G_2 mirror symmetry

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based on [1602.03521],[1701.05202], [1712.06571],[1912.06072],[2312.12104]

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classic story: type II strings on Calabi-Yau threefolds

- Worldsheet theory of type II strings is a non-linear sigma model with $(\mathbf{1},\mathbf{1})$ supersymmetry.
- Covariantly constant p-forms on target space give conserved currents of conformal weight $p/2$ [Howe,Papadopoulos '92].
- The Kähler form J on the Calabi-Yau X gives rise to two conserved currents in the SCFT which enhance this to $(\mathbf{2},\mathbf{2})$.

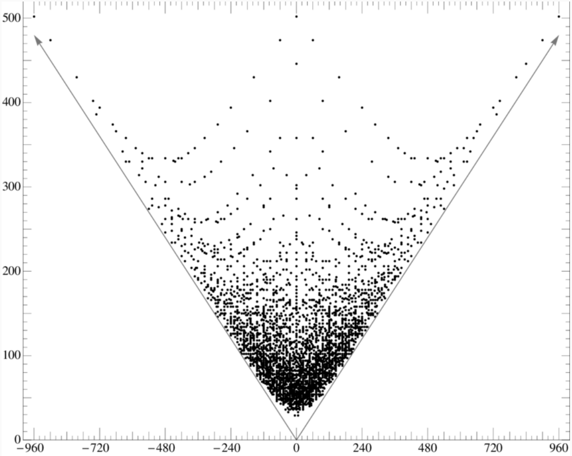
$(2,2)$ susy is very powerful: can reconstruct cohomology of X from 'chiral rings' of SCFT [Lerche, Vafa, Warner '89] up to

$$H^{m,n}(X) \leftrightarrow H^{3-m,n}(X)$$

$$\updownarrow$$

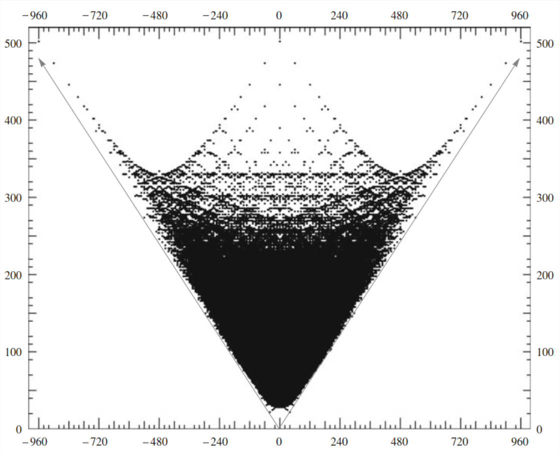
$(\mathbf{2},\mathbf{2})$ superconformal algebra has automorphism $(q_L, q_R) \leftrightarrow (q_L, -q_R)$.

experimental evidence 1: CY hypersurfaces in weighted $\mathbb{P}^4$



[Candelas, Lynker '90][Kreuzer,Skarke '92][Klemm, Schimmrigk '92]

experimental evidence 2: CY hypersurfaces in toric varieties



[Kreuzer,Skarke '00]

Reflexive Polytopes

Let Δ, Δ° be lattice polytopes in $\mathbb{Z}^4 \otimes \mathbb{R}$ such that

$$\langle \Delta, \Delta^\circ \rangle \geq -1$$

For Σ a refinement of the face fan of Δ° by rays through lattice points on Δ°, Δ is the Newton polytope of a family of smooth Calabi-Yau hypersurfaces X_{Δ,Δ° in $\mathbb{P}_\Sigma$.

$$h^{m,n}(X_{\Delta,\Delta^\circ}) = h^{3-m,n}(X_{\Delta^\circ,\Delta})$$

[Batyrev '94]

GLSM perspective

Can realize worldsheet SCFT on X_{Δ,Δ° as IR fixed point of **abelian GLSM**:

- abelian gauge theory in $1+1$ dimensions with $(\mathbf{2},\mathbf{2})$ susy
- charged chiral fields $\phi_i \sim$ homogeneous coordinates of $\mathbb{P}_\Sigma$.
- $U(1)$ charges $Q_a^i \sim$ weights of $\mathbb{C}^*$ action
- vector multiplets V^a
- This works like symplectic reduction with moment maps enforced by D-terms, FI parameters t_a are Kähler parameters.

