6d SCFTs on ALE spaces

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Strings and Geometry 2025
IGAP/ICTP - Trieste

Based on: **2309.00607** and **2311.08462** (with **Michele Del Zotto**) + work in progress with **Daniele Ceppi**

Funded by the European Union

6d SCFTs and solitonic strings

Large zoo of (1,0) SCFTs in six dimensions:

- Brane constructions
  [Hanany-Zaffaroni '97] [Blum-Intriligator '97] [Hayashi-Kim-Lee-Taki-Yagi '15][Zafrir '15] ...
- Bottom up approach, esp. [Bhardwaj '15]
- Geometric classification from F-theory [Heckman-Morrison-Rudelius-Vafa '15], [Bhardwaj-Morrison-Tachikawa-Tomasiello '18], ...

$$\text{Elliptic CY3 } X \longleftrightarrow T_{6d}[X]$$

[... and many other important works!]

- Low-energy description involves two-form fields $B_{\mu\nu}$ that couple to **solitonic strings**
- Strings carry instanton charge wrt 6d gauge algebra due to coupling $S_{6d} \supset \int B \,\mathrm{Tr}\, F \wedge F$
- Good probes to study the properties of 6d theory - global symmetries, massless particle spectrum...

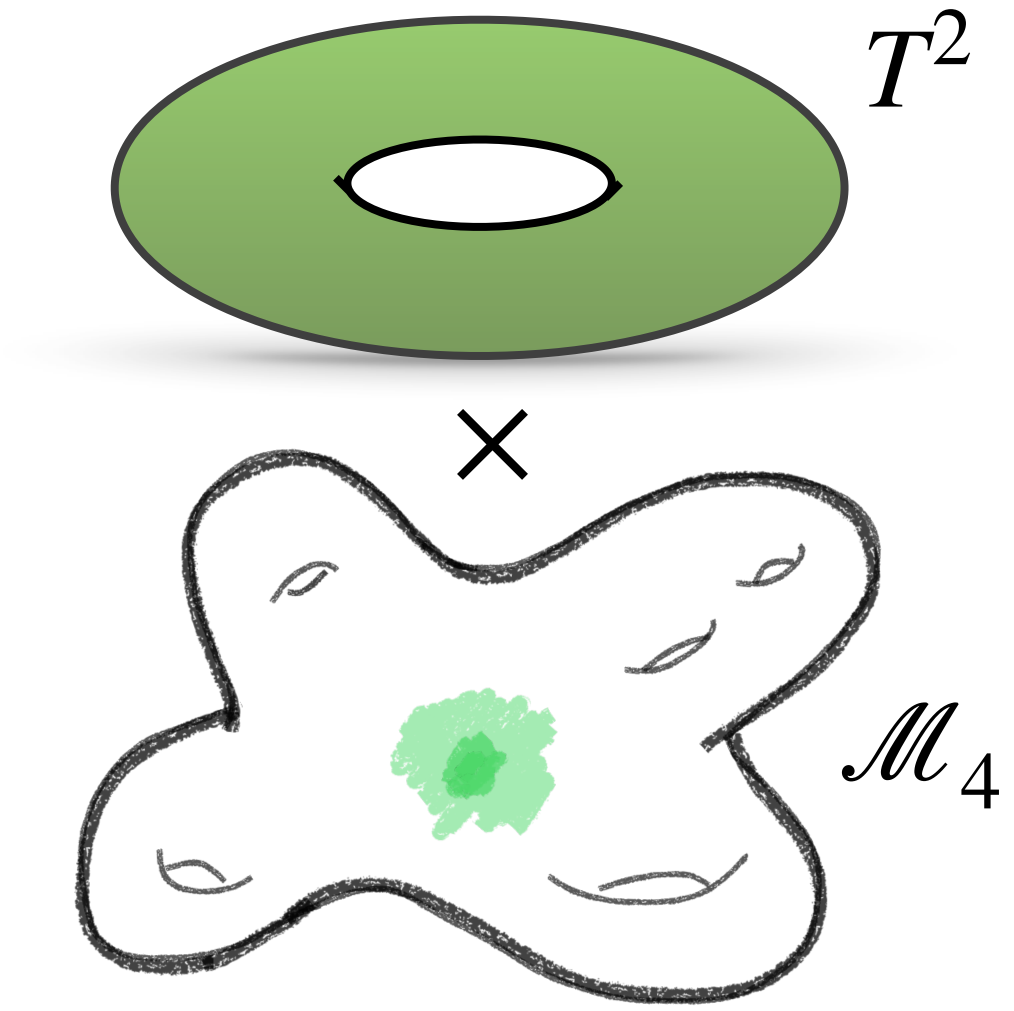
Partition functions and solitonic strings

- Strings should be an important ingredient for computing 6d observables.
- Cleanest example: SUSY partition functions on $T^2 \times \mathscr{M}_4$ (with DW twist along $\mathscr{M}_4$)

Wrapped strings

↓

Instantons on $\mathscr{M}_4$



Sum over topologically distinct field configurations

$$Z_{T^2 \times \mathscr{M}_4} \sim \int_{z.m.} \sum_{\vec{n}} e^{-\vec{n}\cdot\vec{t}}\, \mathbb{E}_{\vec{n}}$$

Integral over zero-modes for compact $\mathscr{M}_4$

- More generally, string expansion should be relevant for fibrations $T^2 \to \mathscr{M}_4$
  - e.g.: superconformal index $S^1 \times S^5$ [Iqbal-Koczaz-Haghighat-GL-Vafa], $S^1 \times$ (Sasaki-Einstein) [Qiu,Tizzano,Winding,Zabzine '14]...
  - Nice framework for computations using fibering operators by [Closset-Magureanu '22]

Equivariant partition functions

- Simplest setting where explicit computations are possible is $\mathscr{M}_4 = \mathbb{C}^2$.

  Lots of computational techniques available, e.g.:
  - 2d quiver gauge theory
  - Topological strings/blow-up equations/brane webs
  - Modularity
  - Gauge theory dualities

  [Agarwal, Cai, Del Zotto, Gadde, Gu, Haghighat, Hayashi, Hohenegger, Huang, Iqbal, Jaramillo Duque, Kashani-Poor, J.Kim, S.Kim, S.-S.Kim, Klemm, K.H. Lee, K.Lee, Ohmori, K.Park, Kozcaz, GL, Maruyoshi, Song, Sun, Vafa, Yoshida, Yun...]

- Noncompact moduli spaces → work in Omega background [Nekrasov '02]

  Equivariant parameters: ϵ_+, ϵ_- for the $SO(4) \sim SU(2)_+ \times SU(2)_-$ isometry of $\mathbb{C}^2$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} e^{2\pi i(\epsilon_+ + \epsilon_-)} z_1 \\ e^{2\pi i(\epsilon_+ - \epsilon_-)} z_2 \end{pmatrix}$$

Partition function for $T_{6d}[X]$ encodes refined DT invariants of elliptic threefold X.

Equivariant partition functions

However, from the perspective of topology $\mathbb{C}^2$ is not terribly interesting.

Can we learn how solitonic strings behave on nontrivial geometric background?

This talk: consider the **equivariant partition function on** $T^2 \times ALE_\Gamma$

$$ALE_\Gamma = \text{hyperkähler resolution of } \mathbb{C}^2/\Gamma.$$

Γ : discrete subgroup of $SU(2)_-$

McKay correspondence:

Γ	$\mathbb{Z}_n$	BD_{2n}	T	O	I
$\mathfrak{g}$	$\mathfrak{su}(n)$	$\mathfrak{so}(2n)$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Geometric counterpart: **higher rank DT** invariants on CY3 X.

[Nekrasov-Okounkov '14, Del Zotto-Nekrasov-Piazzalunga-Zabzine '21]

ALE partition functions

Topological data should appear in partition function:

Cartan matrix for $\mathfrak{g}$

$$H_2(ALE_\Gamma, \mathbb{Z}): \quad \Sigma_i \cdot \Sigma_j = -(C^{\mathfrak{g}})_{ij}$$ → fluxes

$$\pi_1(\partial ALE_\Gamma) = \Gamma$$ → gauge field holonomies

This is familiar from $\mathcal{N} = 4$ $U(r)$ SYM with Vafa-Witten twist: $Z_{VW} = \frac{\chi_\omega^{\hat{\mathfrak{g}}_r}(\tau)}{\eta(\tau)}$

[Vafa-Witten '94]
[Nakajima '94]

ω: highest weight rep for $\hat{\mathfrak{g}}_r \leftrightarrow$ choice of flat connection at infinity.

6d $\mathcal{N} = (2,0)$ origin → Interpretation of d.o.f. as current algebra on torus.

