}{2}\right)^{2g-2} e^{2\pi i k l/N} e^{2\pi i m \beta t}
\end{aligned}
$$

  - $[k] \in \mathbb{Z}_N \ldots$ fractional B-field

- Under these assumptions, we find integer invariants up to genus 11 out of an overdetermined system.

Conclusions

- Summary:
  - We have found a GLSM with two MUM points where one phase is non-regular and has no immediately obvious connection to a geometry.
  - Mirror symmetry indicates that it should be a non-commutative resolution of a singular geometry.
- Open questions
  - What is the physics of the $\zeta \ll 0$-phase and can it be understood in terms of geometry?
  - What about D-branes in non-regular models?

    [Herbst-Hori-Page 08][EHKR asymptoting completion][Guo-Romo-Smith 25]
    - When transported from the $\zeta \gg 0$-phase, do some branes "disappear" into the Coulomb branch?
  - What is Hori duality for non-regular GLSMs? [Hori 11]