Equivariant localization for black holes

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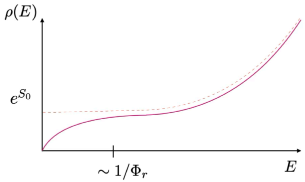
based on work with Jerome Gauntlett and James Sparks, Yusheng Jiao and Alice Lüscher

Non-supersymmetric black holes

3rd law of black hole thermodynamics: the entropy of black holes vanishes at extremality.

"Proof" by inclusion of quantum effects [Iliesiu–Turiaci, ...]



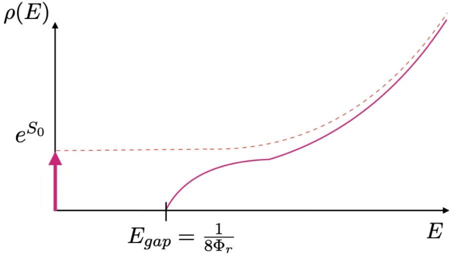
Euclidean "proof" [Hawking–Horowitz–Ross]: Non-extremal black holes have topology $\Sigma \times \mathbb{R}^2$, but at extremality they develop an "infinite throat" and the topology becomes $\Sigma \times S^1_\beta \times \mathbb{R}$. On-shell action is

$$I = \int_{S^1_\beta} \int_{\Sigma \times \mathbb{R}} \mathcal{L} = \beta E \quad \Rightarrow \quad S = -(1 - \beta \partial_\beta) I = 0\,.$$

Supersymmetric black holes

SUSY extremal black holes have large ground state degeneracy [Strominger–Vafa]

"Proof" by inclusion of quantum effects [Boruch–Heydeman–Iliesiu–Turiaci, Heydeman–Toldo²]



Euclidean "proof"? We need to show that $I \neq \beta E$.

Supersymmetric black holes

In the “proof” we broke supersymmetry at non-zero temperature to preserve Lorentzian causality. However, if we could construct a family of supersymmetric non-extremal solutions with topology $\Sigma \times \mathbb{R}^2$ such that

$$I = -\frac{A}{4} + \beta \{\mathcal{Q}, \mathcal{Q}^\dagger\}$$

then we would be done.

These solutions have been constructed in five dimensions minimal gauged supergravity [Cabo-Bizet–Cassani–Martelli–Murthy], and in various other simple models [Cassani–Papini, Bobev–Crichigno, ...].
However, it's still missing a general proof, and even the solutions with all the parameters consistent with supersymmetry turned on

A new technique

We develop a geometric tool to compute physical observables in supersymmetric AdS/CFT from the gravity side *without* solving Einstein's equations.

Can be used in a number of cases:

- $11d$ SUGRA solutions $AdS_5 \times M_6$, $AdS_3 \times M_8$: (off-shell) central charge and scaling dimensions of primary op.s [also Colombo–Faedo–Martelli–Zaffaroni]
- $10d$ massive IIA solutions $AdS_4 \times M_6$ [Couzens–Lüscher]
- gauged SUGRA in $4d$ and $6d$: on-shell action
- black holes gauged and ungauged SUGRA in $4d$ and $5d$, attractor mechanism and non-extremal deformations [also Cassani–Ruipérez–Turetta]

Equivariant localization

Localization has an illustrious history in physics and mathematics. Broadly speaking, it's the idea that in presence of a symmetry, certain integrals only receive contributions from the fixed points of the symmetry.

In particular, in supersymmetric field theories it leads to the exact computation of the path integral on the infinite-dimensional space of fields [Witten, Nekrasov, Pestun, ...]

Here we focus on localization on finite-dimensional spacetime, solution of supergravity theories.

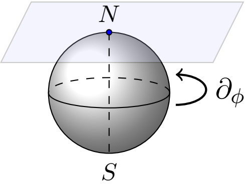
In a space with an isometry generated by ξ, we work with *equivariant forms*, and in particular *equivariant cohomology* [Cartan]
Introduce the operation $\mathrm{d}_\xi \equiv \mathrm{d} - \xi \lrcorner$

- acts on polyforms of multiple degree
- squares to $\mathrm{d}_\xi^2 = -\mathcal{L}_\xi$
- it is a *differential* on equivariant forms

Example: S^2 with metric $\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$.
Isometry generated by $\xi = \partial_\phi$ with isolated fixed points at North and South pole.