Can interpolate between geometric large volume and Landau-Ginzburg regime. [Witten '93]

GLSM and mirror symmetry

Mirror symmetry à la Batyrev is realized in GLSM by dualizing fields $\phi_i \to \phi_i^\vee$ [Hori,Vafa '00].

$$Y_i + \bar{Y}_i = \bar{\Phi}_i e^{2Q_a^i V^a} \Phi_i$$
$$\frac{1}{2} d\left(y_i - \bar{y}_i\right) = * d\psi_i$$

$$e^{-Y_i} = e^{-t_i} \prod_\ell \left(\Phi_\ell^\vee\right)^{\langle m_\ell, \nu_i \rangle + 1}$$

Where $\phi_i = \rho_i e^{i\psi_i}$ and $t_a = \sum Q_a^i t_i$. .
Here ν_i are lattice points on Δ° corresponding to homogeneous coordinates ϕ_i and m_ℓ are lattice points on Δ corresponding to homogeneous coordinates $\phi_\ell^\vee$.

G_2 manifolds

A G_2 manifold is a real 7-dimensional Riemannian manifolds with holonomy group G_2.

- Ricci-flat metric and covariantly constant spinor, good targets for strings/M-Theory
- Closed and co-closed threeform $\Phi \leftrightarrow \Phi$ is covariantly constant
- Φ and $*\Phi$ calibrate associative and co-associative submanifolds
- b_3 deformations keeping metric Ricci-flat

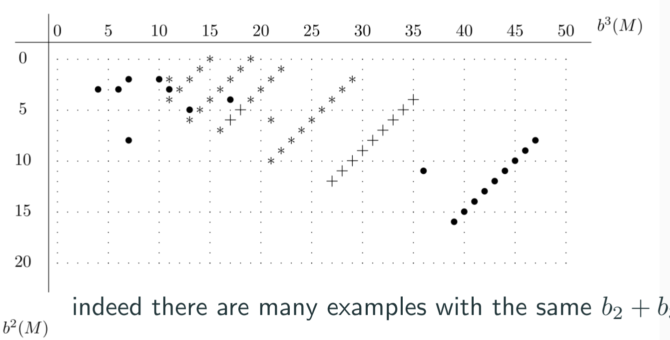
G_2 manifolds give 4D $\mathcal{N} = 1$ theories starting from M-Theory.

type II strings on G_2 manifolds

This is a $(1,1)$ SCFT with extra currents from Φ and $*\Phi$, but it can only detect $b_2 + b_3$. These correspond to exactly marginal deformations.

[Shatashvili,Vafa '94]

Around the same time [Joyce '96] constructed the first compact examples as smoothings of T^7/Γ for $\Gamma \subset G_2$:



indeed there are many examples with the same $b_2 + b_3$

G_2 Mirror Symmetry for orbifolds

For example take $\Gamma = \langle \alpha, \beta, \gamma \rangle$ acting on T^7 as

$$\begin{aligned}
\alpha : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) &\mapsto \quad (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7) \\
\beta : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) &\mapsto \quad (-x_1, \tfrac{1}{2} - x_2, x_3, x_4, -x_5, -x_6, x_7) \\
\gamma : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) &\mapsto \quad (\tfrac{1}{2} - x_1, x_2, -x_3, x_4, -x_5, x_6, -x_7)
\end{aligned}$$

