NUCLEAR INSTRUMENTS AND METHODS 95 (1971) 5–11; © NORTH-HOLLAND PUBLISHING CO.

GEOMETRICAL FACTOR AND DIRECTIONAL RESPONSE OF SINGLE AND MULTI-ELEMENT PARTICLE TELESCOPES*

J. D. SULLIVAN†

Enrico Fermi Institute and Dept. of Physics, The University of Chicago, Chicago, Illinois 60637, U.S.A.

Received 26 February 1971

After a general treatment of the gathering power of particle telescopes, exact formulae are presented for the geometrical factor and directional response of multi-element cylindrically symmetric telescopes with circular or rectangular cross sections. Some useful approximations to these formulae are given. For the gathering power in arbitrary geometries, there is a discussion of applicable digital computer techniques focusing particularly on a Monte-Carlo method.

1. Introduction

The coincidence counting rate of any particle telescope depends upon the effective dimensions and relative positions, i.e. the geometry, of the telescope sensors as well as the intensity of radiation in the surrounding space and the sensor efficiencies. The experimentalist's task is to compute the intensity of radiation given the coincidence counting rate and the parameters (e.g. sensor dimensions) of his telescope. This is the task not only of the space scientist with instruments in an unknown radiation environment but also of the nuclear physicist with his collimated beams.

For an ideal telescope – whose efficiency for detecting particles of a given type is one in a given energy interval and zero otherwise and whose sensors are mathematical surfaces with no thickness – the factor of proportionality relating the counting rate C to the intensity I is defined as the *gathering power* Γ of the telescope. When the intensity is isotropic, i.e., $I = I_0$, the factor of proportionality is called the *geometrical factor* G. That is

$$C = GI_0 .$$

The determination of the telescope gathering power has usually been handled by approximation[1-5]; however, a few explicit formulae for the geometrical factor are known[5-8]. After a general formulation of the problem, exact formulae are obtained below for the geometrical factor and directional response of cylindrically symmetric telescopes with circular or rectangular cross sections. For the gathering power in arbitrary geometries, the discussion centers on an applicable digital computer approximation utilizing a Monte-Carlo method.

2. General formulation

The coincidence counting rate of a particle telescope can be expressed as:

$$C(\boldsymbol{x}, t_0) = (1/T)\int_{t_0}^{t_0+T} \mathrm{d}t \int_S \mathrm{d}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}} \int_\Omega \mathrm{d}\omega \int_0^\infty \mathrm{d}E \times \times \sum_\alpha \varepsilon_\alpha(E,\boldsymbol{\sigma},\omega,t)\, J_\alpha(E,\omega,\boldsymbol{x},t), \quad (1)$$

where

C = coincidence counting rate ($\sec^{-1}$),
α = label for kind of particle,
J_α = spectral intensity of the αth kind of particle ($\sec^{-1}\,\mathrm{cm}^{-2}\,\mathrm{sr}^{-1}\,E^{-1}$),
ε_α = detection efficiency for the αth kind of particle,
t = time,
t_0 = time at start of observation,
T = total observation time,
$\mathrm{d}\boldsymbol{\sigma}$ = element of surface area of the last telescope sensor to be penetrated,
S = total area of the last telescope sensor,
$\mathrm{d}\omega = \mathrm{d}\phi\,\mathrm{d}\cos\theta$ = element of solid angle (θ polar angle, ϕ azimuth),
Ω = domain of ω, this is limited by the other telescope sensors,
$\boldsymbol{x}$ = spatial coordinate of the telescope,
r = unit vector in direction ω, and
$\hat{\boldsymbol{r}}\ \mathrm{d}\boldsymbol{\sigma}$ = effective element of area looking into ω.

This equation just expresses the requirements for the detection of a particle. Although eq. (1) is quite general, still several assumptions are implicit in writing it down. These are:

* This work was supported in part by the National Aeronautics and Space Administration under Grant NGL-14-001-006, NAS-5-9090 and NSF Grants GA-1440 and GA-20565.

† Present address: Department of Physics, University of California, Berkeley, California 94720, U.S.A.

1. that $\mathrm{d}\boldsymbol{\sigma}$, ω, and $\boldsymbol{x}$ are time independent, which would not be the case for a spinning telescope;
2. that no transformation of particle type occurs, other than that included in ε_α;
3. that the particle trajectory is a straight line; and
4. that J_α is independent of $\boldsymbol{\sigma}$ and ε_α of $\boldsymbol{x}$.

