Partition Functions on Supersymmetric Orbifolds

Based on arXiv:
2303.14199 (with M. Inglese & D. Martelli)
2312.17086 (with M. Inglese & D. Martelli)
2403.12318
2404.07173 (with E. Colombo, S. M. Hosseini, D. Martelli & A. Zaffaroni)

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Outline

A spiky index from accelerating black-holes

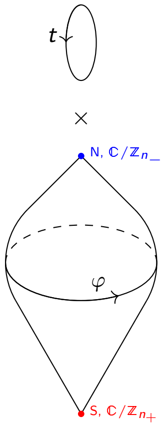
Spindles at the horizon

- spindles $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$ show up at the horizon of accelerating supersymmetric black-holes[a]
- Field theory dual: $\mathcal{N}=2$ QFT on $\Sigma \times S^1$ with $A^{(R)}$ such that $\mathfrak{f}^{(R)} = \frac{1}{2\pi}\int_\Sigma \mathrm{d}A^{(R)} = \frac{1}{2}\left(\frac{1}{n_-} \pm \frac{1}{n_+}\right) = \frac{\chi_\pm}{2}$
- $\mathfrak{f}^{(R)} = \frac{\chi_+}{2} = \frac{\chi_{\mathrm{orb}}}{2} \to Z_{\Sigma\times S^1}$ = twisted index
- $\mathfrak{f}^{(R)} = \frac{\chi_-}{2} \to Z_{\Sigma\times S^1}$ = anti-twisted index (SCI on S^2)

Applications of QFT on orbifolds

Holography, dualities, topological invariants

[a] Ferrero, Gauntlett, Ipiña, Martelli, Sparks, '20



General Procedure: Killing Spinors

Input

Let $x \in [x_1, x_2]$, $\varphi, \psi \in [0, 2\pi)$, $m, n = 1, 2, 3$ and $i, j = 1, 2$, as well as

$$\mathrm{d}s^2 = \delta_{mn} e^m e^n = f^2(x)\mathrm{d}x + c_{ij}(x)\mathrm{d}\varphi_i \mathrm{d}\varphi_j \in \mathbb{C}, \quad \varphi_1 = \varphi, \quad \varphi_2 = \psi,$$

$$e^1 = -f\mathrm{d}x, \qquad e^2 = \sqrt{\frac{\det c}{c_{22}}}\mathrm{d}\varphi, \qquad e^3 = \sqrt{c_{22}}\left(\frac{c_{12}}{c_{22}}\mathrm{d}\varphi + \mathrm{d}\psi\right),$$

$$\exists \mathcal{Q}, \widetilde{\mathcal{Q}} : \left\{\mathcal{Q}, \widetilde{\mathcal{Q}}\right\} \propto K = \zeta\gamma^\mu\widetilde{\zeta}\,\partial_\mu = k_0(\omega\partial_\varphi + \partial_\psi) \in \mathbb{C}, \qquad \mathcal{L}_K g_{\mu\nu} = 0, \quad (1)$$

Output

$\zeta = e^{\frac{\mathrm{i}}{2}(\alpha_2\varphi_2 + \alpha_3\psi)}(u_1, -u_2)^T$, $u_{1,2} = \sqrt{\sqrt{\iota_K K^\flat} \mp k_0\omega\sqrt{\det c/c_{22}}}$ satisfies KSE:

$$(\nabla_\mu - \mathrm{i}A_\mu)\zeta = -2^{-1}H\gamma_\mu\zeta - \mathrm{i}V_\mu\zeta - 2^{-1}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta \qquad (2)$$

Topological twist/anti-twist selected by the value of $\omega \in \mathbb{C}$. Same for $\widetilde{\zeta}$.

Geometry of topologically twisted $\Sigma \times S^1$

We have $x \in [-1,+1]$, $\varphi, \psi \in [0, 2\pi)$, $\lim_{x\to\pm1} f \to n_\pm/\sqrt{2(1\mp x)}$ and