$\mathrm{vol}(S^2) = \sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi$
$\to \; \Psi = \sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi + \cos\theta$

Check: $(\mathrm{d} - \xi\lrcorner)\Psi = \sin\theta\,\mathrm{d}\theta - \sin\theta\,\mathrm{d}\theta = 0.$



Equivariant localization theorem [Berline–Vergne, Atiyah–Bott]
If Ψ is equivariantly closed, then its integral is a sum of contributions from the fixed points of the isometry.

Example In two dimensions: only isolated points

$$\int_M \Psi = \sum_{\text{fixed points } p} \frac{2\pi}{b_p} \Psi_0|_p$$

weight of ξ acting on normal space

On S^2, $\Psi = \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi + \cos\theta$

$$\int_{S^2} \mathrm{vol}(S^2) = \frac{2\pi}{b_N} \cos\theta|_N + \frac{2\pi}{b_S} \cos\theta|_S = 4\pi$$

Example: In four dimensions: "nuts and bolts" [Gibbons–Hawking]

normal to $F \subset M$

$$\int_M \Psi = \sum_{\mathrm{dim}=0} \frac{(2\pi)^2}{b_1 b_2} \Psi_0 + \sum_{\mathrm{dim}=2} \frac{2\pi}{b} \int_F \left[\Psi_2 - \Psi_0 \frac{2\pi}{b} c_1(L) \right]$$

Starting point: $(M, g, F, \cdots, \epsilon)$ with $d = 2n$.
"Supersymmetric" means that ϵ satisfies certain differential and algebraic equations

$$\nabla_\mu \epsilon = \mathcal{O}_\mu(g, F, \cdots)\epsilon\,, \qquad \mathcal{P}_\mu \epsilon = 0\,.$$

Construct forms Φ as bilinears in ϵ

$$\bar{\epsilon}\epsilon,\ \bar{\epsilon}\gamma_\mu\epsilon,\ \bar{\epsilon}\gamma_{\mu\nu}\epsilon, \ldots$$

SUSY equations for ϵ are equivalent to differential and algebraic equations satisfied by the bilinears.
Among them is a vector generating a *symmetry* of the solution

$$\xi^\mu = \bar{\epsilon}\gamma^\mu\gamma_*\epsilon\,, \qquad \mathcal{L}_\xi g = \mathcal{L}_\xi F = \mathcal{L}_\xi(\ldots) = 0\,.$$

An equivariant structure underlies many exact results in holography.

Many BPS quantities (action, black hole entropy, central charges, scaling dimensions, ...) are written as integrals of equivariantly closed polyforms constructed out of the bilinears.

Therefore, they depend only on weights of the ξ action and topology, and can be computed using BVAB *without knowing the solution.*

We consider $4d$ SUGRA with metric $g_{\mu\nu}$, n vectors A^I_μ, $2n$ scalars z^i, $\tilde{z}^{\tilde{i}}$. The model is written in terms of the geometry of the scalar manifold with metric $\mathcal{G}_{i\tilde{j}}$, over which are defined sections $X^I(z)$, $\widetilde{X}^I(\tilde{z})$ and a prepotential $\mathcal{F}(X)$.

Bulk action

$$I = -\frac{1}{16\pi G}\int \Big[\Big(R - 2\mathcal{G}_{i\tilde{j}}\partial^\mu z^i \partial^\nu \tilde{z}^{\tilde{j}} - \mathcal{V}(z,\tilde{z})\Big)\,\mathrm{vol} + \frac{1}{2}\mathcal{I}_{IJ}F^I \wedge *F^J - \frac{\mathrm{i}}{2}\mathcal{R}_{IJ}F^I \wedge F^J\Big]$$

Each supersymmetric solution has a Killing vector $\xi^\mu = -\mathrm{i}\epsilon^\dagger\gamma^\mu\gamma_5\epsilon$