The dropping of these assumptions only complicates eq. (1) and renders an analytic solution difficult.

To simplify the problem further, we consider only ideal telescopes where the efficiency is independent of ω, $\boldsymbol{\sigma}$ and t and is given by:

$$\begin{aligned} \varepsilon_\alpha &= 0, \quad \alpha \neq 1, \\ \varepsilon_1 &= 1, \quad E_1 \leqslant E \leqslant E_\mathrm{u}, \\ &= 0, \quad E < E_\mathrm{u}, \ E > E_1. \end{aligned}$$

Henceforth, we will drop the subscript denoting particle type. If we now assume J is independent of $\boldsymbol{x}$ and t and separates into

$$J(E, \omega) = J_0(E) F(\omega),$$

then eq. (1) becomes

$$C = \left[\int_\Omega \mathrm{d}\omega \int_S \mathrm{d}\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}} F(\omega)\right] I, \tag{2}$$

where

$$I = \int_{E_1}^{E_\mathrm{u}} \mathrm{d}E\, J_0(E).$$

We note that eq. (2) would also result for those non ideal telescopes where the efficiency depends only on the energy; in this case, I would be given by

$$I = \int_0^\infty \mathrm{d}E\, J_0(E)\, \varepsilon(E).$$

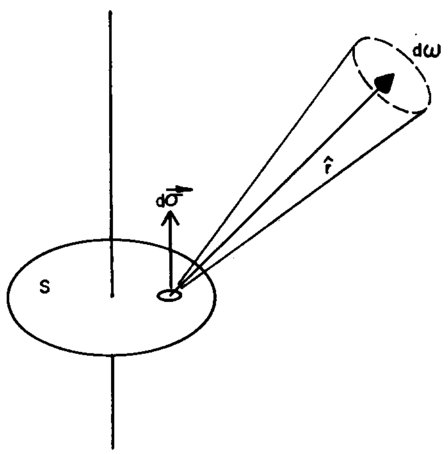

Fig. 1. A telescope with a single plane detector viewing one hemisphere.

The expression in square brackets in eq. (2) is the gathering power of the telescope when the intensity has an angular dependence given by $F(\omega)$. That is

$$\Gamma_F = \int_\Omega \mathrm{d}\omega \int_S \mathrm{d}\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}} F(\omega) = \int_\Omega \mathrm{d}\omega\, F(\omega) \int_S \mathrm{d}\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}}. \tag{3}$$

We can define the directional response function of a telescope, $A(\omega)$, as

$$A(\omega) = \int_S \mathrm{d}\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}}. \tag{4}$$

Thus, eq. (3) may be rewritten as $\Gamma_F = \int_\Omega \mathrm{d}\omega\, F(\omega) A(\omega)$ and the directional response function can be used to facilitate the computation of the gathering power. This is especially useful for numerical calculations.

Considering eq. (3) again we see that if the intensity is isotropic then $F(\omega)$ is unity and the geometrical factor (the gathering power for isotropic flux) depends only on the geometry of the telescope. In other words:

$$G = \int_\Omega \mathrm{d}\omega \int_S \mathrm{d}\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}} = \int_\Omega \mathrm{d}\omega\, A(\omega). \tag{5}$$

3. Explicit formulae

3.1. Single element telescope

For an ideal telescope consisting of a single planar detector (see fig. 1), the geometrical factor is easily evaluated from eq. (5) as

$$\begin{aligned} G = \int_\Omega \mathrm{d}\omega \int_S \mathrm{d}\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}} &= \int_\Omega \int_S \cos\theta\, \mathrm{d}\sigma\, \mathrm{d}\omega \\ &= 2\pi A \int_0^1 \cos\theta \mathrm{d} \cos\theta = \pi A, \end{aligned}$$

where Ω, the domain of ω, is a full hemisphere (particles incident from one side of the detector) and $A = \int_S \mathrm{d}\sigma$ is the surface area of the detector. Thus, the geometrical factor of a single planar detector of area A (with particles incident from one side) is given by

$$G = \pi A. \tag{6}$$

It is clear that if particles are incident from both sides then the area of the detector is doubled, top plus bottom. From eq. (6), it follows that the *geometrical factor of any single detector is π times the total area* provided there exists a tangent plane at every point on the detector (except possibly for a set of points of measure zero) and provided the detector lies entirely on one side of each tangent plane. This includes cylinders, spheres, etc. The directional response function and the gathering power are also easily evaluated from eqs. (3) and (4).