$$
\begin{aligned}
\mathrm{d}s^2 &= f^2\mathrm{d}x + \left(1-x^2\right)(\mathrm{d}\varphi - \Omega\mathrm{d}\psi)^2 + \beta^2\mathrm{d}\psi^2, \\
\chi_{\rm orb} &= \frac{1}{4\pi}\int_\Sigma \mathrm{d}x\mathrm{d}\varphi\sqrt{g_\Sigma}R_\Sigma = \frac{1}{n_+} + \frac{1}{n_-}, \\
\omega &= \Omega \implies \zeta : u_{1,2} = \sqrt{-\frac{k_0\beta}{2}\left(1 \pm \Omega\sqrt{\frac{1-x^2}{\beta^2+\Omega^2(1-x^2)}}\right)}, \\
A^C &= \frac{\alpha_2\mathrm{d}\varphi + \alpha_3\mathrm{d}\psi}{2} - \frac{x(\mathrm{d}\varphi - \Omega\mathrm{d}\psi)}{2f\sqrt{1-x^2}}, \qquad \alpha_3 \in \mathbb{Z}, \\
V &= -\mathrm{i}\beta H\mathrm{d}\psi, \qquad A = A^C + \frac{3}{2}V, \qquad \mathfrak{f}^{(R)} = \int_\Sigma \frac{\mathrm{d}A^C}{2\pi} = \frac{\chi_{\rm orb}}{2} = \frac{\chi_+}{2}, \qquad (3)
\end{aligned}
$$

Same Ω of TTI on S^2. Analogous expressions for anti-twisted $\Sigma \times S^1$.

Geometry of non-trivial fibrations over spindles

$L_{[n_+,n_-]}(n,1)$: $x \in [0,\pi/2]$, $\varphi,\psi \in [0,2\pi)$, $f(0) = -b_1$, $f(\pi/2) = -b_2$ and

$$e^1 = -f\,\mathrm{d}x, \qquad e^2 = b_1 b_2 (2\widetilde{f})^{-1} \sin(2x)\,\mathrm{d}\varphi, \qquad e^3 = \frac{\widetilde{f}}{n}\Big(A^{(1)} + \mathrm{d}\psi\Big),$$

$$A^{(1)} = n\,\widetilde{f}^{-2}\Big(n_- t_- b_1^2 \sin^2 x + n_+ t_+ b_2^2 \cos^2 x\Big)\mathrm{d}\varphi, \qquad n_+ t_- - n_- t_+ = 1,$$

$$\zeta : u_{1,2} = \sqrt{\frac{k_0 b_1 b_2}{2n(t_- b_1 + t_+ b_2)}\Big[1 \pm (n_- b_1 + n_+ b_2)(2\widetilde{f})^{-1}\sin(2x)\Big]},$$

$$A^C = \frac{\alpha_2 \mathrm{d}\varphi + \alpha_3 \mathrm{d}\psi}{2} + \frac{t_+ b_2 - t_- b_1}{2f}\mathrm{d}\varphi + \frac{n_+ b_2 - n_- b_1}{2n\,f}\mathrm{d}\psi,$$

$$\alpha_2 = t_+ - t_-, \quad \alpha_3 = \frac{n_+ - n_-}{n}, \quad \widetilde{f}(0) = n_+ b_2, \quad \widetilde{f}(\pi/2) = n_- b_1,$$

$$V = \text{long expression}, \qquad A = A^C + \frac{3}{2}V, \qquad \frac{1}{2\pi}\int_\Sigma \mathrm{d}A^{(1)} = \frac{n}{n_+ n_-}, \qquad (4)$$

For instance, $\gcd(n\,t_+, n_+) = k_+$ signals a $\mathbb{C}/\mathbb{Z}_{k_+}$ singularity at $x = 0$

Localization and cohomology

Supersymmetry transformations as a cohomological complex

Let $L_X = X^\mu D_\mu$ and $\psi = B\zeta + (C/v)\widetilde{\zeta}$ with $B = \widetilde{\zeta}\psi/v$ and $C = \zeta\psi$. Then,

$$\begin{aligned} \delta\phi &= C, \qquad \delta B = F + \mathrm{i}L_{\widetilde{P}}\phi, & \delta^2 &= -2\mathrm{i}(L_K + \mathcal{G}_{\Phi_G}), \\ L_K &= K^\mu D_\mu = \mathcal{L}_K - \mathrm{i}q_R\Phi_R, & \Phi_R &= \iota_K[A - (V/2)] - \mathrm{i}vH, \\ \mathcal{G}_{\Phi_G}X &= -\mathrm{i}\Phi_G \circ_{\mathfrak{R}_G} X, & \Phi_G &= \iota_K A_G - \mathrm{i}v\sigma, \end{aligned} \tag{5}$$

