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QUANTUM NATURE OF PRIMORDIAL GRAVITATIONAL WAVES

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Introduction

Learning the Deep Mysteries of nature with Cosmology

In my opinion, the most mysterious fact is that
the origin of the large-scale structure of the universe
is quantum fluctuations during inflation.

Remarkably, primordial gravitational waves(PGWs)
are also generated from quantum fluctuations.

Since PGWs are fluctuations of spacetime,
it would give us a hint of quantum theory of gravity.

In this talk, I will show you that

we can probe quantum nature of PGWs
through observations of GWs from binary black holes.

Plan of my talk

1. Inflation
2. Primordial GWs from inflation
3. Quantum description of GWs from binary black holes
4. How to observe quantum nature of PGWs
5. Summary

INFLATION

Inflation = de Sitter spacetime

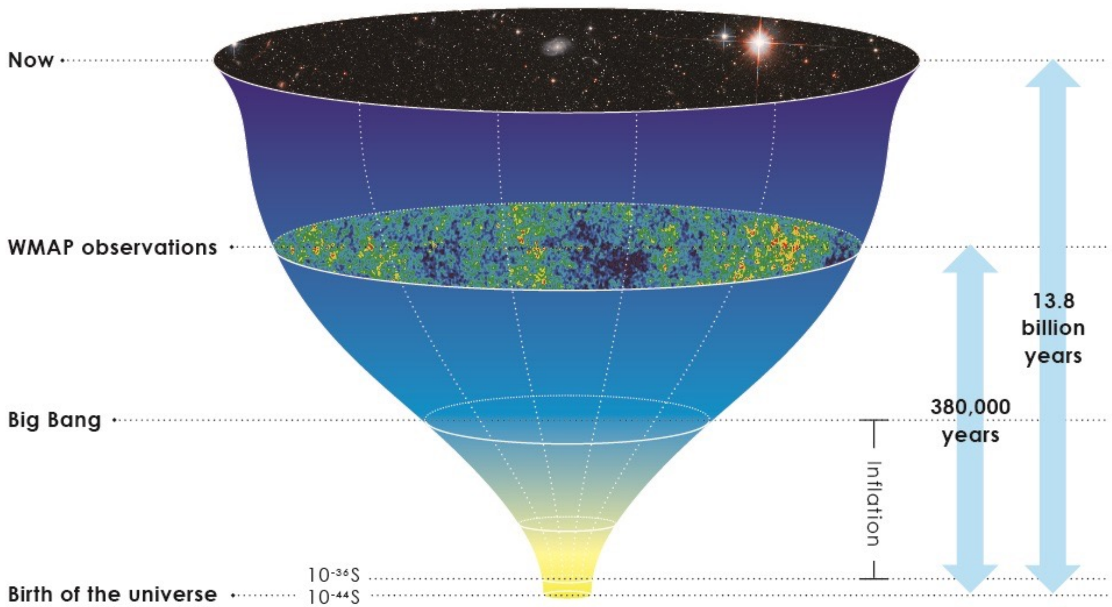
De Sitter expansion solves

$$a(t) = e^{Ht}$$

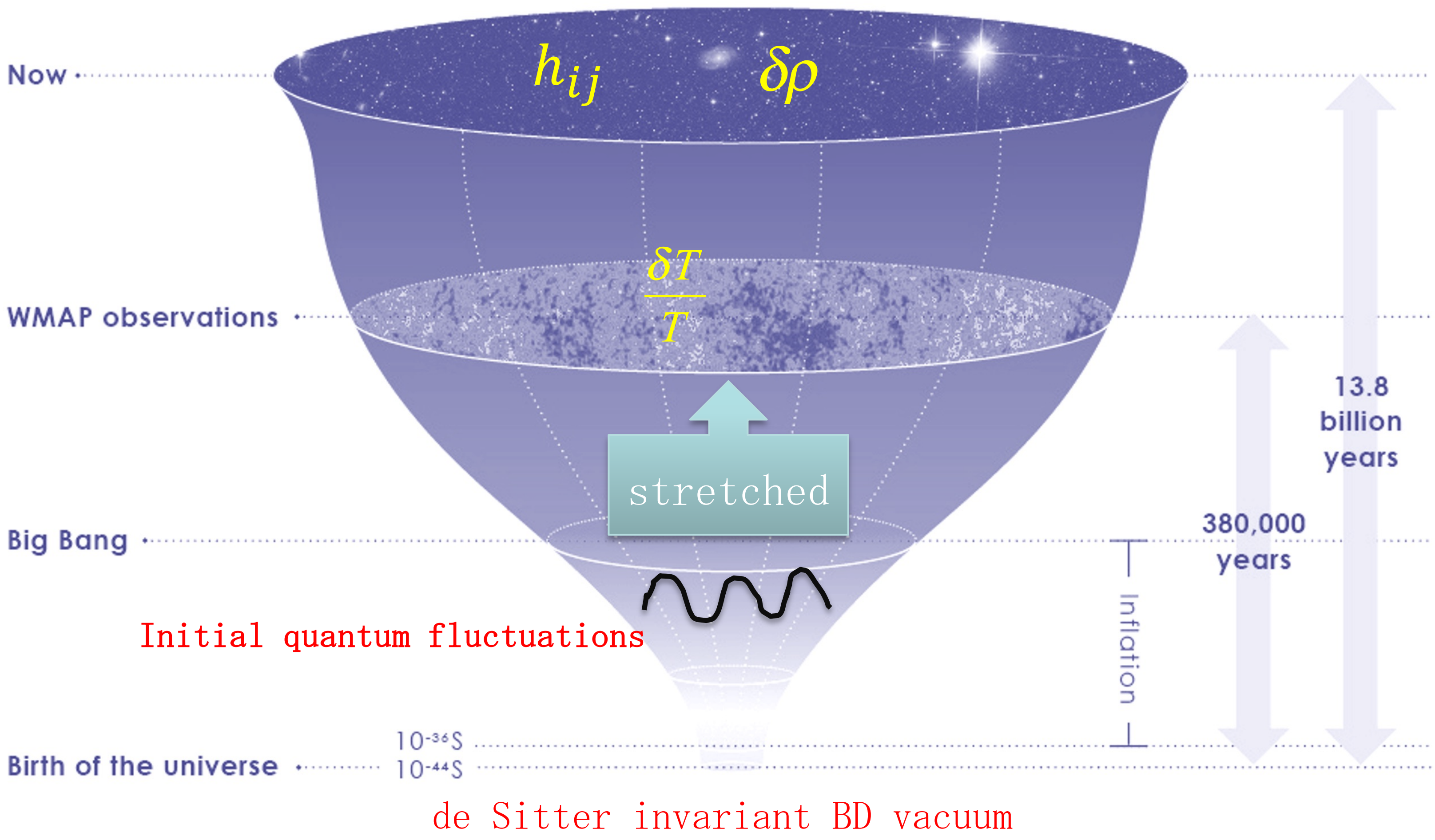
$$ds^2 = -dt^2 + e^{2Ht}\,\delta_{ij}dx^i dx^j$$