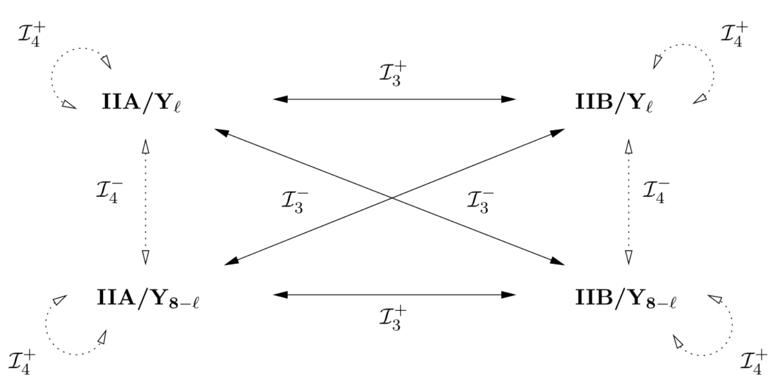
has 9 inequivalent smoothings Y_l with

$$b_2(Y_\ell) = 8 + \ell \qquad b_3(Y_\ell) = 47 - \ell \qquad \ell \in [0, \cdots, 8]$$

Can construct the worldsheet SCFT via free field realization [Acharya '97][Gaberdiel, Kaste '04]

- non-uniqueness of resolution due to discrete torsion phase assignments
- mirrors can be constructed by performing T-dualities along collections of circles

G_2 Mirror Symmetry for orbifolds



G_2 from Calabi-Yau

Here is another construction: take a Calabi-Yau threefold X which admits an anti-holomorphic involution σ and consider [Joyce '96]

$$M = \left(X \times S^1\right)/(\sigma, -)\,.$$

This is very natural for physicists: M is the M-theory lift of a IIA orientifold on X [Kachru,McGreevy '01].

G_2 from Calabi-Yau

Here is another construction: take a Calabi-Yau threefold X which admits an anti-holomorphic involution σ and consider

$$M = \left(X \times S^1\right)/(\sigma, -)\,.$$

Easy version: σ acts freely.

[Harvey,Moore '99][Partouche,Pioline '01]

Find a 'barely G_2' manifold ($\mathrm{hol}(M) = SU(3) \ltimes \mathbb{Z}_2$) with

$$\begin{aligned} b_2(M) &= h^{1,1}_{+}(X) \\ b_3(M) &= 1 + h^{1,1}_{-}(X) + h^{2,1}(X) \end{aligned}$$

so

$$b_2(M) + b_3(M) = 1 + h^{1,1}(X) + h^{2,1}(X)$$

That's very suggestive: swapping $h^{1,1}$ and $h^{2,1}$ leaves $b_2 + b_3$ the same

G_2 from Calabi-Yau

Here is another construction: take a Calabi-Yau threefold X which admits an anti-holomorphic involution σ and consider

$$M = \left(X \times S^1\right)/(\sigma, -)\,.$$

Harder version: σ does not act freely. Fixed locus is a sLag $\mathcal{L}$ and M carries two copies of $\mathbb{R}^4/\mathbb{Z}_2$ singularities over $\mathcal{L}$.

If there is a no-where vanishing harmonic one-form ω taking values in a $\mathbb{Z}_2$ principal bundle L, can glue in a of family Eguchi-Hanson spaces and find G_2 manifold $\tilde{M}$ with

$$\begin{aligned} b_2(\tilde{M}) &= h^{1,1}_{+}(X) + b^0(\mathcal{L}, L) \\ b_3(\tilde{M}) &= 1 + h^{1,1}_{-}(X) + h^{2,1}(X) + b^1(\mathcal{L}, L) \end{aligned}$$

[Joyce, Karigiannis '17]

G_2 SCFTs from Calabi-Yau SCFTs

Using Gepner/Landau-Ginzburg model of X, can determine SCFT spectrum for type II strings on $M = \left(X \times S^1\right)/(\sigma, -)$.

[Blumenhagen, Braun '01] [Eguchi,Sugawara '01][Roiban,Walcher '01]
[Roiban, Römelsberger, Walcher '02]

B-field background (discrete torsion) gives rise to line bundle L, can match LG spectrum to large volume Betti numbers of resolution for some examples.