Cohomological localization

1-loop determinant computed via either eigenvalue pairing or index theorem :

$$Z = \sum_{\text{fluxes}} \int_{\text{holonomies}} Z_{\text{classical}} Z_{\text{1-L}}, \qquad Z_{\text{classical}} = e^{-S|_{\text{BPS}}},$$

$$Z_{\text{1-L}} = \frac{\det_{\text{Ker}\, L_P}(L_K + \mathcal{G}_{\Phi_G})}{\det_{\text{Ker}\, L_{\widetilde{P}}}(L_K + \mathcal{G}_{\Phi_G})} \quad \longleftrightarrow \quad \text{equivariant index of } L_{\widetilde{P}} \text{ wrt } \widehat{g} = e^{-\mathrm{i}\epsilon\delta^2}.$$

Localization on topologically twisted $\Sigma \times S^1$

BPS locus: $\delta\psi = \delta\widetilde{\psi} = \delta\lambda = \delta\widetilde{\lambda} = 0$

BPS gauge field: $\mathcal{A} = \mathcal{A}_\varphi(x)\mathrm{d}\varphi + \mathcal{A}_\psi(x)\mathrm{d}\psi$. Let $\varphi_G \in \mathbb{C}$ and $\mathcal{A}_\varphi(\pm 1) = \mathfrak{m}_\pm / n_\pm$, with $n_+\mathfrak{m}_- - n_-\mathfrak{m}_+ = \mathfrak{m} \in \mathbb{Z}$. Then

$$\int_\Sigma \frac{\mathrm{d}\mathcal{A}}{2\pi} = \frac{\mathfrak{m}}{n_+ n_-}, \quad e^{\mathrm{i}\oint_{S^1}\mathcal{A}} \in U(1), \quad \phi|_{\mathrm{BPS}} = \widetilde{\phi}|_{\mathrm{BPS}} = F|_{\mathrm{BPS}} = \widetilde{F}|_{\mathrm{BPS}} = 0,$$

$$\sigma|_{\mathrm{BPS}} = (\mathrm{i}/\beta)[-\varphi_G + \omega\mathcal{A}_\varphi + \mathcal{A}_\psi], \quad D|_{\mathrm{BPS}} = -\frac{\mathcal{A}'_\varphi}{f\sqrt{(1-x^2)}}, \tag{6}$$

Eigenfunctions of L_K in $\mathrm{Ker}\, L_{\widetilde{P}}$ contributing to $Z_{1\text{-L}}$

Regularity of $\Phi_{n_\varphi,n_\psi}(x,\varphi,\psi) = e^{\mathrm{i}n_\varphi\varphi + \mathrm{i}n_\psi\psi}\phi_{n_\varphi,n_\psi}(x) \in \mathrm{Ker}\, L_{\widetilde{P}}$ requires:

$$\lim_{x\to 1}\phi_{n_\varphi,n_\psi} \to (1-x)^{\left(2\mathfrak{m}_+ - 2n_\varphi n_+ - r\right)/4} \implies n_\varphi \le \left\lfloor \frac{\mathfrak{m}_+ - (r/2)}{n_-} \right\rfloor, \tag{7}$$

1-loop determinant on $\Sigma \times S^1$ with arbitrary twist

Chiral-multiplet 1-loop determinant

$$Z^{\mathrm{CM}}_{\text{1-L}} = \prod_{\rho \in \mathfrak{R}_G} \left(-y^{-\rho}\right)^{(2\mathfrak{b}-1+\mathfrak{s})/4} q^{(1-\mathfrak{s})(\mathfrak{b}-1)/8} \frac{\left(q^{(\mathfrak{b}+1)/2} y^{-\rho}; q\right)_\infty}{\left(q^{\mathfrak{s}(\mathfrak{b}-1)/2} y^{-\mathfrak{s}\rho}; q\right)_\infty}, \qquad (8)$$

with $\mathfrak{s} = \pm 1$ corresponding to twisted and anti-twisted spindle, respectively, and

$$\mathfrak{p}_+ = \mathfrak{m}_+ - \mathfrak{s}(r/2), \qquad \mathfrak{p}_- = \mathfrak{m}_- + (r/2),$$
$$\mathfrak{b} = 1 + \mathfrak{s}\left\lfloor \mathfrak{s}\frac{\mathfrak{p}_+}{n_+} \right\rfloor + \left\lfloor -\frac{\mathfrak{p}_-}{n_-} \right\rfloor, \qquad \mathfrak{c} = \frac{[\![-\mathfrak{p}_-]\!]_{n_-}}{n_-} - \mathfrak{s}\frac{[\![\mathfrak{s}\mathfrak{p}_+]\!]_{n_+}}{n_+},$$
$$\gamma_R = \frac{\omega}{4}\chi_{-\mathfrak{s}} - \frac{\alpha_3}{2}, \qquad \gamma_G = -\varphi_G + \frac{\omega}{2}\left(\frac{\mathfrak{m}_-}{n_-} + \frac{\mathfrak{m}_+}{n_+}\right),$$
$$q = e^{2\pi i \omega}, \qquad y = q^{\mathfrak{c}/2} e^{2\pi i (r\gamma_R + q_G \gamma_G)}, \qquad (9)$$