The gravitational free energy leads to another equivariantly closed form, which can be evaluated to

chirality of ϵ at fixed set

$$F_{\text{grav}} = \frac{\pi}{G}\Bigg[\sum_{\text{nuts}_\pm} \mp \frac{(b_1 \mp b_2)^2}{b_1 b_2} \mathrm{i}\mathcal{F}(u^J_\pm) + \sum_{\text{bolts}_\pm} \left(-\kappa \partial_I \mathrm{i}\mathcal{F}(u^J_\pm)\mathfrak{p}^I_\pm \pm \mathrm{i}\mathcal{F}(u^J_\pm) \int_{\Sigma_\pm} c_1(L) \right)\Bigg]$$

$$u^I_+ \equiv \frac{\widetilde{X}^I}{\zeta_J \widetilde{X}^J} \qquad u^I_- \equiv \frac{X^I}{\zeta_J X^J} \qquad \mathfrak{p}^I_\pm \equiv \int_{\Sigma_\pm} \frac{F^I}{4\pi}$$

This is expressed in terms of "IR" bulk data. To obtain "UV" field theory data, note that each F^I gives an equivariantly closed form, which can be evaluated on compact and non-compact submanifolds giving UV-IR relations generalizing [Bobev–Charles–Min].

Euclidean $4d$ SUGRA with vectors

Static topologically twisted black holes

To discuss Wick-rotations of black holes, we want to look for supersymmetric solutions with $\Sigma \times \mathbb{R}^2$ topology.

Suppose $\Sigma \cong \Sigma_g$ with $g > 1$. Then ξ rotates $\mathbb{R}^2$ and there's a single bolt

$$F_{\text{grav}} = -\frac{\pi}{G} \mathrm{i}\mathcal{F}_I(u_\pm)\mathfrak{p}^I(u_\pm) \qquad \zeta_I \mathfrak{p}^I_\pm = 2(g-1)\,,\ \zeta_I u^I_\pm = 1$$

Simplest case: zero mass deformations

- UV-IR gives $u^I_\pm = \pm\Delta^I = \int_{S^1_{\text{UV}}} \frac{A^I}{4\pi}$
- grand-canonical free energy $F_{\text{grav}} = -\frac{\pi}{G}\mathrm{i}\mathcal{F}_I(\Delta)\mathfrak{p}^I(u_\pm)$
- (constrained) Legendre transform gives entropy $S = -F_{\text{grav}} - \mathrm{i}q_I\Delta^I$
- for X^0X^1 and STU reproduces known analytic solutions deforming static TT black holes [Bobev–Charles–Min, PBG–Toldo]
- For general model there is formula for S [Halmagyi]. Can one recover it?

Suppose $\Sigma \cong S^2$. Then $\xi = \varepsilon\partial_{\varphi_1} + \varphi_2$ and there are two isolated fixed points at the poles. Regularity and UV-IR relation give

$$F_{\text{grav}} = \frac{\pi}{G}\frac{1}{\varepsilon}\big[\chi_N \mathrm{i}\mathcal{F}(y_S - \varepsilon\mathfrak{p}) - \chi_S \mathrm{i}\mathcal{F}(y_S)\big]$$

where $\chi_{N,S}$ are the chiralities at the poles, and $y_S^I = \Delta_S^I - 2\pi\mathrm{i}\sigma_S^I$ constrained by

$$\zeta_I \mathfrak{p}^I = -(1 + \chi_N\chi_S)\,, \quad \zeta_I y_S^I = \chi_S(1 - \chi_S\varepsilon)\,.$$

There are different ways of preserving supersymmetry depending on $\chi_N\chi_S$.

Euclidean $4d$ SUGRA with vectors

Rotating black holes: twist

Suppose $\Sigma \cong S^2$, $\xi = \varepsilon\partial_{\varphi_1} + \varphi_2$ and $\chi_N = \chi_S$. Then

$$F_{\text{grav}} = \chi_S \frac{\pi}{G}\frac{1}{\varepsilon}\left[\mathrm{i}\mathcal{F}(y_S - \varepsilon\mathfrak{p}) - \mathrm{i}\mathcal{F}(y_S)\right]$$

with

$$\zeta_I \mathfrak{p}^I = -2\,, \quad \zeta_I y_S^I = \chi_S(1 - \chi_S\varepsilon)\,.$$