G_2 mirrors from Calabi-Yau mirrors

Let's consider the behaviour of the 'vanilla' anti-holomorphic involution $\sigma : \phi \to \bar{\phi}$ under the dualization

$$\begin{aligned} Y_i + \bar{Y}_i &= \bar{\Phi}_i e^{2Q^i_a V^a} \Phi_i \\ \tfrac{1}{2} d\left(y_i - \bar{y}_i\right) &= * d\psi_i \qquad\qquad\qquad (\phi_i = \rho_i e^{i\psi_i}) \\ e^{-Y_i} &= e^{-t_i} \prod_\ell \left(\Phi_\ell^\vee\right)^{\langle m_\ell, \nu_i \rangle + 1} \end{aligned}$$

In terms of the dual fields, we also have $\sigma^\vee : \phi_\ell \to \bar{\phi}_\ell$. Hence using mirror symmetry/T-duality we have equivalences between

$$\begin{array}{cc} \left(X \times S^1, \sigma\right) & \left(X \times (S^1)^\vee, \sigma\right) \\ \left(X^\vee \times S^1, \sigma^\vee\right) & \left(X^\vee \times (S^1)^\vee, \sigma^\vee\right) \end{array}$$

[AB,Dadhley'24]

G_2 mirrors from Calabi-Yau mirrors

Quotients of equivalent models by equivalent involutions must give the same thing.

Need to be a little bit more careful however:

- discrete torsion/B-field data needed as well ... but this freedom must show up as freedom to add different sets of twisted sectors
- Need defining equation/superpotential to be invariant, which disconnects moduli space/leads to different fixed loci $\mathcal{L}$:

$$\pm x_1^8 + x_2^8 + x_3^8 + x_4^8 + x_5^2 = 0 \subset \mathbb{P}^4_{1114}$$

Depending on the sign, $\mathcal{L} = \emptyset$ or $\mathcal{L} = S^3$. Can't deform between the two models without going through a singularity.

Hence there must be mirror maps between elements of the four sets

$$\left\{ \left(X \times S^1\right)/(\sigma, -) \right\} \qquad \left\{ \left(X \times (S^1)^\vee\right)/(\sigma, -) \right\}$$
$$\left\{ \left(X^\vee \times S^1\right)/(\sigma^\vee, -) \right\} \qquad \left\{ \left(X^\vee \times (S^1)^\vee\right)/(\sigma^\vee, -) \right\}$$

How can we get a handle on elements of these sets?

K3 fibred Calabi-Yau threefolds and tops

Lets assume X_{Δ,Δ° is such that

$$\Delta^\circ_F := \Delta^\circ \cap F$$

is again reflexive for F a 3dim hyperplane. This implies (with some mild assumptions) that X_{Δ,Δ° carries a fibration by K3 surfaces from the algebraic family $S_{\Delta_F,\Delta^\circ_F}$ over $\mathbb{P}^1$.

[Candelas,Font '96][Klemm,Lerche,Mayr '95][Hosono,Lian,Yau '96]
[Avram,Kreuzer,Mandelberg,Skarke '96]

F cuts Δ° into two halves, $\Diamond^\circ_a$ and $\Diamond^\circ_b$: 'tops'.

Letting $\partial\Diamond^\circ$ be the face that contains the origin we can write $\partial\Diamond^\circ_a = \partial\Diamond^\circ_b = \Delta^\circ_F$.