- ✓ Flatness problem
- ✓ Horizon problem
- ✓ Monopole problem
- ✓ …..

It is a maximally symmetric spacetime.



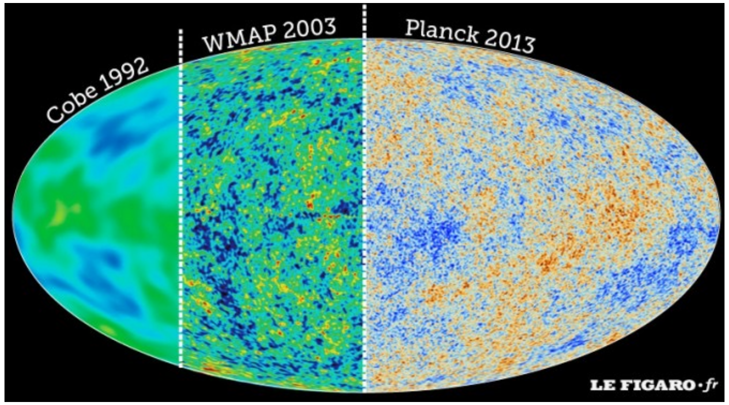
The origin of PGWs is quantum



Classical nature of primordial fluctuations

The de Sitter symmetry completely determines
the classical nature of primordial GWs.

- statistically homogeneous
- statistically isotropic
- Gaussian
- scale invariant

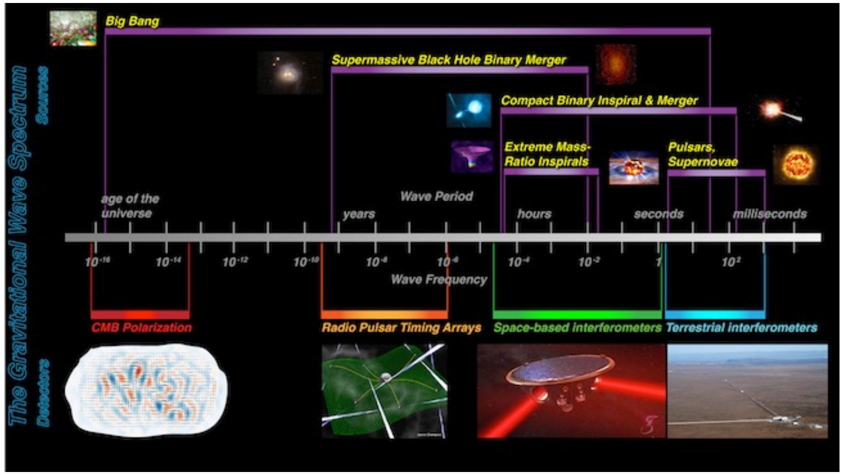


Recent precision observations forced us to look at the fine structures.

- statistically homogeneous
- statistically isotropic. -> statistical anisotropy
- Gaussian -> non-gaussianity
- scale invariant -> tilt of the spectrum

Quantum nature of primordial fluctuations

There are many projects.



We may be able to go further.