3.2. Two-element telescopes

An ideal cylindrically symmetric telescope with two planar detectors is shown in fig. 2. As ever, the geometrical factor is given by eq. (5) with the domain Ω limited by the top detector. That is,

$$G = \int_{\Omega} \int_{S_2} (\mathrm{d}\boldsymbol{\sigma}_2 \cdot \hat{\boldsymbol{r}})\,\mathrm{d}\omega\,.$$

In this case, however, $\mathrm{d}\omega$ may be expressed as

$$\mathrm{d}\omega = \frac{\hat{\boldsymbol{r}} \cdot \mathrm{d}\boldsymbol{\sigma}_1}{r^2}\,,$$

where r is the distance between $\mathrm{d}\sigma_1$, and $\mathrm{d}\sigma_2$. Consequently,

$$G = \int_{S_1} \int_{S_2} \frac{(\hat{\boldsymbol{r}} \cdot \mathrm{d}\boldsymbol{\sigma}_1)(\hat{\boldsymbol{r}} \cdot \mathrm{d}\boldsymbol{\sigma}_2)}{r^2} \leqslant$$

$$\leqslant \int_{S_1} \int_{S_2} \frac{\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2}{l^2} = \frac{A_1 A_2}{l^2}\,,$$

where A_1, A_2 are the areas of the detectors and l is their separation. Thus,

$$\frac{A_1 A_2}{l^2} \geqslant G\,. \tag{7}$$

This inequality is strictly valid for any two element telescope.

1. *Circular symmetry:* For a telescope with two circular detectors of radii R_1 and R_2 respectively

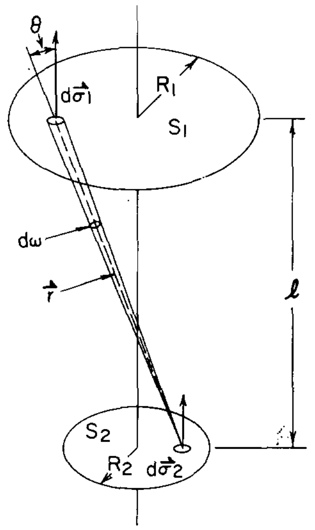

Fig. 2. An ideal cylindrically symmetric telescope with two circular detectors.

(cf. fig. 2), the geometrical factor can be evaluated by direct integration of eq. (5). Whence,

$$G = \tfrac{1}{2}\pi^2 \left[R_1^2+R_2^2+l^2-\{(R_1^2+R_2^2+l^2)^2-4R_1^2R_2^2\}^{\frac{1}{2}}\right]. \tag{8}$$

For quick estimation, this exact result can be expanded yielding to the first order:

$$G \geqslant \frac{A_1 A_2}{R_1^2+R_2^2+l^2}\,. \tag{9}$$

It should be noted that eq. (7) holds for all telescopes, whereas eq. (9) is applicable only to two circular-detector telescopes. Further, we can evaluate the directional response function from eq. (4). Evaluation of this function can be visualized as the overlapped area between the detectors when one is parallel-projected onto the other from direction ω. Thus, it is possible to write $A(\omega)$ by inspection. The result is

$$\begin{aligned} A(\omega) = A(\theta,\phi) &= \pi R_s^2 \cos\theta, \quad \theta_c \geqslant \theta \geqslant 0 \\ &= \cos\theta\left[\tfrac{1}{2}R_1^2(2\Psi_1-\sin 2\Psi_1)+\right. \\ &\quad \left.+\tfrac{1}{2}R_2^2(2\Psi_2-\sin 2\Psi_2)\right], \\ &\qquad\qquad \theta_m \geqslant \theta \geqslant 0_c \\ &= 0, \quad \theta \geqslant \theta_m\,; \end{aligned} \tag{10}$$

where
$$R_s = \text{smaller of } (R_1, R_2),$$
$$\theta_c = \tan^{-1}\frac{|R_1-R_2|}{l},$$
$$\theta_m = \tan^{-1}\frac{R_1+R_2}{l},$$
$$\Psi_1 = \cos^{-1}\left[\frac{R_1^2+l^2\tan^2\theta-R_2^2}{2\,lR_2\tan\theta}\right], \text{ and}$$
$$\Psi_2 = \cos^{-1}\left[\frac{R_2^2+l^2\tan^2\theta-R_1^2}{2\,lR_2\tan\theta}\right].$$

Note $A(\omega)=A(\theta)$ with no ϕ dependence. Whenever the intensity is nonisotropic with an angular dependence given by

$$F(\omega) = F(\theta,\phi) = \cos^n\theta\,,$$

the gathering power can be directly evaluated by integrating* eq. (3) with $A(\omega)$ given by eq. (10). If n is even, the result contains only elementary functions

* The integration proceeds in a straight forward manner by parts.