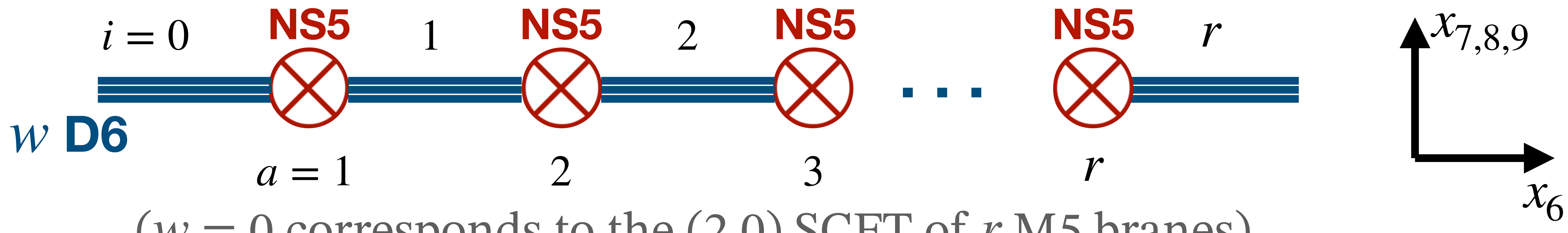
Natural to expect interesting interactions with solitonic strings wrapped on T^2

$\tau = \mathbb{C}$-structure of T^2

Brane setup

Focus on a class of 6d SCFTs with simple brane realization:

Type IIA:



($w = 0$ corresponds to the (2,0) SCFT of r M5 branes)

Field content: **NS5 branes** → tensor multiplets $T_a^{(1,0)}$, $a = 1,\ldots,r$

D6 branes → vector multiplets $V_i^{(1,0)}$, $i = 1,\ldots,r-1$+ bifundamental hypers

Gauge symmetry: $\prod_{i=1}^{r-1} G_i = SU(w)^{(i)} \times U(1)^{(i)}$

Flavor symmetry: $SU(w)^{(0)} \times U(1)^{(0)} \times SU(w)^{(r)} \times U(1)^{(0)}$

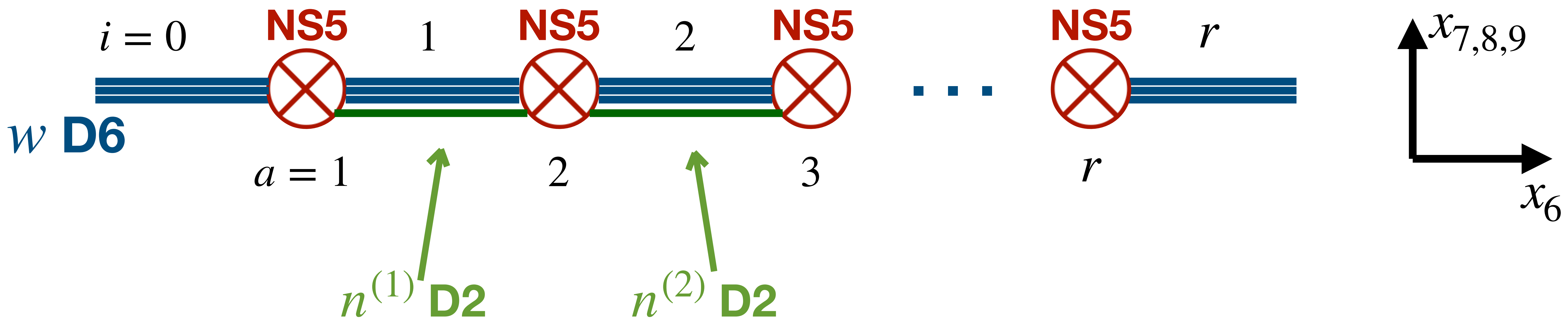
Stuckelberg mechanism: most abelian factors massive. Only global $U(1)_{diag} = \mathrm{diag}(U(1)^{(0)} \times \ldots \times U(1)^{(r)})$ survives.

[Douglas-Moore'96,Berkooz-Leigh-Polchinski-Scwharz-Seiberg-Wittenl'96,Hanany-Zaffaroni '97]

Instanton strings

Type IIA brane realization:

[Haghighat-Kozcaz-GL-Vafa '13]



Stretch $n^{(i)}$ D2 branes between the i and $(i+1)$-th NS5 brane → bound state of instanton strings

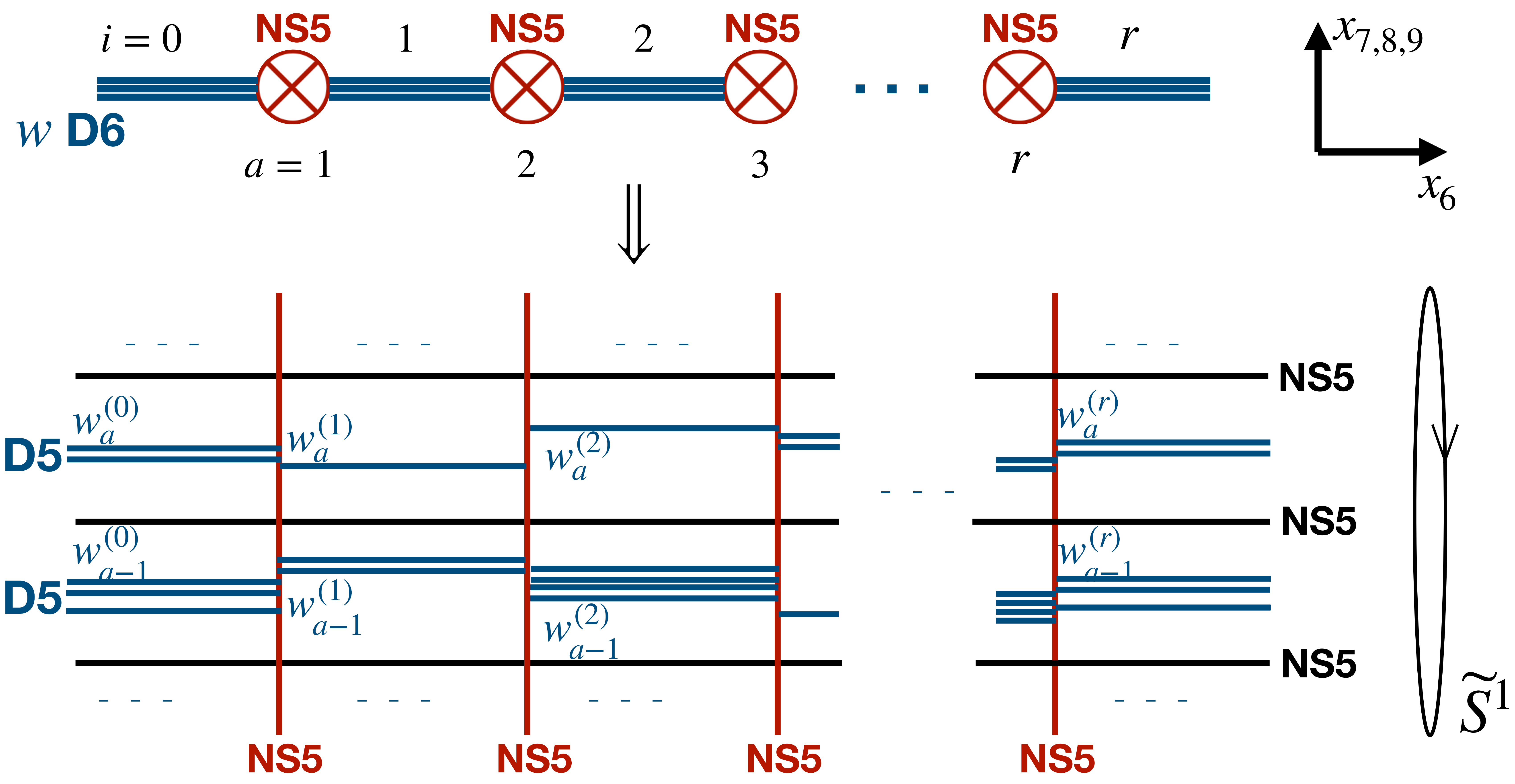
$n^{(i)}$: instanton charge for gauge group $SU(w)^{(i)}$

SCFT on $\mathbb{C}^2/\mathbb{Z}_n$: T-dual picture

Type IIA brane realization:

- Replace $\mathbb{C}^2/\mathbb{Z}_n$ with TN_n
- T-dualize along TN circle

Type IIB realization:



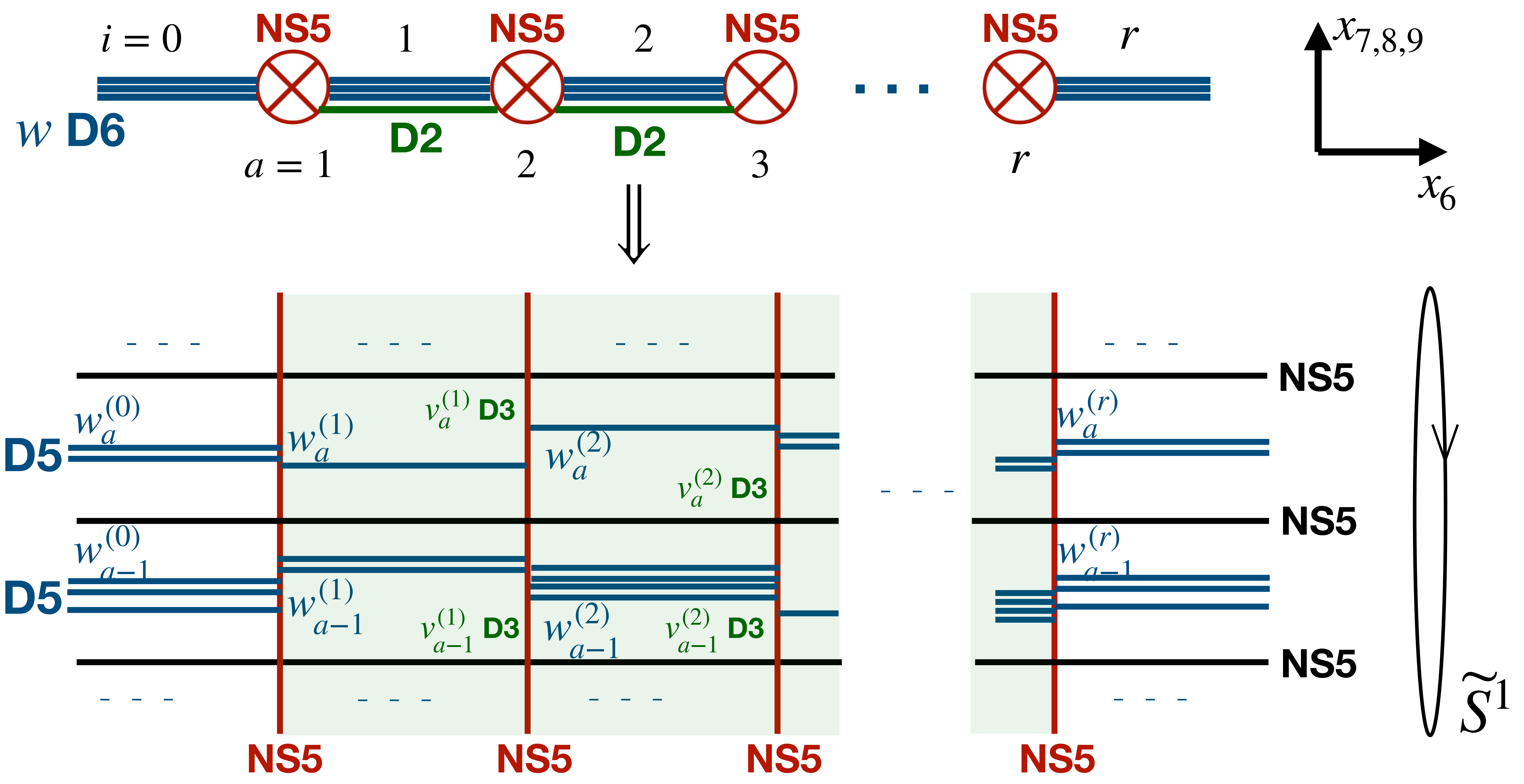
Nontrivial holonomies on $\partial(ALE) \Longrightarrow$ $U(w)^{(a)} \to U(w_1^{(a)}) \times \ldots \times U(w_n^{(a)})$

SCFT on $\mathbb{C}^2/\mathbb{Z}_n$: T-dual picture

Instanton strings

↓

Suspended D3 branes



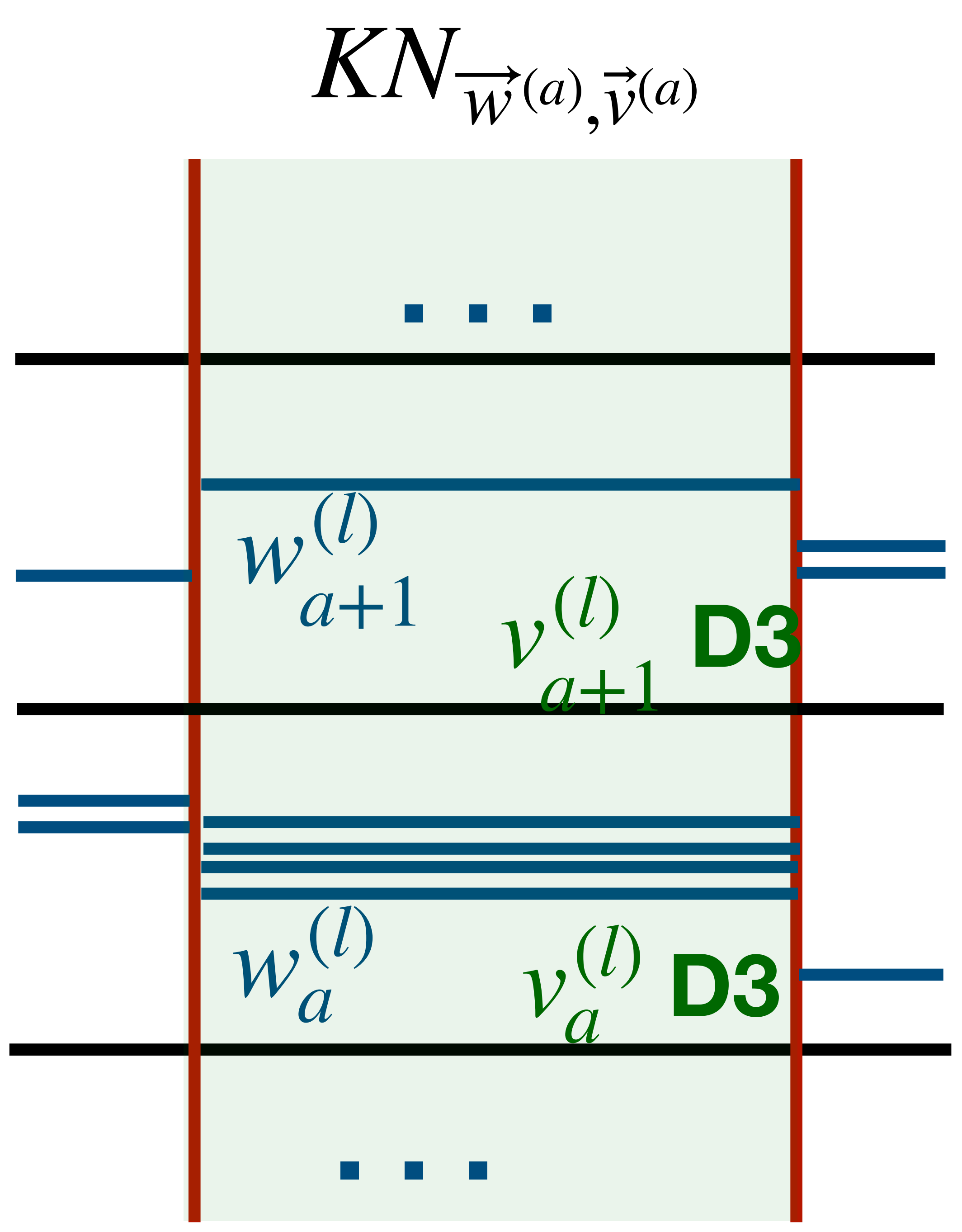
Stuckelberg constraint: all 6d abelian gauge factors identified → all $c_1(U(w)^{(a)})$ are identical.