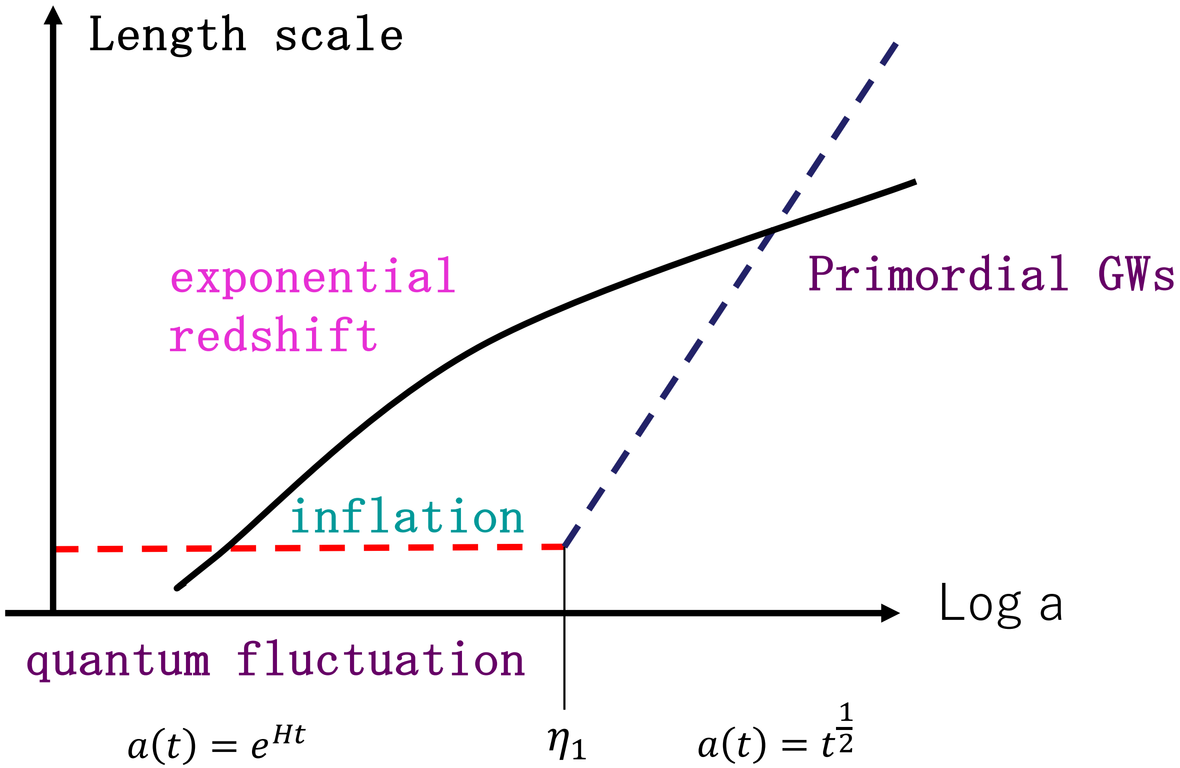
In fact, the de Sitter symmetry also determines the quantum nature of primordial GWs.

- statistically homogeneous
- statistically isotropic. -> statistical anisotropy
- Gaussian -> non-gaussianity
- scale invariant -> tilt of the spectrum
- quantum entangled vacuum -> ???????????????????

The aim of this talk is to propose one way to probe quantum nature of PGWs.

PRIMORDIAL GWS FROM INFLATION

PGWs as Quantum fluctuations of spacetime!



$$ds^2 = a^2(\eta)\left[-d\eta^2 + \delta_{ij}dx^i dx^j\right]$$

$$a(\eta) = \begin{cases} \dfrac{1}{-H(\eta - 2\eta_1)} & \text{inflation} \\ \dfrac{\eta}{H\eta_1^2} & \text{radiation} \end{cases}$$

Canonical quantization of gravitational waves

GWs in cosmological background: $S = \frac{M_p^2}{4}\int d\eta d^3x\, a^2(\eta)\left[\frac{1}{2}h'_{ij}h'_{ij} - \frac{1}{2}h_{ij,k}h_{ij,k}\right]$

canonical variable

$$\psi_{ij} = \frac{M_p}{2} a h_{ij} = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \tilde{\psi}_{ij}(k)\, e^{i\vec{k}\cdot\vec{x}} \qquad S = \int d\eta d^3k \left[\frac{1}{2}\tilde{\psi}'_{ij}\tilde{\psi}'_{ij} - \frac{1}{2}\left(k^2 - \frac{a''}{a}\right)\tilde{\psi}_{ij}\tilde{\psi}_{ij}\right]$$

canonical commutation relations

$$\pi^{ij} = \psi'_{ij} \qquad \left[\psi_{ij}(x), \pi^{kl}(y)\right] = i\left(P_i^k P_j^l + P_i^l P_j^k - P_{ij}P^{kl}\right)\delta(x-y) \qquad P_i^j = \delta_i^j - \frac{\partial_i \partial^j}{\partial^2}$$

Fock space representation

$$h_{ij}(x^i, \eta) = \frac{2}{a(\eta)M_p}\int \frac{d^3k}{(2\pi)^{3/2}} \sum_P \left[e_{ij}^P(k^i)u_k(\eta)a_P(k^i) + e_{ij}^P(-k^i)u_k^*(\eta)a_P^\dagger(-k^i)\right]e^{i\vec{k}\cdot\vec{x}}$$

$$e_{ij}^P(k^i)e_{ij}^Q(k^i) = \delta^{PQ}$$

$$\left[a_P(k^i), a_Q^\dagger(p^i)\right] = \delta_{PQ}\delta(k^i - p^i)$$

$$iu_k^*\partial_\eta u_k - iu_k\, \partial_\eta u_k^* = 1 \qquad \left(\frac{d^2}{d\eta^2} + k^2 - \frac{a''}{a}\right)u_k(\eta) = 0$$

Choice of mode functions

In curved spacetime, the mode function is not uniquely determined.

$$h_k^P(\eta) = U_k(\eta) a_P(k^i) + U_k^*(\eta) a_P^\dagger(-k^i)$$

Positive frequency mode of Minkowski vacuum at $k \to \infty$

$$u_k(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k(\eta - 2\eta_1)}\right) e^{-ik(\eta - 2\eta_1)} \xleftarrow[\eta<\eta_1]{} U_k(\eta) \xrightarrow[\eta>\eta_1]{} ????$$

inflation → radiation

BD vacuum: $a_P(k^i)|BD\rangle = 0$

$$h_k^P(\eta) = V_k(\eta) b_P(k^i) + V_k^*(\eta) b_P^\dagger(-k^i)$$

$$???? \xleftarrow[\eta<\eta_1]{} V_k(\eta) \xrightarrow[\eta>\eta_1]{} v_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta}$$

inflation → radiation

vacuum for an observer in RD: $b_P(k^i)|0_R\rangle = 0$

Bogoliubov transformation

Since both mode functions are linearly independent and span a complete set, we can define