[as eq. (8) for $n=0$]; whereas, if n is odd, the result contains complete elliptic integrals, which are available in tables.

2. *Rectangular symmetry:* For a telescope with two rectangular detectors with sides (a_1,b_1) and (a_2,b_2) and where a_1 and b_1 are parallel to a_2 and b_2 respectively (cf. fig. 3), the geometrical factor is still obtained by integrating eq. (5). Whence

$$\begin{aligned} G = {} & l^2 \ln \frac{l^2+\alpha^2+\delta^2}{l^2+\alpha^2+\beta^2}\,\frac{l^2+\gamma^2+\beta^2}{l^2+\gamma^2+\delta^2} + \\ & + 2\alpha(l^2+\beta^2)^{\frac{1}{2}} \tan^{-1} \frac{\alpha}{(l^2+\alpha^2)^{\frac{1}{2}}} + \\ & + 2\beta(l^2+\alpha^2)^{\frac{1}{2}} \tan^{-1} \frac{\beta}{(l^2+\beta^2)^{\frac{1}{2}}} - \\ & - 2\alpha(l^2+\delta^2)^{\frac{1}{2}} \tan^{-1} \frac{\alpha}{(l^2+\delta^2)^{\frac{1}{2}}} - \\ & - 2\beta(l^2+\gamma^2)^{\frac{1}{2}} \tan^{-1} \frac{\beta}{(l^2+\gamma^2)^{\frac{1}{2}}} - \\ & - 2\gamma(l^2+\beta^2)^{\frac{1}{2}} \tan^{-1} \frac{\gamma}{(l^2+\beta^2)^{\frac{1}{2}}} - \\ & - 2\delta(l^2+\alpha^2)^{\frac{1}{2}} \tan^{-1} \frac{\delta}{(l^2+\alpha^2)^{\frac{1}{2}}} + \\ & + 2\gamma(l^2+\delta^2)^{\frac{1}{2}} \tan^{-1} \frac{\gamma}{(l^2+\delta^2)^{\frac{1}{2}}} + \\ & + 2\delta(l^2+\gamma^2)^{\frac{1}{2}} \tan^{-1} \frac{\delta}{(l^2+\gamma^2)^{\frac{1}{2}}}, \end{aligned} \tag{11}$$

where

$$\alpha = \tfrac{1}{2}(a_1+a_2),\ \beta = \tfrac{1}{2}(b_1+b_2),$$
$$\gamma = \tfrac{1}{2}(a_1-a_2) \text{ and } \delta = \tfrac{1}{2}(b_1-b_2).$$

This expression simplifies if the detectors are either square or identical and simplifies even more if the detectors are identical squares. One useful algebraic approximation for G valid whenever $l^2 >$ maximum sum of any pair $(\alpha^2,\beta^2,\gamma^2,\delta^2)$ is

$$\frac{A_1 A_2}{l^2} \geqslant G \geqslant \frac{A_1 A_2}{l^2}\left[1-\frac{a_1^2+a_2^2+b_1^2+b_2^2}{6\,l^2}\right]. \tag{12a}$$

Otherwise, for small l,

$$G \approx \frac{4\pi A_1 A_2}{2\,l^2+a_1^2+a_2^2+b_1^2+b_2^2}. \tag{12b}$$

The directional response function is again obtained by visualizing the projected overlapped area but depends in this case on ϕ as well as θ. A functional form useful for computer calculations is[9])

$$A(\theta,\phi) = X\cdot H(X)\cdot Y\cdot H(Y), \tag{13}$$

where

$$\begin{aligned} X &= a_2-(\gamma+\xi)H(\gamma+\xi)-(\gamma-\xi)H(\gamma-\xi),\\ Y &= b_2-(\delta+\eta)H(\delta+\eta)-(\delta-\eta)H(\delta-\eta),\\ \xi &= -\,l\tan\theta\cos\phi,\ \eta=-\,l\tan\theta\sin\phi,\ \text{and}\\ H(Z) &= 1,\ Z>0\\ &= 0,\ Z\leqslant 0. \end{aligned}$$