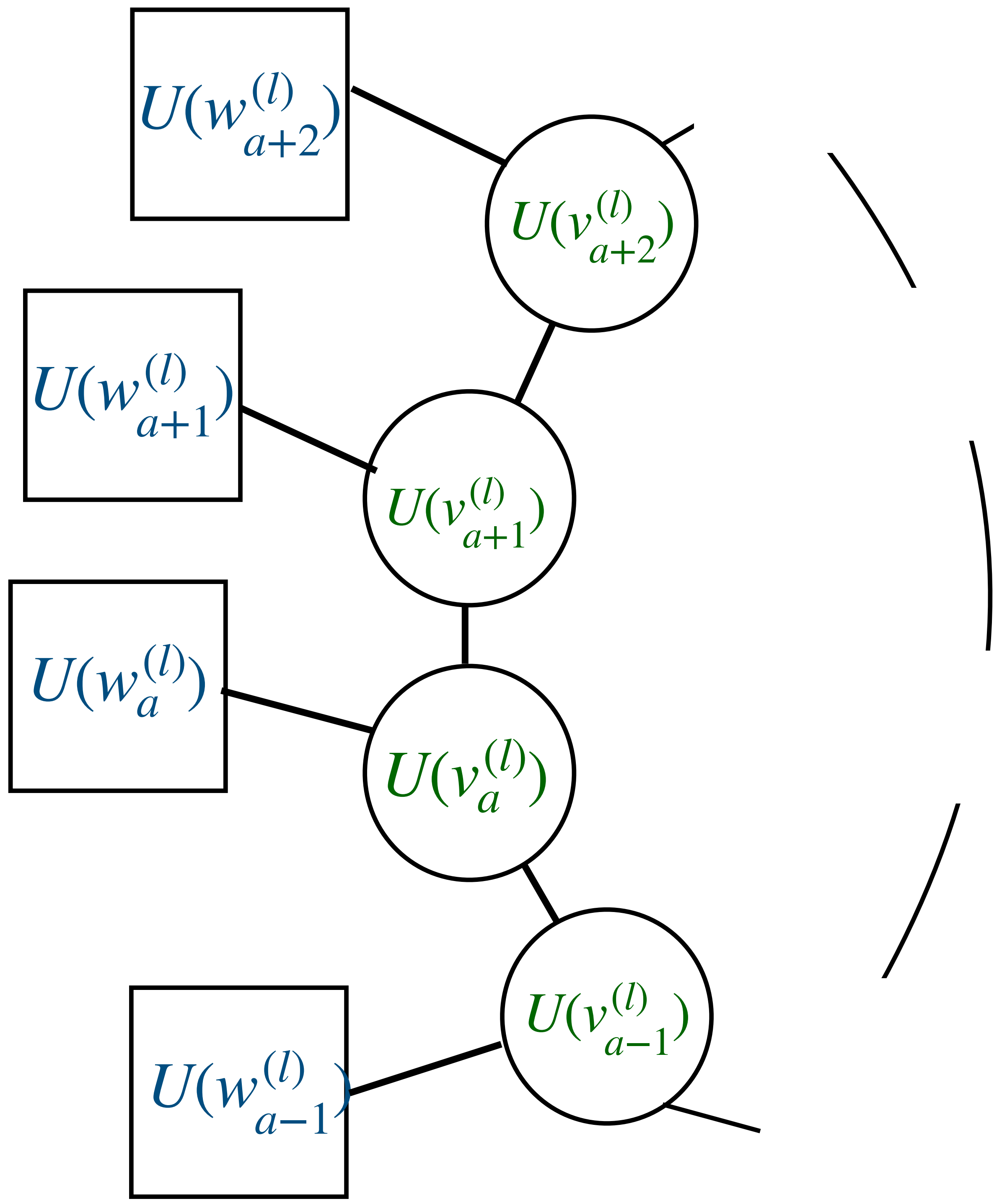
$$\Downarrow$$

$$\vec{w}^{(0)} = \vec{w}^{(1)} - C^{\widehat{A}_{n-1}} \cdot \vec{v}^{(1)} = \vec{w}^{(2)} - C^{\widehat{A}_{n-1}} \cdot \vec{v}^{(2)} = \ldots \qquad = \vec{w}^{(r)}$$

The instanton string quiver

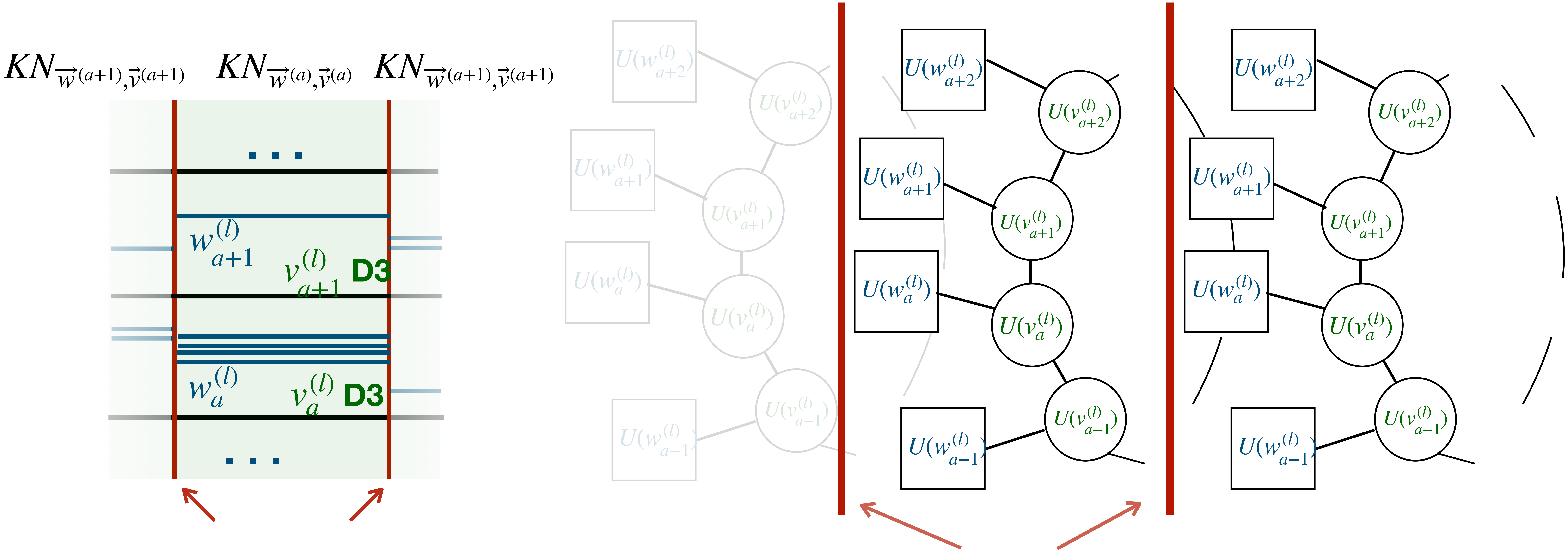
The skeleton of the quiver are 3d $\mathcal{N} = 4$ **Kronheimer-Nakajima** quiver gauge theories describing $U(w)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$.