SCI: $\mathfrak{s} = -1$ and $n_- = n_+ = 1$. TTI on S^2: $\mathfrak{s} = +1$ and $n_- = n_+ = 1$.

1-loop determinant on general fibrations over spindles

Chiral-multiplet 1-loop determinant

On $L_{[n_+,n_-]}(n,1)$ with vortex lines $\mathfrak{m}_\pm/n_\pm$ and fiber holonomy $e^{2\pi \mathrm{i} h/n} \in \mathbb{Z}_n$:

$$Z^{\mathrm{CM}}_{\text{1-L}} = \prod_{j_1=0}^{n-1} \prod_{k,\ell\in\mathbb{N}} \frac{\prod_{j_2\in\mathbb{J}_-(j_1,\mathfrak{h})}\left[\frac{1}{b}\left(k+\frac{j_2}{n}\right)+b\left(\ell+\frac{j_1}{n}\right)+\frac{Q}{2n}+\frac{\mathrm{i}u}{n}\right]}{\prod_{j_3\in\mathbb{J}_+(j_1,\mathfrak{h})}\left[\frac{1}{b}\left(k+\frac{j_1}{n}\right)+b\left(\ell+\frac{j_3}{n}\right)+\frac{Q}{2n}-\frac{\mathrm{i}u}{n}\right]}, \qquad (10)$$

where $b=\sqrt{b_1/b_2}$, $Q=\left(b^{-1}+b\right)$, $\mathfrak{h}=h-\mathfrak{m}$ and

$$u = q_G\sqrt{b_1 b_2}\left[\sigma_c + \mathrm{i}\left(\frac{\mathfrak{m}_+}{b_1}+\frac{\mathfrak{m}_-}{b_2}\right)\right] + \mathrm{i}(r-1)\frac{Q}{2},$$

$$\mathbb{J}_\mp(j_1,\mathfrak{h}) = \left\{j_0 = 0,\dots,(n-1) : [\![n_\mp j_0]\!]_n = [\![n_\pm j_1 + \mathfrak{h}]\!]_n\right\}, \qquad (11)$$

with

- $n_\pm = 1$ and $\mathfrak{m}=0$: 1-loop on the squashed Lens space[1]
- $n=1$ and $\mathfrak{m}=0$: squashed sphere's double-sine function

[1] Imamura and Yokoyama, '12

Large-N limit and holography

Entropy function for accelerating and rotating black holes

$S_{\mathrm{BH}} = A_{\mathrm{horizon}}/(4G_{\mathrm{4d}}) = \mathscr{S}|_{\mathrm{critical}}$ wrt flavour fugacities φ_α, where

$$\mathscr{S} = \log Z_{\Sigma\times S^1} + \mathrm{i}\sum_{\alpha=1}^{\mathrm{rank}(G_F)} \varphi_\alpha Q_\alpha - \mathrm{i}\epsilon J, \qquad \sum_{\alpha=1}^{\mathrm{rank}(G_F)} \varphi_\alpha + \epsilon\frac{\chi_{-\mathfrak{s}}}{2} = 2\pi, \tag{12}$$

with Q_α=flavour charges, J=angular momentum, $\epsilon = 2\pi\omega$

Large-N factorization in gravitational blocks

For a quiver theory with $U(N)$ gauge groups[2], such as ABJM:

$$\lim_{N\to\infty} \log Z_{\Sigma\times S^1} \to \frac{F_-}{\epsilon} - \mathfrak{s}\frac{F_+}{\epsilon}, \qquad F_\pm = \frac{2\sqrt{2}}{3}N^{3/2}\sqrt{\Delta_1^\pm\Delta_2^\pm\Delta_3^\pm\Delta_4^\pm}, \tag{13}$$

where $\sqrt{2}N^{3/2}/3 \sim G_{\mathrm{4d}}^{-1}$ and $\Delta_\alpha^\pm = \varphi_\alpha \pm \epsilon\,\mathfrak{f}^{F\times R}/2$.