Bogoliubov transformation $$U_k(\eta) = \alpha_k V_k(\eta) + \beta_k V_k^*(\eta)$$

$$u_k(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k(\eta - 2\eta_1)}\right)e^{-ik(\eta - 2\eta_1)} \xleftarrow[\eta<\eta_1]{} U_k(\eta) \xrightarrow[\eta>\eta_1]{} \alpha_k v_k(\eta) + \beta_k v_k^*(\eta)$$

BD vacuum $a_P(k^i)|BD\rangle = 0$

radiation

inflation

$b_P(k^i)|0_R\rangle = 0$

$$\xleftarrow[\eta<\eta_1]{} V_k(\eta) \xrightarrow[\eta>\eta_1]{} v_k(\eta) = \frac{1}{\sqrt{2k}}e^{-ik\eta}$$

At the end of inflation, we can deduce the conditions

$$u_k(\eta_1) = \alpha_k v_k(\eta_1) + \beta_k v_k^*(\eta_1) \qquad u_k'(\eta_1) = \alpha_k v_k'(\eta_1) + \beta_k v_k'^*(\eta_1)$$

$$\Longrightarrow \quad \alpha_k = \left(1 + \frac{i}{k\eta_1} - \frac{1}{2k^2\eta_1^2}\right)e^{2ik\eta_1} \qquad \beta_k = \frac{1}{2k^2\eta_1^2}$$

Spectrum of PGWs

We have two ways to represent the same field

$$h_k^P(\eta) = U_k(\eta)a_P(k^i) + U_k^*(\eta)a_P^\dagger(-k^i) \qquad h_k^P(\eta) = V_k(\eta)b_P(k^i) + V_k^*(\eta)b_P^\dagger(-k^i)$$

Using the Bogoliubov transformation, we have

$$h_k^P(\eta) = (\alpha_k V_k(\eta) + \beta_k V_k^*(\eta))a_P(k^i) + (\alpha_k^* V_k^*(\eta) + \beta_k^* V_k(\eta))a_P^\dagger(-k^i)$$

$$= V_k(\eta)b_P(k^i) + V_k^*(\eta)b_P^\dagger(-k^i)$$

Simple comparison gives rise to the relation

$$b_P(k^i) = \alpha_k a_P(k^i) + \beta_k^* a_P^\dagger(-k^i)$$

Thus, we have the number of gravitons

$$N_k = \langle BD|b_P^\dagger(k^i)b_P(k^i)|BD\rangle = |\beta_k|^2 = \frac{1}{4k^4\eta_1^4} = \frac{a_1^4H^4}{4k^4} = \frac{H_0^2H^2}{4k^4}\Omega_R \qquad a_1 = \frac{1}{H\eta_1}$$

and the energy density

$$\rho_g = 2\int\frac{d^3k}{(2\pi)^3}2\pi f N_k = \int_0^\infty d\log f\, f^4 N_k$$

Hence, the density parameter

$$\Omega_{GW}(f) = \frac{1}{\rho_0}\frac{d\rho_g}{dlogf} = \frac{k^4N_k}{\pi^2\rho_c} = \frac{1}{12\pi^2}\left(\frac{H}{M_p}\right)^2\Omega_R = 10^{-14}\left(\frac{H}{10^{-4}M_p}\right)^2$$

Quantum state of PGWs

Grishchuk 1998

The Bogoliubov transformation can be inverted as

$$b_P(k^i) = \alpha_k a_P(k^i) + \beta_k^* a_P^\dagger(-k^i) \longrightarrow a_P(k^i) = \alpha_k^* b_P(k^i) - \beta_k^* b_P^\dagger(-k^i)$$

The BD vacuum is the solution of the following equation

$$a_P(k^i)|BD\rangle = \left[\alpha_k^* \frac{\partial}{\partial b_P^\dagger(k^i)} - \beta_k^* b_P^\dagger(-k^i)\right]|BD\rangle = 0$$

Thus, we obrain

$$|BD\rangle = \frac{1}{\cosh r_k}\prod_{k,P} e^{\frac{1}{2}\tanh r_k b_P^\dagger(k) b_P^\dagger(-k)}|0_R\rangle \qquad \tanh r_k = \frac{\beta_k^*}{\alpha_k^*}$$

$$\propto |0_{\mathbf{k}}\rangle \otimes |0_{-\mathbf{k}}\rangle + \tanh r_k |1_{\mathbf{k}}\rangle \otimes |1_{-\mathbf{k}}\rangle + \tanh^2 r_k |2_{\mathbf{k}}\rangle \otimes |2_{-\mathbf{k}}\rangle + \cdots$$

This is a highly entangled two-mode squeezed state.

The squeezing parameter r_k can be calculated as

$$k_c = 2\pi f_c = \frac{1}{|\eta_1|}$$

$$\sinh r_k = \frac{1}{2}\left(\frac{f_c}{f}\right)^2 \qquad f_c = 10^9 \sqrt{\frac{H}{10^{-4} M_p}}\ \text{Hz}$$

We assume the present graviton state is kept to be squeezed.