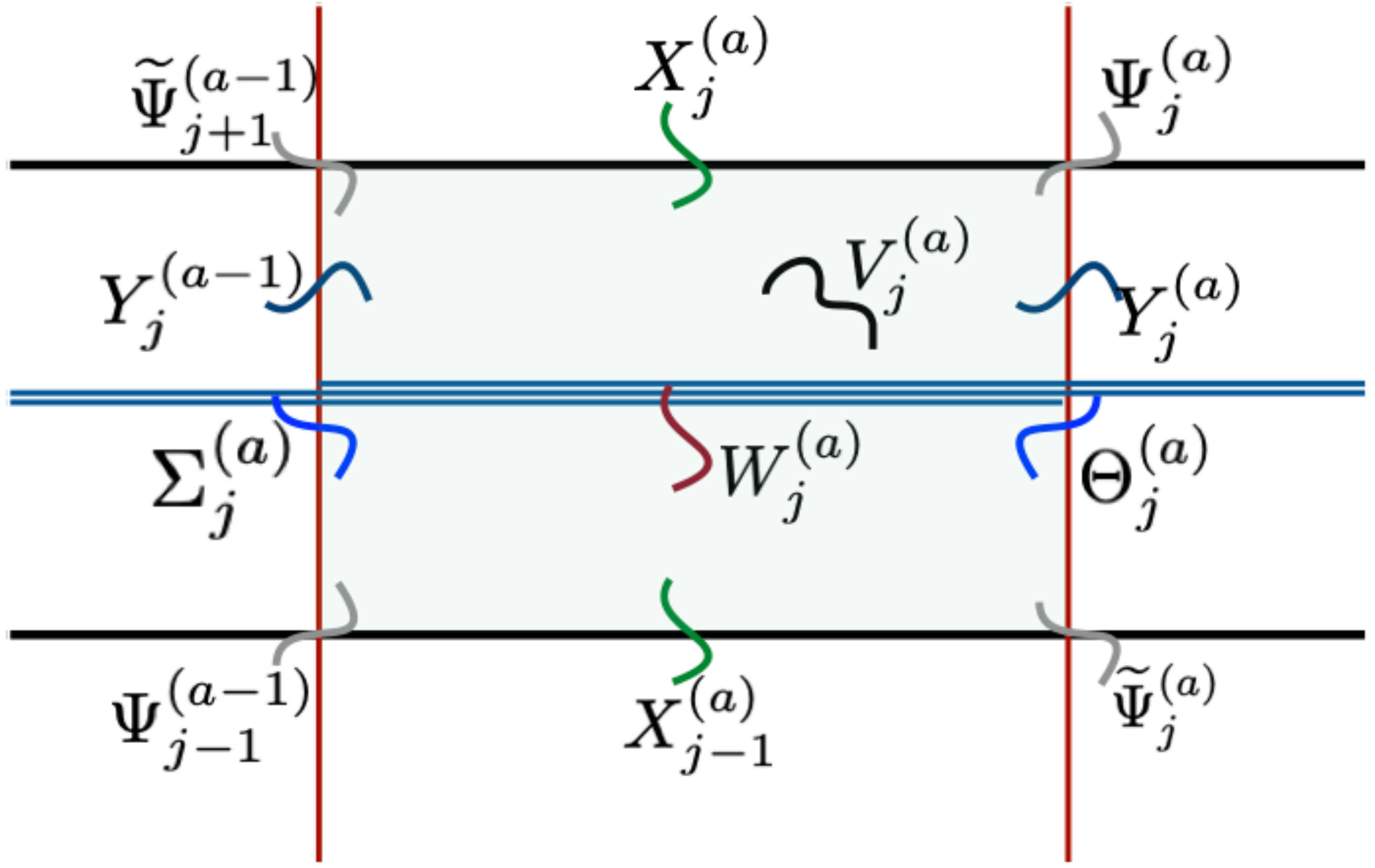
The instanton string quiver

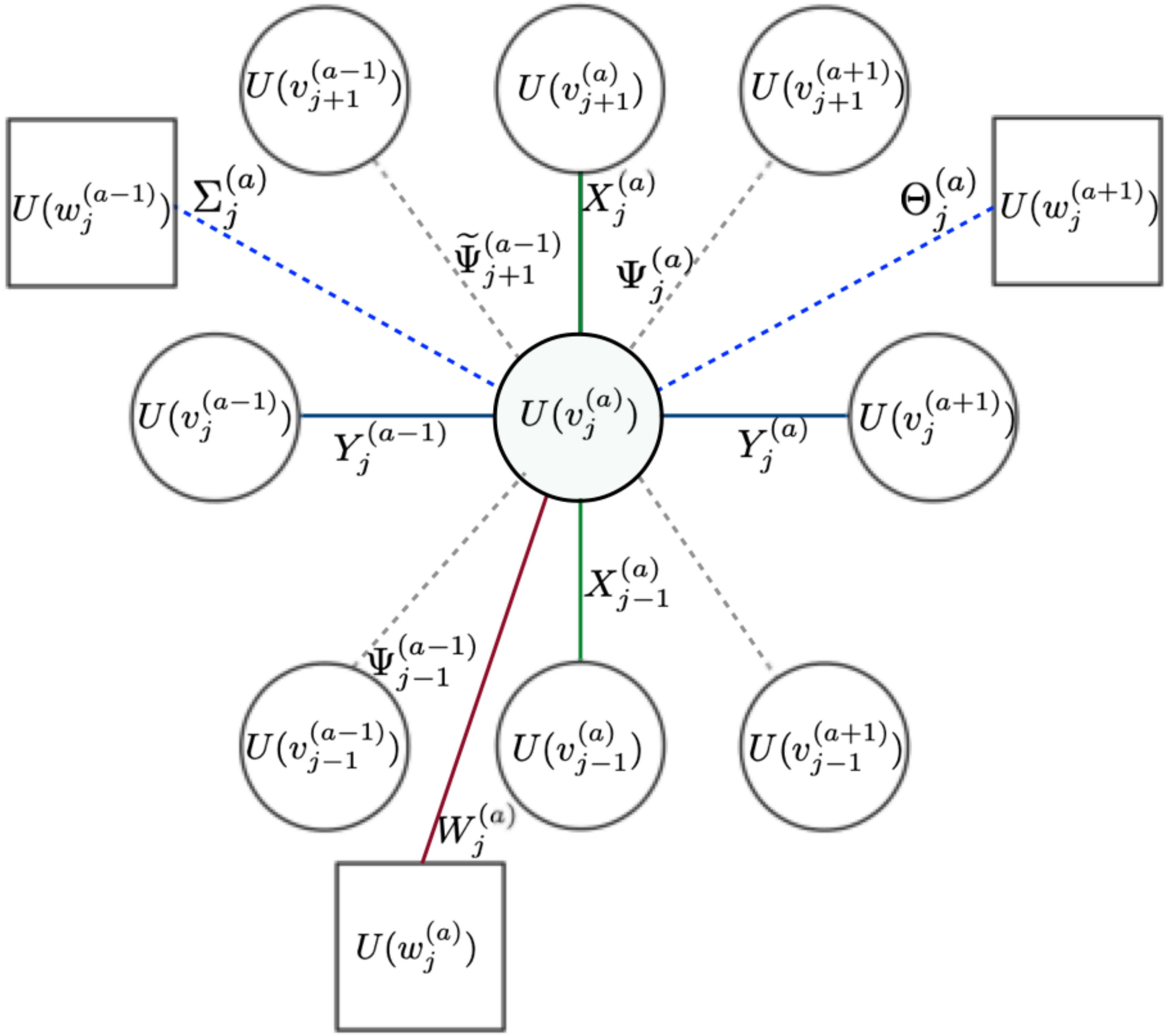
The skeleton of the quiver are 3d $\mathcal{N} = 4$ **Kronheimer-Nakajima** quiver gauge theories describing $U(w)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$. Stacking them between interfaces leads to a 2d $\mathcal{N} = (0,4)$ theory.



Gauge anomalies

Focus on a single gauge node:





- $SU(v_j^{(a)})$ gauge anomaly cancelation on all nodes

  equivalent to imposing the Stuckelberg constraint

$$\vec{w}^{(0)} = \vec{w}^{(1)} - C^{\widehat{A}_{n-1}} \cdot \vec{v}^{(1)} = \vec{w}^{(2)} - C^{\widehat{A}_{n-1}} \cdot \vec{v}^{(2)} = \dots \qquad = \vec{w}^{(r)}$$

Gauge anomalies

However: Abelian gauge anomalies don't cancel...

The solution: $n \cdot r$ complex fermions $\psi_j^{(a)}$ localized at NS5-NS5 intersections

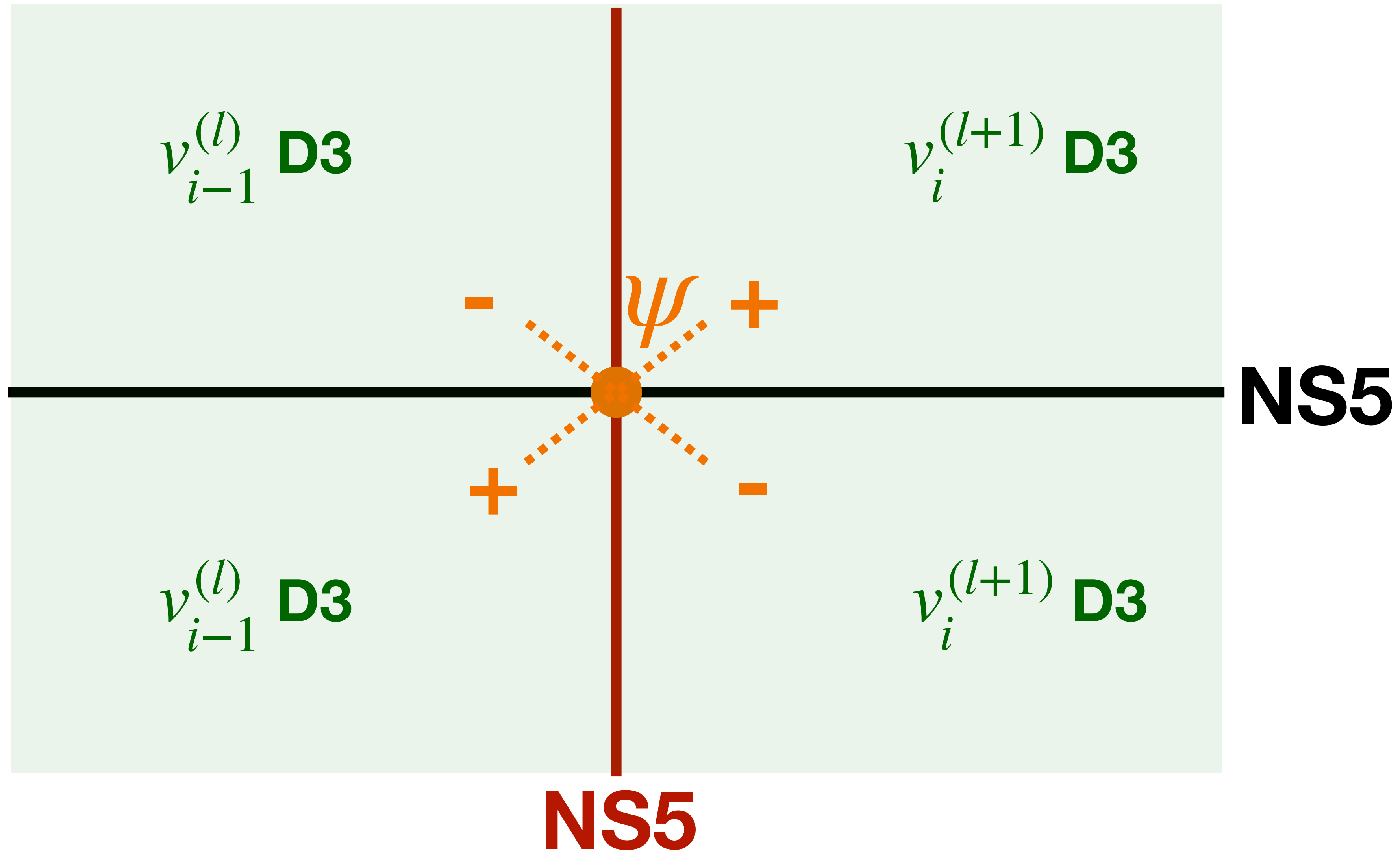
Fermions give rise to $\mathfrak{u}(r\,n)_1$ current algebra

[Itzakhi-Kutasov-Seiberg '05, Dijkgraaf -Hollands -Sulkowski-Vafa '07]

On tensor branch → $\mathfrak{u}(n)_1^{\oplus r}$ current algebra

Coupling to D3 branes ensures anomaly cancelation!