[2] Colombo, Hosseini, Martelli, Pittelli, Zaffaroni '24

1-loop determinant on topologically twisted $\mathbb{\Sigma} \times T^2$

Chiral-multiplet 1-loop determinant

4d spindle index on twisted $\mathbb{\Sigma} \times T^2$:

$$Z_{\text{1-L}}^{\text{CM}} = \prod_{\rho \in \mathfrak{R}_G} e^{2\pi i \Psi} \frac{\Gamma_e\left(q^{(1-\mathfrak{b})/2} y^{-\rho}; p, q\right)}{\Gamma_e\left(q^{(1+\mathfrak{b})/2} y^{-\rho}; p, q\right)}, \tag{14}$$

with $\Psi = \text{poly}^3(\gamma_R, \gamma_G)/(\omega\tau)$, $y =$ gauge/flavour fugacity, $\alpha_3, \alpha_4 \in \mathbb{Z}$ and

$$\Gamma_e(u; p, q) = \frac{(pq/u; p, q)_\infty}{(u; p, q)_\infty}, \qquad \gamma_R = \frac{\omega}{4}\chi_- + \tau\frac{\alpha_3}{2} - \frac{\alpha_4}{2},$$
$$p = e^{2\pi i \tau}, \qquad q = e^{2\pi i \omega}. \tag{15}$$

$S^2 \times T^2$ corresponds to $n_- = n_+ = 1$. What about anti-twist on $\mathbb{\Sigma} \times T^2$?

Refined Seifert index in 4d: 1-loop determinant on $L_{[n_+,n_-]}(n,1)\times S^1$

Chiral-multiplet 1-loop determinant

On $L_{[n_+,n_-]}(n,1)\times S^1$ with vortex lines $\mathfrak{m}_\pm/n_\pm$ and holonomy $e^{2\pi i h/n}\in\mathbb{Z}_n$:

$$Z^{\mathrm{CM}}_{\text{1-L}}=\prod_{\rho\in\mathfrak{R}_G} e^{2\pi i\Psi^{\mathrm{CM}}_\rho}\prod_{j=0}^{n-1}\frac{\prod_{i\in\mathbb{J}_+(j,\rho(\mathfrak{h}))}\left(q_1^{(1+j)/n}q_2^{(1+i)/n}z^{-\rho};q_1,q_2\right)_\infty}{\prod_{k\in\mathbb{J}_-(j,\rho(\mathfrak{h}))}\left(q_1^{k/n}q_2^{j/n}z^{\rho};q_1,q_2\right)_\infty},\qquad(16)$$

where $z=e^{2\pi i(q_G\gamma_G+r\gamma_R)}$, $\alpha_4\in\mathbb{Z}$, $\mathfrak{h}=h-\mathfrak{m}$, $\mathfrak{m}=n_+\mathfrak{m}_--n_-\mathfrak{m}_+$ and

$$q_{1,2}=e^{2\pi i\omega_{1,2}}=e^{2\pi n\beta/b_{1,2}},\qquad \gamma_R=\frac{\omega_1+\omega_2}{2n}-\frac{\alpha_4}{2},$$
$$\gamma_G=-\frac{\omega_1\mathfrak{m}_++\omega_2\mathfrak{m}_-}{2n}-a_0,\qquad e^{2\pi i a_0}\in U(1),\qquad(17)$$

generalizing the refined Lens index ($n_\pm=1$ and $\mathfrak{m}=0$) and the 4d SCI ($n=1$ and $\mathfrak{m}=0$)

Discussion

Results

- General solution of KSE for a large class of three-orbifolds
- Exact partition functions on any S^1 fibration over Σ with any twist
- Exact partition function on topologically twisted $\Sigma \times T^2$
- Refined Seifert index in 4d
- Large-N limit of the spindle index $= S_{\mathrm{BH}}$ of accelerating black holes

Outlook

- Large-N limit of the 3d refined Seifert partition function $\rightarrow$ NUTs and spindle bolts in AdS_4?
- Large-N limit of the 4d spindle index $\rightarrow S_{\mathrm{BH}}$ of accelerating black strings?
- Large-N limit of the 4d refined Seifert index $\rightarrow$ orbifolded sugra solution in AdS_5?

The end?

THANK YOU!