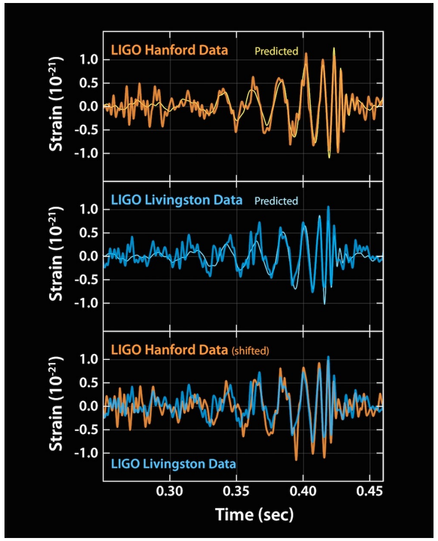
QUANTUM DESCRIPTION OF GWS FROM BINARY BLACK HOLES

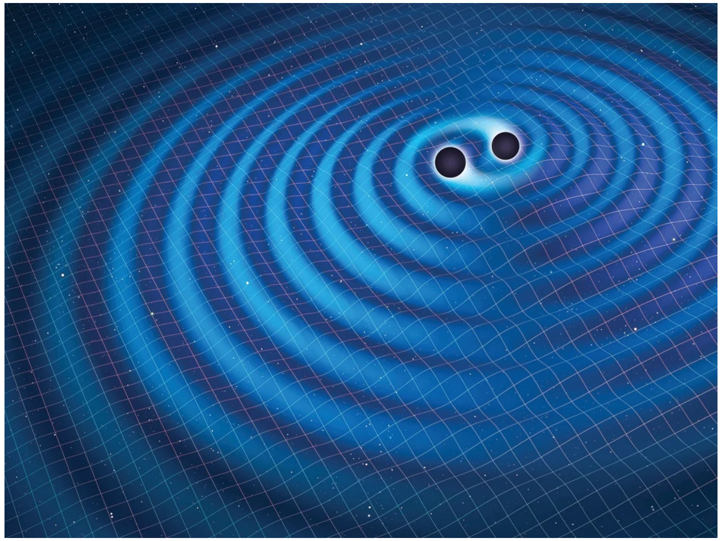
S.Kanno, J.S, A.Taniguchi,
Quantum nature of gravitational waves from binary black holes,
arXiv:2508.17947 [gr-qc].

Gravitational waves have been detected!

In 2015, gravitational waves from binary BHs are discovered.

GW150914





Quantum mechanics is fundamental.

Any macroscopic phenomenon should have a quantum description.

Quantum theory of GWs

How to describe classical radiation in quantum theory?
It is commonly believed,
electromagnetic waves are described by a coherent state of photon
generated by the operator

$$\mathrm{Exp}\left[-i\int_{-\infty}^{t} H_I(t)\,dt\right] = \exp\left[-i\int_{-\infty}^{t}\int d^3x\ \mathbf{j}(x^i,t)\cdot\hat{\boldsymbol{A}}(x^i,t)\right]$$

Here, the current is assumed classical.

How about quantum theory of gravitational waves?
The point is that the gravitational interaction is generically nonlinear

$$\mathrm{Exp}\left[-i\int_{-\infty}^{t} H_I(t)\,dt\right] = \exp\left[-i\int_{-\infty}^{t}\int d^3x\, T_{ij}(x^i,t)\,\hat{h}_{ij}(x^i,t) + \Lambda_{ijkl}\,\hat{h}_{ij}(x^i,t)\hat{h}_{kl}(x^i,t) + \cdots\right]$$

The first term generates a coherent state.
In contrast to photon, the second term generates a squeezed state.

Thus, we can expect the deviation from the coherent state.

Quantum GWs from binary BHs

The binary black holes are described by

$$S = S_{\mathrm{EH}} + S_1 + S_2 = \frac{M_{\mathrm{p}}^2}{2} \int d^4x \sqrt{-g} R - m_1 \int_{\zeta_1} d\tau - m_2 \int_{\zeta_2} d\tau$$

We assume that backreaction is negligible

$$-d\tau^2 = ds^2 = -dt^2 + (\delta_{ij} + h_{ij})\, dx^i dx^j$$

We treat the system perturbatively. The action of the free part is given by

$$S_{\mathrm{EH}} = \frac{M_{\mathrm{p}}^2}{8} \int d^4x \left(\dot{h}_{ij}\dot{h}_{ij} - h_{ij,k}h_{ij,k} \right) \qquad \psi_{ij} = \frac{M_p}{2} a h_{ij}$$

The canonical quatization procedure is parallel to the de Sitter case.

$$\left[\psi_{ij}(t, \boldsymbol{x}), \dot{\psi}^{k\ell}(t, \boldsymbol{x}')\right] = \frac{i}{2} \left(P_i{}^k P_j{}^\ell + P_i{}^\ell P_j{}^k - P_{ij} P^{k\ell} \right) \delta(\boldsymbol{x} - \boldsymbol{x}') \qquad P_{ij} = \delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}$$

$$h_{ij}(t, \boldsymbol{x}) = \frac{2}{M_{\mathrm{p}}} \sum_{P=+,\times} \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} \left[\frac{e^{-i\omega_{\boldsymbol{k}} t}}{\sqrt{2\omega_{\boldsymbol{k}}}} e_{ij}^{(P)}(\boldsymbol{k}) a^{(P)}(\boldsymbol{k}) + \frac{e^{i\omega t}}{\sqrt{2\omega_k}} e_{ij}^{(P)}(-\boldsymbol{k}) a^{(P)\dagger}(-\boldsymbol{k}) \right] e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\left[a^{(P)}(\boldsymbol{k}), a^{(Q)\dagger}(\boldsymbol{k}')\right] = \delta(\boldsymbol{k} - \boldsymbol{k}')\delta^{PQ}$$