Conversely, whenever $\partial\Diamond^\circ_a = \partial\Diamond^\circ_b$ for two tops we can form a reflexive polytope if these tops are 'projecting' (the projection of $\Diamond^\circ$ to F is contained in $\partial\Diamond^\circ$). [Candelas,Constantin,Skarke '12]

K3 fibred Calabi-Yau threefolds and tops

Geometric picture: can define dual top $\Diamond$ and construct compact threefold $Z_{\Diamond,\Diamond^\circ}$ in analogy to Batyrev. [AB '16]

$X_{\Diamond,\Diamond^\circ} := Z_{\Diamond,\Diamond^\circ} \setminus S^0$ is an asymptotically cylindrical Calabi-Yau threefold, i.e. it asymptotes to a cylinder times a K3 surface S_0, and we can glue

$$X_{\Delta,\Delta^\circ} = X_{\Diamond_a,\Diamond_a^\circ} \sharp X_{\Diamond_b,\Diamond_b^\circ}$$

if $S_0^a = S_0^b$.

There is a also degeneration limit

$$X_{\Delta,\Delta^\circ} \to Z_{\Diamond_a,\Diamond_a^\circ} \cup Z_{\Diamond_b,\Diamond_b^\circ}$$

which metrically means stretching the $\mathbb{P}^1$ base separating singular K3 fibres at the ends. Furthermore for the mirror we have

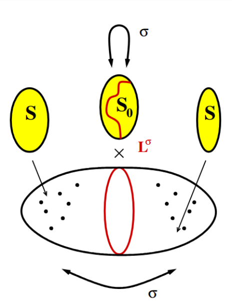
$$X_{\Delta^\circ,\Delta} \to Z_{\Diamond_a^\circ,\Diamond_a} \cup Z_{\Diamond_b^\circ,\Diamond_b}$$

Nice Calabi-Yau manifolds with involutions

Let's assume $\Diamond_a^\circ = \Diamond_b^\circ = \Diamond$ (after reflecting on F) and form a reflexive polytope Δ.

X_{Δ,Δ° has an anti-holomorphic involution σ swapping the two copies of $X_{\Diamond,\Diamond^\circ}$ it is glued from.

The fixed locus of σ is a circle in the $\mathbb{P}^1$ base times S_0/σ_F.



G_2s from nice Calabi-Yau manifolds with involutions

M becomes a twisted connected sum G_2 manifold à la [Kovalev '03][Corti, Haskins, Nordström, Pacini '13] glued as

$$M = X_{\diamondsuit,\diamondsuit^\circ} \times S^1 \sharp X_{r,a,\delta} \times S^1$$

with 'Voisin-Borcea' building block $X_{r,a,\delta} = \left(\mathbb{P}^1 \times S_0\right) / \sigma_{r,a,\delta} \setminus S_0$.

Thinking of M as M-theory lift of IIA orientifold on X, the two glued pieces encode closed/open string sectors, smoothing $\tilde{X}_{r,a,\delta} \to X_{r,a,\delta}$ means moving D6-branes off O6-planes etc..

[AB '19].

Analogue of weak coupling degeneration limit of elliptic CY fourfolds [Sen '96][Clingher,Donagi,Wijnholt '12].

G_2 mirrors again

We can access different twisted sectors corresponding to smooth geometries by constructing different smoothings $\tilde{X}_{r,a,\delta} \to X_{r,a,\delta}$.

For $M = \left(X \times S^1\right)/(\sigma, -)$ we argued that trading $X \to X^\vee$ gives a G_2 mirror. So for

$$M = X_{\Diamond,\Diamond^\circ} \times S^1 \sharp \tilde{X}_{r,a,\delta} \times S^1$$

let

$$M^\vee = X^\vee_{\Diamond,\Diamond^\circ} \times S^1 \sharp \tilde{X}_{r,a,\delta} \times S^1$$

with

$$X^\vee_{\Diamond,\Diamond^\circ} = X_{\Diamond^\circ,\Diamond} \,.$$

which is what we considered in [AB,del Zotto '17] with the result that

$$H^\bullet(M) = H^\bullet(M^\vee)$$

implying $b_2(M) + b_3(M) = b_2(M^\vee) + b_3(M^\vee)$.

Thank you!

Questions:

- multiple mirrors?
- Mirror map and moduli spaces?
- Can we come up with enumerative predictions?
- Spin(7) version?