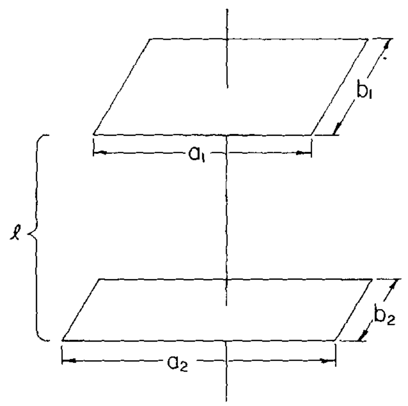

Fig. 3. An ideal cylindrically symmetric telescope with two rectangular detectors.

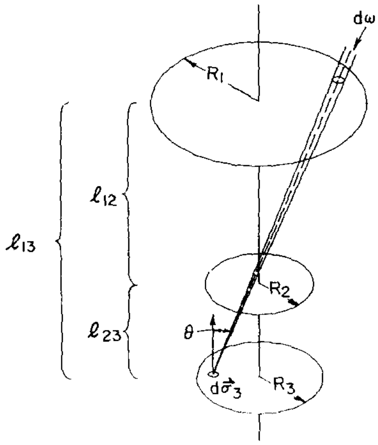

Fig. 4. An ideal cylindrically symmetric telescope with three circular detectors.

3.3. Multi-element telescopes

For a telescope with more than two sensors, determination of the geometrical factor becomes somewhat tedious; eq. (5) is still correct but now the domain Ω is limited in general by each of the medial detectors. We demonstrate the method by computing the geometrical factor for the telescope shown in fig. 4 of three circular detectors of radii R_1, R_2 and R_3 and inter-separations l_{12} and l_{23}. The telescope detectors are labeled such that $R_1 \geq R_3$ with detector 2 in the middle. Again we will integrate eq. (5), but instead of obtaining $A(\omega)$ by integration, we write it down by inspection and then do the ω integration. Define $A_{123}(\omega)$ as the directional response function of the three element telescope. Define $A_{ij}(\omega)$ as the directional response function of the two element telescope (ij) given by eq. (10). Also define

$$\theta_{\mathrm{m}}^{ij} = \tan^{-1}\frac{R_i+R_j}{l_{ij}}, \quad l_{13} = l_{12}+l_{23},$$

$$\theta_{\mathrm{c}}^{ij} = \tan^{-1}\frac{R_i-R_j}{l_{ij}}, \quad \text{and}$$

$$\theta_{\mathrm{a}} = \tan^{-1}\frac{(l_{23}R_1^2+l_{12}R_3^2-l_{13}R_2^2)^{\frac{1}{2}}}{l_{12}l_{23}l_{13}}.$$

The angle θ_{a} corresponds to that parallel-projection angle for which the three circles bounding the detectors intersect in exactly 2 points. Before writing down $A_{123}(\omega)$ it should be noted that all three detectors may not be active in defining the telescope [for instance, if $R_2 > R_1, R_3$, then detector 2 does not limit the response of the two-element telescope (13), in which case the respective two-element telescope geometrical factor is A_{123}]. These cases are

$$\begin{aligned} A_{123} &= A_{13}, \quad \theta_{\mathrm{c}}^{12} \geqslant \theta_{\mathrm{c}}^{13}, \\ &= A_{12}, \quad \theta_{\mathrm{m}}^{13} \geqslant \theta_{\mathrm{m}}^{12}, \\ &= A_{23}, \quad \theta_{\mathrm{c}}^{12} \geqslant \theta_{\mathrm{m}}^{13}. \end{aligned}$$

The directional response function for the distinct three-element telescope is

$$\begin{aligned} A_{123}(\omega) &= 0, \qquad \theta \geqslant \theta_{\mathrm{m}}^{13} \\ &= A_{13}(\omega), \quad \theta_{\mathrm{m}}^{13} \geqslant \theta \geqslant \theta_{\mathrm{a}} \\ &= A_{12}(\omega)+A_{23}(\omega)-\pi R_2^2\cos\theta, \ \theta_{\mathrm{a}} \geqslant \theta \geqslant \theta_{\mathrm{c}}^{12} \\ &= A_{23}(\omega), \quad \theta_{\mathrm{c}}^{12} \geqslant \theta \geqslant 0. \end{aligned} \tag{14}$$