Interaction Hamiltonian

The action for the geodesic motion of particles contains the interaction terms

$$S_{\rm int} = -\sum_{N=1,2} m_N \int_{\zeta_N} \sqrt{dt^2 - \delta_{ij}dx^i dx^j - h_{ij}(t,\bar{\boldsymbol{x}}_N(t))dx^i dx^j}$$
$$= -\sum_{N=1,2} m_N \int_{\zeta_N} dt\, \frac{1}{\gamma_N}\sqrt{1-\gamma_N^2 h_{ij}(t,\bar{\boldsymbol{x}}_N(t))v_N^i v_N^j}$$

$$v_N^i = d\bar{x}_N^i/dt$$
$$\gamma_N = 1/\sqrt{1-v_N^2}$$

After performing Legendre transformation,
the interaction Hamiltonian up to the second order takes the form

$$H_{\rm int}(t,\bar{\boldsymbol{x}}) = \sum_{N=1,2}\left[\frac{\gamma_N^3 m_N}{2} h_{ij}(t,\bar{\boldsymbol{x}}_N(t))v_N^i v_N^j + \frac{3}{8}\gamma_N^5 m_N h_{ij}(t,\bar{\boldsymbol{x}}_N(t))\; h_{lm}(t,\bar{\boldsymbol{x}}_N(t))\; v_N^i v_N^j v_N^l v_N^m\right]$$

In the interaction picture, the time evolution of a quantum state is given by operating the unitary operator

$$\hat{U}(t,\bar{\boldsymbol{x}}) = \mathcal{T}\left[\exp\left(-i\int^t dt' \hat{H}_{\rm int}(t')\right)\right]$$

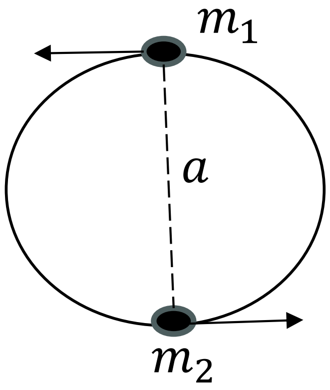
on the initial state.

Coherent state description

For simplicity, we consider circular motion

$$x_1 = \frac{m_2}{M} a\cos(\Omega t)\ , \qquad y_1 = \frac{m_2}{M} a\sin(\Omega t)\ , \qquad z_1 = 0 \qquad M = m_1 + m_2$$

$$x_2 = \frac{m_1}{M} a\cos(\Omega t + \pi)\ , \quad y_2 = \frac{m_1}{M} a\sin(\Omega t + \pi)\ , \quad z_2 = 0$$



At the linear order in h, we have

$$\begin{aligned}\hat{U}(t, \bar{\boldsymbol{x}}_N) &= \mathcal{T}\left[\exp\left(-i\int^t dt' \hat{H}_{\rm int}(t')\right)\right] \\ &= \exp\Bigg[-i\frac{2}{M_{\rm p}}\sum_{N=1,2}\frac{\gamma_N^3 m_N}{2}\sum_{P=+,\times}\int^t dt' \int\frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} \\ &\qquad \times\left(\frac{e^{-i\omega_{\boldsymbol{k}}t'}}{\sqrt{2\omega_{\boldsymbol{k}}}}e_{ij}^{(P)}(\boldsymbol{k})a(\boldsymbol{k}) + \frac{e^{i\omega_{\boldsymbol{k}}t'}}{\sqrt{2\omega_{\boldsymbol{k}}}}e_{ij}^{(P)}(-\boldsymbol{k})a^\dagger(-\boldsymbol{k})\right)e^{i\boldsymbol{k}\cdot\bar{\boldsymbol{x}}_N}v_N^i v_N^j\Bigg]\end{aligned}$$

Comparing the above expression with the displacement operator

$$\hat{D}(\alpha) = \prod_P \exp\left[\int d^3\boldsymbol{k}\left(\alpha^{(P)}(\boldsymbol{k})a^{(P)\dagger}(\boldsymbol{k}) - \alpha^{(P)*}(\boldsymbol{k})a^{(P)}(\boldsymbol{k})\right)\right]$$

We can read off the coherent parameter as

$$\alpha^{(P)}(\boldsymbol{k}) = -\frac{i}{(2\pi)^{3/2}}\sum_{N=1,2}\int^t dt' \frac{\gamma_N^3 m_N}{M_{\rm p}}\frac{e^{i\omega_{\boldsymbol{k}}t'}}{\sqrt{2\omega_{\boldsymbol{k}}}}e_{ij}^{(P)}(\boldsymbol{k})v_N^i v_N^j e^{-i\boldsymbol{k}\cdot\bar{\boldsymbol{x}}_N}$$

Explicit formula

Given the wave vector $\vec{k} = k(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

We can write down the explicit form of polarization tensors as

$$e_{ij}^{(+)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos^2\theta\cos^2\varphi - \sin^2\varphi & (1+\cos^2\theta)\sin\varphi\cos\varphi & -\frac{1}{2}\sin 2\theta\cos\varphi \\ (1+\cos^2\theta)\sin\varphi\cos\varphi & \cos^2\theta\sin^2\varphi - \cos^2\varphi & -\frac{1}{2}\sin 2\theta\sin\varphi \\ -\frac{1}{2}\sin 2\theta\cos\varphi & -\frac{1}{2}\sin 2\theta\sin\varphi & \sin^2\theta \end{pmatrix} \qquad e_{ij}^{(\times)} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\cos\theta\sin 2\varphi & \cos\theta\cos 2\varphi & \sin\theta\sin\varphi \\ \cos\theta\cos 2\varphi & \cos\theta\sin 2\varphi & -\sin\theta\cos\varphi \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & 0 \end{pmatrix}$$