[Hanany-Okazaki '18], see also [Gaiotto-Costello '18]

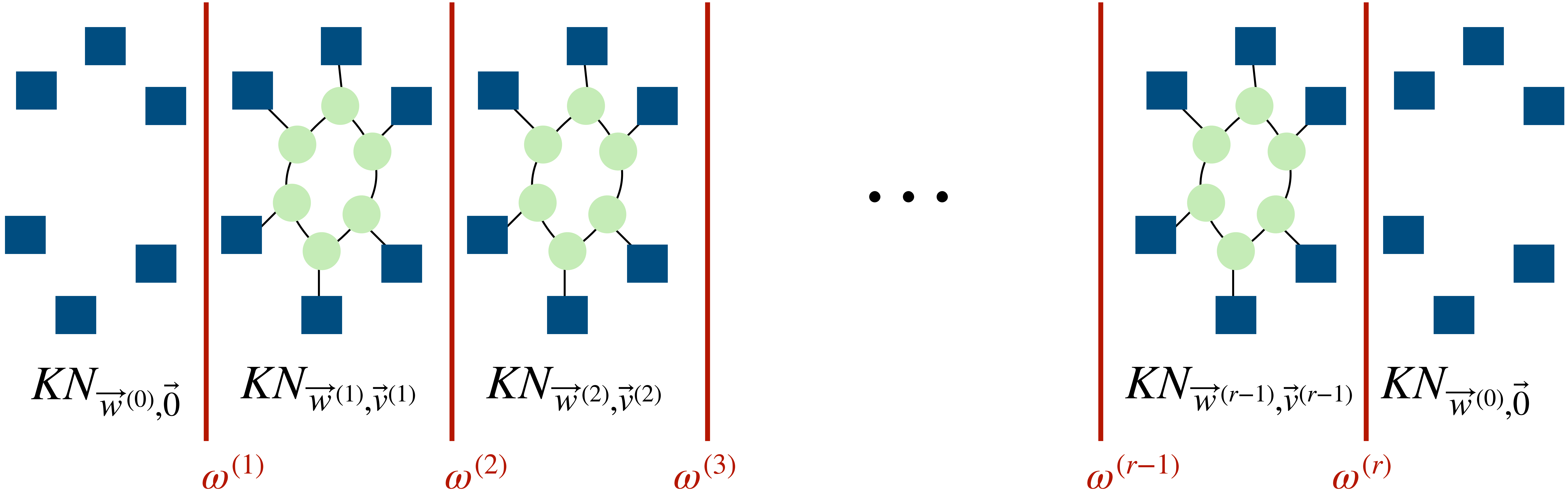


On $\mathbb{C}^2/\mathbb{Z}_n$, anomalies cured by coupling to $(\widehat{\mathfrak{su}(n)}_1)^r$ currents supported on interfaces

The A-type string quiver

Schematically we can represent the soliton string quivers as follows:

$$KN^{2d}\begin{bmatrix} \{\omega^{(a)}\} \\ \{\vec{w}^{(a)}\},\ \{\vec{v}^{(a)}\} \end{bmatrix} =$$



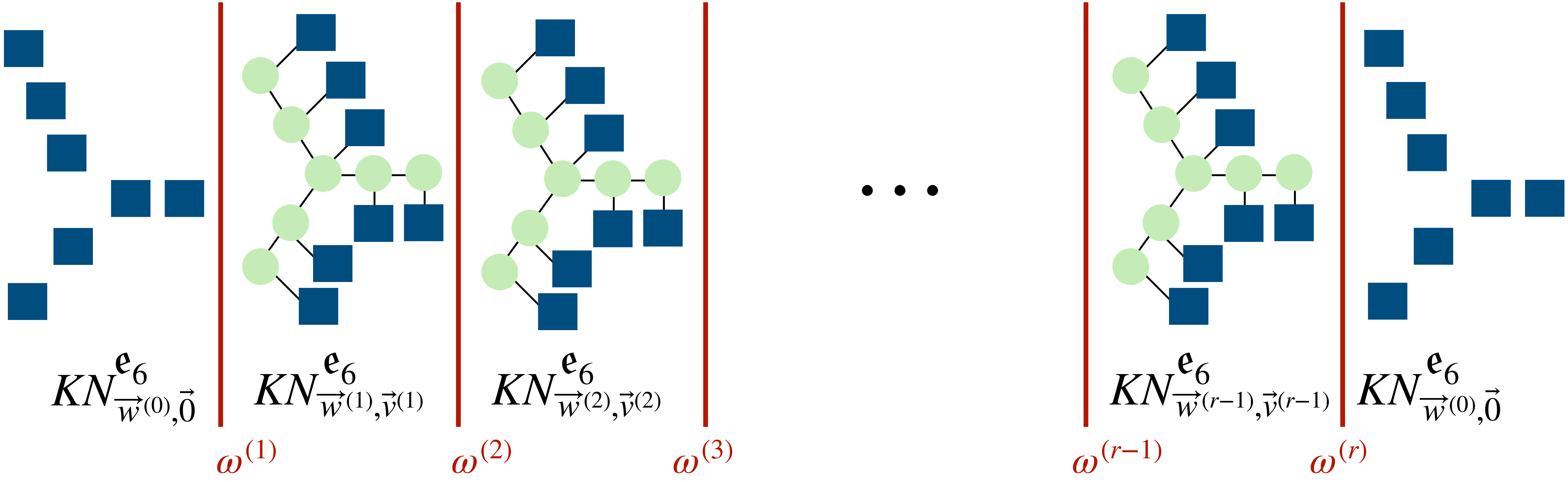
$\omega^{(a)}$: integrable h.w. representations of $\widehat{\mathfrak{su}}(n)_1$

The D,E-type string quivers

A-type quiver generalizes on the nose to the D and E cases, e.g. for $\mathbf{e}_6$:

$$KN^{2d}_{\mathbf{e}_6}\left[\begin{array}{c}\{\omega^{(a)}\}\\ \{\vec{w}^{(a)}\},\{\vec{v}^{(a)}\}\end{array}\right]$$

$$=$$



Abelian gauge anomalies killed by current algebras at interfaces

$\omega^{(a)}$: integrable h.w. representations of $\widehat{\mathbf{e}}_{6,1}$

The elliptic genus

$$\mathbb{E}^{\Gamma}\begin{bmatrix} \{\omega^{(a)}\} \\ \{\overrightarrow{w}^{(a)}\},\{\vec{v}^{(a)}\}\end{bmatrix} = \mathrm{Tr}_R\left[(-1)^F q^{L_0-\frac{c_L}{24}}\overline{q}^{\overline{L}_0-\frac{c_R}{24}} e^{\epsilon_+(J^{SU(2)_+}+J^{SU(2)_R})} e^{\vec{\xi}\cdot\vec{J}}\right] \qquad q = e^{2\pi i\tau}$$

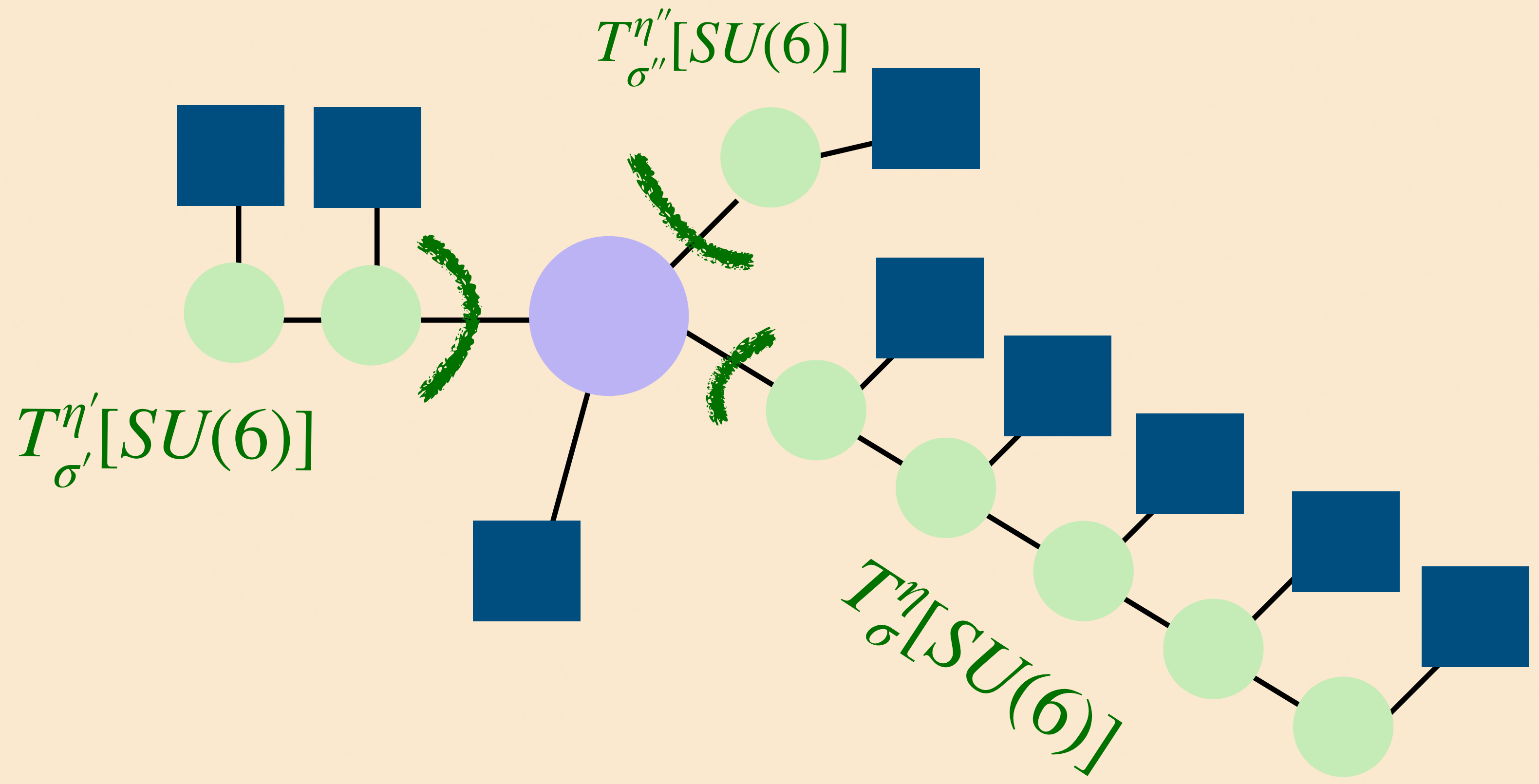
Compute by localization [Benini-Eager-Hori-Tachikawa et al. '13]:

$$\mathbb{E}^{\Gamma}\begin{bmatrix} \{\omega^{(a)}\} \\ \{\overrightarrow{w}^{(a)}\},\{\vec{v}^{(a)}\}\end{bmatrix} = \int \left(\prod_{a=1}^{r}\frac{d\vec{z}^{(a)}}{2\pi i}\right)\cdot Z^{1-loop}_{\{\overrightarrow{w}^{(a)}\},\{\vec{v}^{(a)}\}}\cdot\prod_{a=1}^{r}\widehat{\chi}^{\mathfrak{g}}_{\omega^{(a)}}$$

Integrals given by summing over Jeffrey-Kirwan residues. For $\Gamma = \mathbb{Z}_n$, we obtain exact combinatorial expressions.

For D,E-type:

- Quiver tails described by 2d version of $T[SU(n)]$ theories, coupled to current algebras $\widehat{\mathfrak{g}}^1 \oplus \widehat{\mathfrak{g}}^2 \oplus \widehat{\mathfrak{g}}^3 \subset \widehat{\Gamma}$
- Elliptic genus obtained by gauging central node + summing over h.w. reps of $\widehat{\mathfrak{g}}^{1,2,3}$.



Character decomposition

The elliptic genus can be decomposed as follows:

$$\mathbb{E}^{\Gamma,\{\omega^{(a)}\}}_{\{\vec{w}^{(a)}\},\{\vec{v}^{(a)}\}} = \sum_{i,j,k} c_{ijk}\, \chi_i^{\hat{\mathfrak{g}}_r} \chi_j^{D_r \hat{\mathfrak{g}}}\, \chi_k^{\mathscr{V}_{\{\vec{w}^{(a)}\},\{\vec{v}^{(a)}\}}}$$

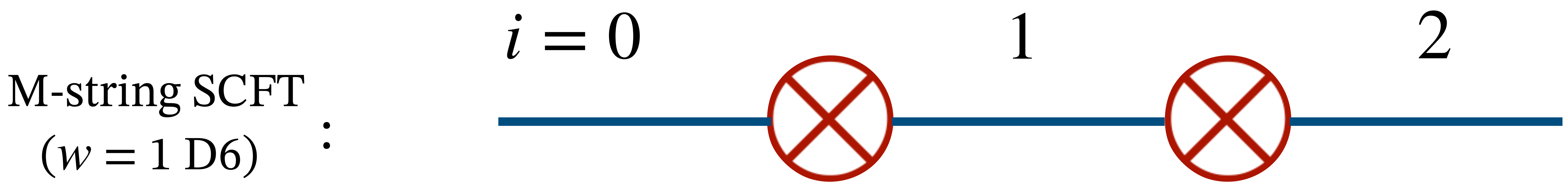
$c_{ijk} \in \mathbb{Z}$

$\widehat{\mathfrak{g}}_r$:

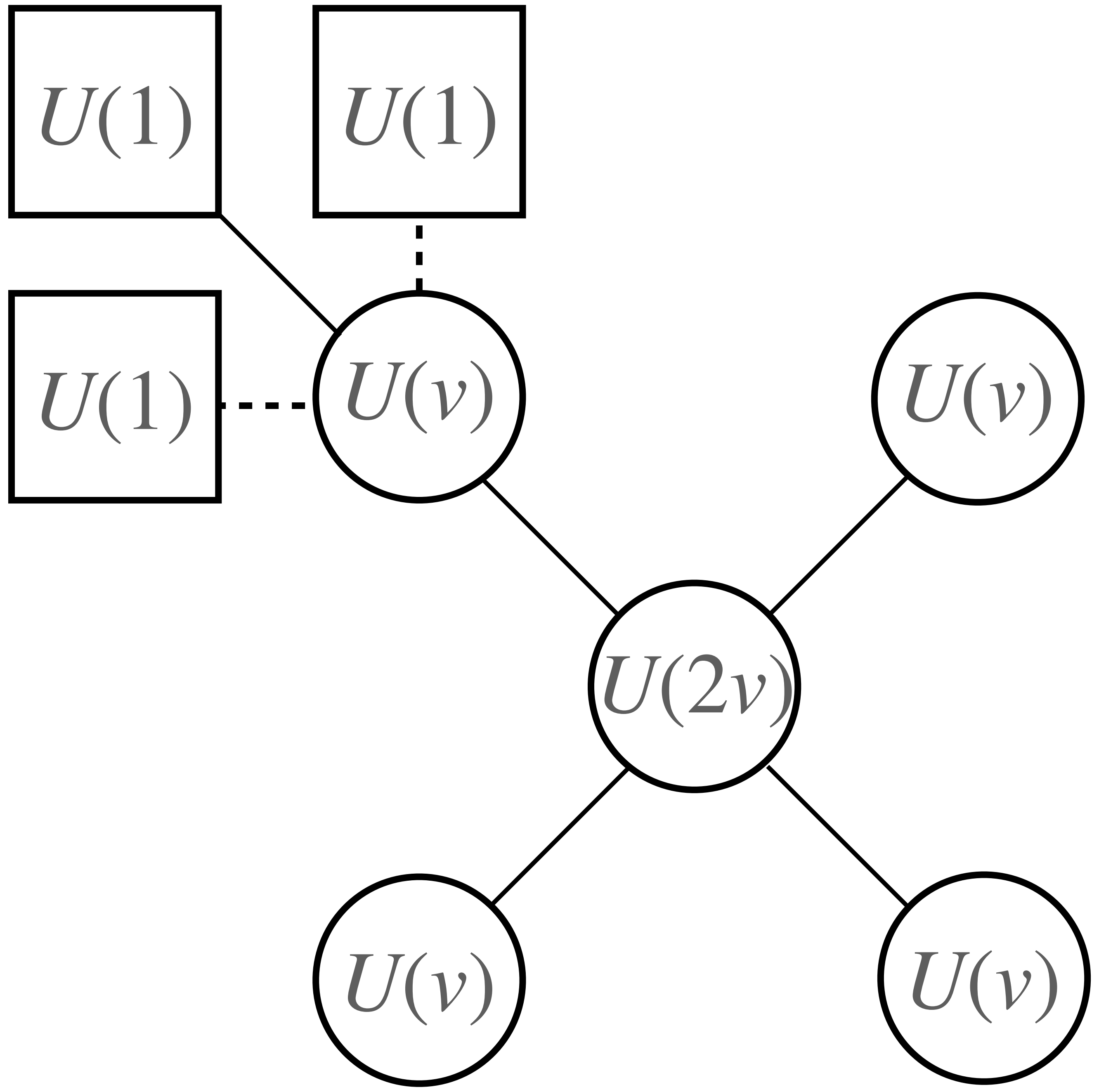
$$D_r \hat{\mathfrak{g}} = \frac{\hat{\mathfrak{g}}_1^r}{\hat{\mathfrak{g}}_r}$$

Example: M-string SCFT on $\mathbb{C}^2/BD_2$

BD_2: binary dihedral group of order 8 $\rightarrow \mathfrak{g} = SO(8)$.