Now the integration of eq. (5) is immediate

$$G_{123} = G_{13} - \pi^2 R_2^2 \sin^2\theta_{\mathrm{a}} + Z_{23}(\theta_{\mathrm{a}}) + Z_{12}(\theta_{\mathrm{a}}) - Z_{13}(\theta_{\mathrm{a}}), \tag{15}$$

where

$$Z_{ij}(\alpha) = \int_0^{\alpha} 2\pi \sin\theta A_{ij}\,\mathrm{d}\theta. \tag{16}$$

Eq. (16) is almost identical to eq. (5) except that the limit of integration is no longer fixed as θ_m^{ij}; thus $Z_{ij}(\alpha)$ can be considered the incomplete geometrical factor function. Explicitly, it is

$$\begin{aligned} Z_{ij}(\alpha) &= \frac{\pi^2 l^2}{2}\left[\frac{a+b}{2} - (ab)^{\frac{1}{2}}\right], \quad \theta_{\mathrm{m}}^{ij} \leqslant \alpha \\ &= \frac{\pi l^2}{2}\Bigg[\frac{(a+b)}{2}\cos^{-1}\frac{a+b-2\kappa}{a-b} - \\ &\quad - (ab)^{\frac{1}{2}}\cos^{-1}\frac{2ab-\kappa(a-b)}{\kappa(a-b)} + \\ &\quad + \frac{[(a-\kappa)(\kappa-b)]^{\frac{1}{2}}}{\kappa} + \\ &\quad + \frac{\sin^2\theta}{2}\Bigg\{(\lambda+\mu)^2\cos^{-1}\frac{\cot\alpha}{\lambda+\mu}(\tan^2\alpha+\lambda\mu) + \\ &\quad + (\lambda-\mu)^2\cos^{-1}\frac{\cot\alpha}{\lambda-\mu}(\tan^2\alpha-\lambda\mu)\Bigg\}\Bigg], \\ &\qquad \theta_{\mathrm{c}}^{ij} \leqslant \alpha \leqslant \theta_{\mathrm{m}}^{ij} \\ &= \pi^2 R_2^2 \sin^2\alpha, \quad 0 \leqslant \alpha \leqslant \theta_{\mathrm{c}}^{ij}, \end{aligned} \tag{17}$$

where

$$a = 1+\lambda^2, \ b = 1+\mu^2, \ \kappa = 1+\tan^2\alpha, \ \lambda = (R_i+R_j)/l \ \text{and} \ \mu = (R_i-R_j)/l.$$

The remaining symbols were defined above.

The gathering power can be computed by integrating eq. (3) but will be expressed for odd powers of $\cos\theta$ in terms of incomplete elliptic integrals. For a circular detector telescope with n sensors, the geometrical factor can be evaluated analogously in terms of the incomplete geometrical factor function.

Thus, given a telescope with sufficient symmetry, the directional response function can be written down by inspection and the geometrical factor and gathering power explicitly integrated.

4. Computer approximations

Analytical computation of the gathering power becomes increasingly difficult for more complex telescope geometries. In these cases, numerical solution (or approximation) on a digital computer is usally easier. The solution again starts with eq. (3) for the gathering power Γ_F:

$$\Gamma_F = \int_\Omega \mathrm{d}\omega F(\omega) \int_S \mathrm{d}\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}} = \int_\Omega \mathrm{d}\omega F(\omega) A(\omega). \qquad (3)$$

One approach is to integrate eq. (3) numerically using any of the standard textbook methods. Another approach, which is quite general and efficient, is to use a Monte-Carlo technique to 'integrate' eq. (3).

The basis of this technique is the definition of the gathering power as a factor of proportionality between the incident flux and the observed coincidence counting rate. The general procedure may be outlined as follows:

1. Choose a random* point on the opening (planar) aperture and a random trajectory through this point in such a way that a large number of such choices would correspond to the intensity incident on the aperture.
2. Follow this trajectory to see if it passes through the detector(s) of interest – this might, for instance, include the passage of the particle through a magnetic field.
3. Tally the results of step 2, and, if applicable, calculate any item of interest, e.g., the angle of intersection of the trajectory with respect to a $\mathrm{d}E/\mathrm{d}x$ detector.
4. Repeat steps 1–3 enough times to let a statistical pattern emerge.