The coherent parameter can be deduced as

$$\alpha^{(+)}(\boldsymbol{k}) = \frac{i}{(2\pi)^{3/2}}\frac{\mu(a\Omega)^2}{\sqrt{2}M_{\rm p}}\int^t dt' \frac{e^{i\omega_{\boldsymbol{k}}t'}}{\sqrt{2\omega_{\boldsymbol{k}}}}\left(\frac{\sin^2\theta}{2} + \frac{1+\cos^2\theta}{2}\cos(2\Omega t' - 2\varphi)\right) \times \left[\gamma_1^3\left(\frac{m_2}{M}\right)e^{-i(k_x x_1 + k_y y_1)} + \gamma_2^3\left(\frac{m_1}{M}\right)e^{-i(k_x x_2 + k_y y_2)}\right]$$

$$\mu = \frac{m_1 m_2}{M}$$

$$\alpha^{(\times)}(\boldsymbol{k}) = \frac{i}{(2\pi)^{3/2}}\frac{\mu(a\Omega)^2}{\sqrt{2}M_{\rm p}}\int^t dt' \frac{e^{i\omega_{\boldsymbol{k}}t'}}{\sqrt{2\omega_{\boldsymbol{k}}}}\cos\theta\sin(2\Omega t' - 2\varphi) \times \left[\gamma_1^3\left(\frac{m_2}{M}\right)e^{-i(k_x x_1 + k_y y_1)} + \gamma_2^3\left(\frac{m_1}{M}\right)e^{-i(k_x x_2 + k_y y_2)}\right]$$

where

$$k_x x_1 + k_y y_1 = \frac{m_2}{M}ak(\sin\theta\cos\varphi\cos(\Omega t') + \sin\theta\sin\varphi\sin(\Omega t'))$$
$$k_x x_2 + k_y y_2 = -\frac{m_1}{M}ak(\sin\theta\cos\varphi\cos(\Omega t') + \sin\theta\sin\varphi\sin(\Omega t'))$$

GWs in a coherent state

Coherent state $$a^{(P)}(\boldsymbol{k})\,|\alpha\rangle = \alpha^{(P)}(\boldsymbol{k})\,|\alpha\rangle$$

$$|\alpha\rangle = \hat{D}(\alpha)\,|0\rangle$$

Note that we have used a Minkowski vacuum as an initial state.

Now, the expectation value is given by

$$\langle\alpha|\, h_{ij}(t,\boldsymbol{x})\,|\alpha\rangle = \frac{2}{M_{\rm p}}\sum_{P=+,\times}\int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}}\frac{e_{ij}^{(P)}(\boldsymbol{k})}{\sqrt{2\omega_{\boldsymbol{k}}}}\left[\alpha^{(P)}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega_{\boldsymbol{k}}t} + \alpha^{(P)*}(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}+i\omega_{\boldsymbol{k}}t}\right]$$

Substituting the explicit form of coherent parameters, we obtain

$$\begin{aligned}\langle\alpha|\, h_{ij}\,|\alpha\rangle \;=\; & \frac{\mu(a\Omega)^2}{2\sqrt{2}\pi M_{\rm p}^2}\frac{1}{r}\int_0^\infty r dk\;\frac{1}{2}\int_0^\pi \sin\theta d\theta\;\frac{1}{2\pi}\int_0^{2\pi} d\varphi\;\frac{1}{\pi}\int_{-\infty}^t k dt' \\ & \times\left\{ i\left[\gamma_1^3\left(\frac{m_2}{M}\right)e^{-i(k_x x_1+k_y y_1)} + \gamma_2^3\left(\frac{m_1}{M}\right)e^{-i(k_x x_2+k_y y_2)}\right]e^{-ikr\cos\theta+i\omega_k t}e^{i\omega_k t'}\right. \\ & \left.\times\left[e_{ij}^{(+)}\left(\frac{\sin^2\theta}{2}+\frac{1+\cos^2\theta}{2}\cos(2\Omega t'-2\varphi)\right)+e_{ij}^{(\times)}\cos\theta\sin\left(2\Omega t'-2\varphi\right)\right]+\text{c.c.}\right\}\end{aligned}$$

Comparison with quadrapole formula

The integration can be performed analytically.

For $r\Omega \gg 1$, the result reads

$$\langle\alpha|\, h_{xx}\, |\alpha\rangle = \frac{2G\mu(a\Omega)^2}{r}\left[\cos(2\Omega(t+r)) - \cos(2\Omega(t-r))\right]$$

Curiously, there are both outgoing and ingoing waves.

Compared to the quadrapole formula, the amplitude is just a half.

The maximum value shows complete agreement with the quadrapole formula.

For your reference, I give a number

$$\langle\alpha|\, h_{ij}(t,\boldsymbol{x})\, |\alpha\rangle \simeq 10^{-21}\left(\frac{\mu}{16\ M_\odot}\right)\left(\frac{a\Omega}{0.41}\right)^2\left(\frac{410\ \text{Mpc}}{r}\right)$$

HOW TO OBSERVE QUANTUM NATURE OF PGWs

S.Kanno, J.S, A.Taniguchi,
Quantum nature of gravitational waves from binary black holes,
arXiv:2508.17947 [gr-qc].