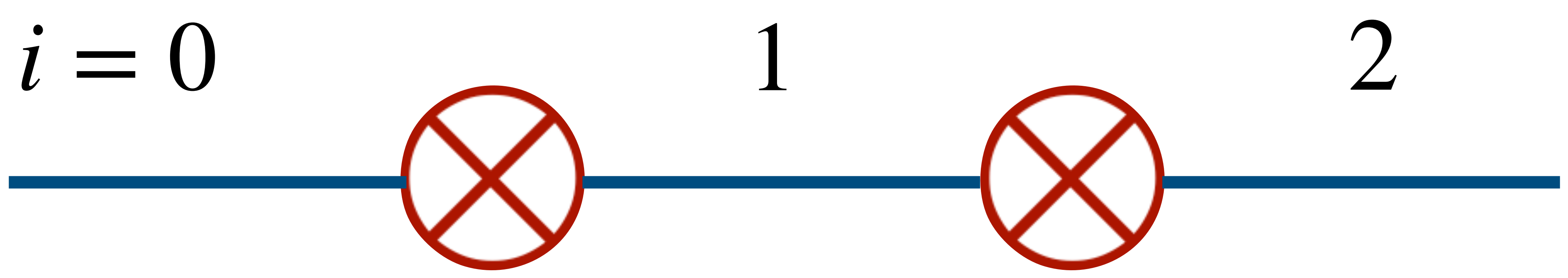


We obtain the following 2d quiver, coupled to $\widehat{\mathfrak{so}(8)}_1 \times \widehat{\mathfrak{so}(8)}_1$



$v =$ instanton number

Example: M-string SCFT on $\mathbb{C}^2/BD_2$



For one string:

$$\mathbb{E}_{1\,string}^{\mathbb{C}^2/BD_2,\{\omega^{(a)}\}} = \sum_{i,j,k} c_{ijk}\, \chi_i^{\widehat{SO(8)}_2} \chi_j^{D_r\widehat{SO(8)}}\ \chi_k^{\mathcal{V}_{1\,string}}$$

We find a residual VOA $\mathcal{V}_{1\,string}$ with 5 inequivalent characters:

$$\chi_1^{\mathcal{V}_{1\,string}} = A \cdot \frac{\theta_1(1/4+4\epsilon_+,\tau)\theta_2(1/4+4\epsilon_+,\tau)}{\theta_1(2\epsilon_+,\tau)\theta_1(4\epsilon_+,\tau)}$$

$$\chi_2^{\mathcal{V}_{1\,string}} = A \cdot \frac{\theta_3(1/4+4\epsilon_+,\tau)\theta_4(1/4+4\epsilon_+,\tau)}{\theta_1(2\epsilon_+,\tau)\theta_1(4\epsilon_+,\tau)}$$

$$\chi_3^{\mathcal{V}_{1\,string}} = A \cdot \frac{\theta_3(4\epsilon_+,\tau)\theta_4(4\epsilon_+,\tau)}{\theta_1(2\epsilon_+,\tau)\theta_1(4\epsilon_+,\tau)}$$

$$\chi_4^{\mathcal{V}_{1\,string}} = A \cdot \frac{\theta_1(2\epsilon_+,\tau)\theta_2(2\epsilon_+,\tau)}{\theta_3(2\epsilon_+,\tau)\theta_4(2\epsilon_+,\tau)} \frac{\theta_1(4\epsilon_+,\tau)\theta_2(4\epsilon_+,\tau)}{\theta_1(2\epsilon_+,\tau)\theta_1(4\epsilon_+,\tau)}$$

$$\chi_5^{\mathcal{V}_{1\,string}} = A \cdot \frac{\theta_3(2\epsilon_+,\tau)\theta_4(2\epsilon_+,\tau)}{\theta_1(2\epsilon_+,\tau)\theta_2(2\epsilon_+,\tau)} \frac{\theta_1(4\epsilon_+,\tau)\theta_2(4\epsilon_+,\tau)}{\theta_1(2\epsilon_+,\tau)\theta_1(4\epsilon_+,\tau)}$$

$\theta_i(z,\tau) =$ Jacobi theta functions $\qquad A = \dfrac{\theta_1(m+\epsilon_+,\tau)\theta_1(m-\epsilon_+,\tau)}{\eta(\tau)^2}$

The partition function

Summing over string contributions → **6d partition functions**

$$Z^{T^2\times\mathbb{C}^2/\Gamma}_{\{w^{(a)}\},\{\omega^{(a)}\}} = Z^{pert}_{\{w^{(a)}\},\{\omega^{(a)}\}} \times \sum_{\{\vec{v}^{(a)}\}} e^{-n^{(a)}t^{(a)}}\mathbb{E}\left[\begin{matrix}\{\vec{v}^{(a)}\}\\ \{w^{(a)}\},\{\omega^{(a)}\}\end{matrix}\right]$$

Tensor of partition functions labeled by $\omega^{(1)},\ldots,\omega^{(r)}$.

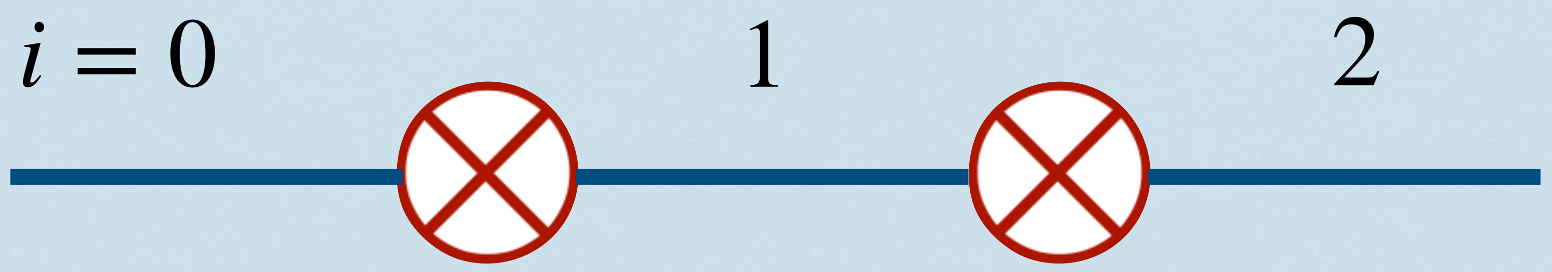
Modular transformation:

$$Z^{T^2\times\mathbb{C}^2/\Gamma}_{\{w^{(a)}\},\{\omega^{(a)}\}}(-1/\tau,\epsilon_+/\tau,\vec{\xi}/\tau) = \prod_{a=1}^{r}\sum_{\tilde{\omega}^{(a)}\in rep(\hat{\mathfrak{g}}_1)} \mathcal{S}_{\omega^{(a)}\tilde{\omega}^{(a)}}\, Z^{T^2\times\mathbb{C}^2/\Gamma}_{\{w^{(a)}\},\{\tilde{\omega}^{(a)}\}}(\tau,\epsilon_+,\vec{\xi})$$

$\hat{\mathfrak{g}}_1$ modular S-matrix

A check using 6d/5d duality

6d:

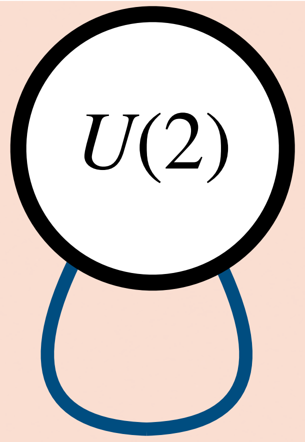


'M-strings' SCFT

$$Z^{T^2\times\mathbb{C}^2/\Gamma}_{\{w^{(a)}\},\{\omega^{(a)}\}} = Z^{pert}_{\{w^{(a)}\},\{\omega^{(a)}\}} \times \sum_{\{\vec{v}^{(a)}\}} e^{-n^{(a)}t^{(a)}}\,\mathbb{E}\left[\begin{matrix}\{\vec{v}^{(a)}\}\\ \{w^{(a)}\},\{\omega^{(a)}\}\end{matrix}\right]$$

$$\Longleftrightarrow$$

5d:



$\mathcal{N} = 1^*$, $G = U(2)$ SYM

$$Z^{5d,S^1\times\mathbb{C}^2/\mathbb{Z}_2}_{\omega^{(1)},\omega^{(2)}} = Z^{5d,pert}_{\omega^{(1)},\omega^{(2)}} \times \sum_{n\geq 0} q^n\, Z^{5d,\,n\,inst}_{\omega^{(1)},\omega^{(2)}}$$

Can compute Z_{5d} on $\mathbb{C}^2/\mathbb{Z}_2$ by uplifting results of [Bruzzo-Pedrini-Sala-Szabo '13] in 4d.

(see also e.g. [Fucito-Morales-Poghossian '04, Bonelli-Maruyoshi-Tanzini-Yagi '12, Ito-Maruyoshi-Okuda '13])

$Z^{5d,\,n\,inst} \sim$ integrals over moduli stacks of framed sheaves over a (stacky) compactification $\mathscr{X}_2$ of $\mathbb{C}^2/\mathbb{Z}_2$

The result:

- Perfect match for the choice $\omega^{(1)} = \omega^{(2)}$!
- For $\omega^{(1)} \neq \omega^{(2)}$, perfect match after making a specific change to the 5d setup. Suggests an extension of the mathematical framework is required.

Summary

- The ALE_Γ partition functions for an infinite family of 6d SCFTs can be computed in terms of 2d KN quivers.

- Solitonic strings are described by relative 2d theories due to interaction with current algebras.

Open questions

- Extension to other 6d SCFTs?
  - Fairly clear how to proceed for the E-string SCFT
  - Far from obvious in most other cases

- What about partition functions on compact $\mathscr{M}_4$? E.g. it would be very interesting to study equivariant partition functions on compact toric spaces *à la* [Bonelli,Fucito,Morales,Ronzani,Sysoeva '20]

Thank you!