Simplest case: zero mass deformation

- grand-canonical free energy $F_{\text{grav}}(\Delta, \epsilon, \mathfrak{p})$
- (constrained) Legendre transform gives entropy
- for STU, it leads to the entropy of the analytic BPS rotating TT black holes [Hristov–Katmadas–Toldo]
- proves "gravitational blocks" formula [Hosseini–Hristov–Zaffaroni]
- as $\varepsilon \to 0$, $F_{\text{grav}} = -\chi_S \frac{\pi}{G}\mathrm{i}\mathcal{F}_I(y)\mathfrak{p}^I$: gauged attractor mechanism [Cacciatori–Klemm, Dall'Agata–Gnecchi]

Euclidean $4d$ SUGRA with vectors

Rotating black holes: anti-twist

Suppose $\Sigma \cong S^2$, $\xi = \varepsilon\partial_{\varphi_1} + \varphi_2$ and $\chi_N = -\chi_S$. Then

$$F_{\text{grav}} = -\chi_S \frac{\pi}{G}\frac{1}{\varepsilon}\left[\mathrm{i}\mathcal{F}(y_S - \varepsilon\mathfrak{p}) + \mathrm{i}\mathcal{F}(y_S)\right]$$

with

$$\zeta_I \mathfrak{p}^I = 0\,, \quad \zeta_I y_S^I = \chi_S(1 - \chi_S\varepsilon)\,.$$

Simplest case: zero mass deformation

- grand-canonical free energy $F_{\text{grav}}(\Delta, \epsilon, \mathfrak{p})$
- (constrained) Legendre transform gives entropy
- for STU, it leads to the entropy of the analytic BPS rotating KN_4 black holes [Hristov–Katmadas–Toldo]
- proves "gravitational blocks" formula [Hosseini–Hristov–Zaffaroni]
- cannot take $\varepsilon \to 0$

A Euclidean "proof" of the 3rd law

Generically localization theorems give us $F_{\rm grav} = F_{\text{fixed set}} + F_{\rm bdry}$. However, we showed that $F_{\rm bdry} = 0$. For spaces M with topology $\Sigma \times \mathbb{R}^2$ and $\partial M \cong \Sigma \times S^1_\beta$, we can write *even without knowing the explicit solution*

$$F_{\rm bdry} = \beta\{\mathcal{Q}, \mathcal{Q}^\dagger\}\,, \qquad \partial_\beta F_{\text{fixed set}} = 0$$

Therefore, we evade the argument by [Hawking–Horowitz–Ross].

For holographic models, we knew that this had to be the case, since $F_{\rm grav}[M] = -\log Z_{\rm SCFT}[\partial M]$, but $Z_{\rm SCFT}[\Sigma \times S^1_\beta]$ has anti-periodic fermions around S^1_β and thus it's a supersymmetric index [Witten, Kinney–Maldacena–Minwalla–Raju, Benini–Zaffaroni, Inglese–Martelli–Pittelli]

What I haven't told you

- For static TT black holes, we have *metric* $AdS_2 \times_w \Sigma$. Reduce the $4d$ action on AdS_2, obtaining a functional on Σ: $\mathcal{S}_{\mathrm{BH}}$. This is the integral of an equivariantly closed form, which can be evaluated using localization theorems. The result matches what we've just found, but now $\mathcal{S}_{\mathrm{BH}}$, being an action *off-shell*, should be extremized, leading to the black hole entropy [Sen]
- Everything goes through if you take $\Sigma \cong \mathbb{WP}_{[n_N, n_S]}$ (spindle). The result is the entropy of accelerating black holes
- You can turn off the gauging, obtaining results for ungauged SUGRA: OK for $AdS_2 \times \Sigma$ solutions (attractor mechanism [Ferrara–Kallosh–Strominger, ...]), subtler for general solution (for simple topology [Boruch–Iliesiu–Murthy–Turiaci])
- Everything goes through if you take $AdS_3 \times \Sigma$ in $5d$: attractor and black strings and rings

Today I summarized one use of equivariant localization in gravity, but it has been found everywhere people looked, and it underlies the computations of many BPS observables.

- How far does equivariant cohomology go in supergravity? Is that a universal property of solutions with a Killing vector?
- Does it also hold for higher derivative corrections? Yes for $4d$ minimal [Bobev–Charles–Hristov–Reys, PBG–Richmond] [Bobev–David–Hong–Reys–Zhang]
- How does it work in odd dimensions? Hints in [Cassani–Ruipérez–Turetta]
- Why does it hold at all?

THANK YOU

FOR YOUR ATTENTION