Squeezing of GWs from a binary system

At the second order, we have

$$U(t,\bar{\boldsymbol{x}}) = \exp\left[-i\sum_{N=1,2}\int^t dt'\frac{3\gamma_N^5 m_N}{2M_{\rm p}^2}\sum_{P,Q}\int\frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}}\int\frac{d^3\boldsymbol{k}'}{(2\pi)^{3/2}}\right.$$
$$\left.\times\left(\frac{e^{-i(\omega_k+\omega_{k'})t'}}{2\sqrt{\omega_k\omega_{k'}}}e_{ij}^{(P)}(\boldsymbol{k})e_{lm}^{(Q)}(\boldsymbol{k}')v_N^i v_N^j v_N^l v_N^m a^{(P)}(\boldsymbol{k})a^{(Q)}(\boldsymbol{k}')e^{i\boldsymbol{k}\cdot\bar{\boldsymbol{x}}_N}e^{i\boldsymbol{k}'\cdot\bar{\boldsymbol{x}}_N} + \text{h.c.}\right)\right]$$

Comparing the above expression with the squeezing operator

$$\hat{S}(\beta) = \prod_P \exp\left[\int d^3\boldsymbol{k}\int d^3\boldsymbol{k}'\left(\beta_{kk'}^{(P)}a^{(P)\dagger}(\boldsymbol{k})a^{(P)\dagger}(\boldsymbol{k}') - \beta_{kk'}^{(P)*}a^{(P)}(\boldsymbol{k})a^{(P)}(\boldsymbol{k}')\right)\right]$$

we find the squeezing parameter

$$\beta_{kk'}^{(P)} = -\frac{3i}{(2\pi)^3 2M_{\rm p}^2}\sum_{N=1,2}\gamma_N^5 m_N\int^t dt'\frac{e^{i(\omega_k+\omega_{k'})t'}}{2\sqrt{\omega_k\omega_{k'}}}e_{ij}^{(P)}(\boldsymbol{k})e_{lm}^{(P)}(\boldsymbol{k}')v_N^i v_N^j v_N^l v_N^m e^{-i\boldsymbol{k}\cdot\bar{\boldsymbol{x}}_N}e^{-i\boldsymbol{k}'\cdot\bar{\boldsymbol{x}}_N}$$

Explicit formula

More concretely, the parameter is expressed as

$$\beta^{(+)}_{kk'} = -\frac{3i}{64\pi^3 M_{\rm p}^2}\mu(a\Omega)^4$$
$$\times \int^t dt' \left[\gamma_1^5\left(\frac{m_2}{M}\right)^3 e^{-i(k_x+k'_x)x_1-i(k_y+k'_y)y_1} + \gamma_2^5\left(\frac{m_1}{M}\right)^3 e^{-i(k_x+k'_x)x_2-i(k_y+k'_y)y_2}\right]$$
$$\times \frac{e^{i(\omega_k+\omega_{k'})t'}}{\sqrt{\omega_k\omega_{k'}}}\left[\frac{\sin^2\theta}{2}+\frac{1+\cos^2\theta}{2}\cos(2\Omega t'-2\varphi)\right]\left[\frac{\sin^2\theta'}{2}+\frac{1+\cos^2\theta'}{2}\cos(2\Omega t'-2\varphi')\right]$$

$$\beta^{(\times)}_{kk'} = -\frac{3i}{64\pi^3 M_{\rm p}^2}\mu(a\Omega)^4$$
$$\times \int^t dt' \left[\gamma_1^5\left(\frac{m_2}{M}\right)^3 e^{-i(k_x+k'_x)x_1-i(k_y+k'_y)y_1} + \gamma_2^5\left(\frac{m_1}{M}\right)^3 e^{-i(k_x+k'_x)x_2-i(k_y+k'_y)y_2}\right]$$
$$\times \frac{e^{i(\omega_k+\omega_{k'})t'}}{\sqrt{\omega_k\omega_{k'}}}\cos\theta\sin(2\Omega t'-2\varphi)\cos\theta'\sin(2\Omega t'-2\varphi')\ .$$

The order of magnitude is estimated using the formula

$$\zeta \simeq \frac{4\pi}{3}(2\Omega)^3|\beta| \simeq \frac{1}{8\pi M_{\rm p}^2}\mu(a\Omega)^4 f$$

as

$$\zeta \simeq 2\times 10^{-3}\left(\frac{\mu}{16\ M_\odot}\right)\left(\frac{a\Omega}{0.41}\right)^4\left(\frac{f}{68\ {\rm Hz}}\right)$$

PGW Squeezed state detection with Binary BHs

S.Kanno, J.S, A.Taniguchi, in preparation.

Our main observation is that
we can utilize binary BHs to gain information of PGWs.

squeezing from binary BHs

$$S(\zeta) = \exp\left[\frac{1}{2}\zeta a^\dagger a^\dagger - \frac{1}{2}\zeta^* aa\right] \qquad \zeta = r e^{i\varphi}$$

This is quite small, however,

PGWs has a squeezing $|\xi\rangle = S(\xi)\,|0\rangle$ $\qquad S(\xi) = \exp\left[\frac{1}{2}\xi a^\dagger a^\dagger - \frac{1}{2}\xi^* aa\right]$

The point is that binary black holes emit GWs with a squeeing

$$S(\zeta)\,S(\xi)\,|0\rangle \approx S(\xi+\zeta)\,|0\rangle$$

PGWs at the 100Hz has a squeezing 280dB corresponding to $r \approx 30$. $\qquad \sinh r_k = \frac{1}{2}\left(\frac{f_c}{f}\right)^2$

In quantum optics, we can detect this squeezing using the intensity interferometry.

It is interesting to perform intensity interferometry in the context of Gravitational waves.

Summary

- Inflation is necessary for understanding the Universe.
- Inflation produces PGWs from quantum fluctuations of spacetime.
- The quantum state of PGWs is a squeezed state.
- We studied quantum nature of GWs from binary black holes.
- We showed that we can probe squeezing of PGWs using GWs from binary BHs.