The gathering power for the telescope is then given by

$$\Gamma = \frac{\text{number of trajectories detected}}{\text{total number of trajectories chosen}} \times$$
$$\times \text{ gathering power of the opening aperture.} \qquad (18)$$

The heart of the problem is step 1, simulating the intensity on the opening aperture. This is done as follows:

a. In choosing a random point on the opening aperture, equal areas should have equal weights. In cartesian coordinates $\mathrm{d}\sigma = \mathrm{d}x\mathrm{d}y$. Thus, choose x and y random. This method can be used for any arbitrarily shaped aperture by enclosing it within a rectangle, choosing random points within the rectangle, and then using them if they fall within the arbitrary aperture. For a circular aperture $\mathrm{d}\sigma = \frac{1}{2}\mathrm{d}r^2\mathrm{d}\phi$ thus choose r^2 and ϕ random†.

* The availability of a random number generator in the computer software is assumed.

† Choosing random r^2 is quite different from choosing random r and squaring.

b. To choose trajectories corresponding to the intensity incident on the aperture, consider the element of area $\mathrm{d}\sigma$ centered on the point picked in a. The incident directions will now be weighted not only by $F(\omega)$ but also by a factor $\cos\theta$ from $\mathrm{d}\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}} = \cos\theta\, \mathrm{d}\sigma$. Thus, the weighted solid angle becomes

$$\cos\theta\, F(\omega)\, \mathrm{d}\cos\theta\, \mathrm{d}\phi = \tfrac{1}{2}\, F(\omega)\, \mathrm{d}\cos^2\theta\, \mathrm{d}\phi.$$

The method of choosing trajectories is now apparent: if $F(\omega) = 1$, i.e., an isotropic intensity, choose random $\cos^2\theta$ and random ϕ; if $F(\omega) = \cos^n\theta$, choose random $\cos^{n+2}\theta$ and random ϕ; etc.

In general most of the random trajectories through the opening aperture will miss the other telescope elements. This can make computer programs somewhat inefficient. However, if sufficient symmetry is present, some improvements are possible. For instance, for telescopes with circular cylindrical symmetry it is more efficient to limit the range of θ to between 0 and θ_{m}. The conversion to an absolute scale is provided simply by the ratio of the number of trajectories used to the number that would have been used without the range limitation.

This Monte-Carlo method is easily applied. It is also useful for approximating eq. (1) and treating the problem in general.

5. Conclusion

We have presented an analytic treatment of telescope gathering power and have given explicit formulae for the geometrical factor in several general cases. In applying these formulae it must be recalled that the analysis is for ideal telescopes. For real telescopes, there exist many other factors which will affect the problem (cf. sec. 2). Two of these, not mentioned previously, are finite detector thickness and variable or ill-defined sensitive area. These introduce uncertainties in the telescope geometry. When such other factors are important or when the telescope geometry is complex, the Monte-Carlo approach discussed in section 4 may be used.

I appreciate the support and encouragement of Prof. J. A. Simpson during the performance of this work. I would like to thank Dr. J. J. O'Gallagher for originally introducing me to this problem. Discussions with G. M. Mason have proven helpful. I thank Dr. I. Lerche for many useful discussions of the mathematics involved. I would also like to express my appreciation to B. G. Cartwright, P. J. Schmidt and C. Fulks for their critical readings of the manuscript.

References

1) H. E. Newell, Jr., Rev. Sci. Instr. **19** (1948) 384.
2) H. E. Newell, Jr. and E. C. Pressly, Rev. Sci. Instr. **20** (1949) 568.
3) H. E. Newell, Jr. and E. C. Pressly, Rev. Sci. Instr. **21** (1950) 918.
4) E. R. Witmer and M. A. Pomerantz, J. Franklin Instr. **246** (1948) 293.
5) C. R. Gillespie, Rev. Sci. Instr. **41** (1970) 42.
6) C. L. Critchfield, E. P. Ney and Sophie Oleska, Phys. Rev. **85** (1952) 461.
7) G. M. Mason and J. D. Sullivan, Notes on the geometry of particle telescopes (March 1968) unpublished. [An edited version of this work has been included by G. Gloecker in his lectures in *Introduction to experimental techniques of high-energy astrophysics*, NASA SP-243 (eds. H. Ögelman and J. R. Wayland; 1970).]
8) M. A. Perkins, J. J. Kristoff, G. M. Mason and J. D. Sullivan, Nucl. Instr. and Meth. **68** (1969) 149.
9) P. J. Schmidt, private communications.