Applied SUSY QFT & BHPT: one loop corrections to near extremal Kerr thermodynamics

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talk @ Black holes, Holography and de Sitter spacetimes

Based on

- G.B., C. Iossa, D. Panea Lichtig, A. Tanzini "Exact solution of Kerr black hole perturbations via CFT2 and instanton counting. Greybody factor, Quasinormal modes and Love numbers"; PRD 105 (2022) 4
- P. Arnaudo, G.B. and A. Tanzini, “One loop effective actions in Kerr-(A)dS Black Holes,” PRD 110 (2024) 10
- P. Arnaudo, G.B. and A. Tanzini, "One-loop corrections to near extremal Kerr thermodynamics from semiclassical Virasoro blocks" [arXiv: 2412.16057 [hep-th]]

Abstract

• The study of Black-Hole perturbation theory [BHPT] is a classical problem in General Relativity and crucial to study gravitational waves etc..

• As the high order of symmetry of the BH gravitational field implies separation of variables, BHPT reduces to the study of 2^{nd} order linear ordinary differential equations (ODE).

• These ODEs are of generalized Fuchsian type and are **[A.M.Polyakov]** solved exactly in terms of classical Virasoro conformal blocks.

• The latter have an explicit expression because of AGT duality (BPS sector in susy gauge theory <=> CFT2) and the classical limit of crossing relations generate the explicit solution of the connection problem of the ODE.

• Applying the solution back to BHPT one gets a powerful computational technique.

• Among other results, combining with a generalisation of the Gelfand-Yaglom formula for singular potentials one gets an exact expression for the one loop BH effective action and in particular its universal scaling properties at low Hawking temperature for the near extremal Kerr BH as

$$S_{1-loop} = -\frac{3}{2}\ln(T_H) + \mathcal{O}(T_H^0)\,.$$

Introduction

We put in contact:

- General Relativity in D=4 $\rightarrow$ BH physics
- Exact BPS partition functions in $\mathcal{N}$=2, D=4 susy gauge theories $\rightarrow$ surface observables
- Conformal Field Theories in D=2 $\rightarrow$ Liouville conformal blocks

The equations we work on are respectively:

- Einstein Equations linearised around Black Holes solutions [BHPT]
- Renormalisation Group equation for BPS defects [Quantum Seiberg-Witten curve]
- Null states equation at level 2 [BPZ equation, classical limit]

All these reduce, after the proper dictionaries, to the very same linear 2^{nd} order ODEs, known as **Heun Equations**.

- Instanton counting $\xrightarrow[\text{correspondence}]{\text{AGT}}$ Liouville CB $\xrightarrow[\text{limit}]{\text{classical}}$ BHPT

Teukolsky equation / GR side

- We consider spinning BH solutions of the Einstein equation /w c.c.

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0$$

- The Einstein equation linearised around the BH metric is called Teukolsky equation
- This is important because it describes gravitational waves emitted by the Black Hole
- After a small perturbation or at the end of a resilient process, the BH spacetime is described by an approximate metric $g = g^{BH} + \psi$. (E.g. early inspiral and late ringdown phases of binary BH)
- Up to higher order corrections the metric perturbation ψ solves the Teukolsky equation

$$\mathrm{Lin}\mathcal{G}_{\mu\nu}[g^{BH}](\psi) = 0$$

[Concretely, ψ are the linearisation of some components of the curvature]

- Because of symmetries, Teukolsky equation is completely separable.
- This separation of variables + GY theorem are crucial also to study the spectral properties of $\mathrm{Lin}\mathcal{G}_{\mu\nu}[g^{BH}]$ (gravity one loop effective action).

Teukolsky equation

- The (A)dS Kerr BH metric in cylindrical coordinates $(x^\mu) = (t, r, \phi, \theta)$

$$ds^2 = g^{BH}_{\mu\nu}(\theta, r; M, a_{BH}, \Lambda) dx^\mu dx^\nu$$

  where Λ is the cosmological constant, M the BH mass and a_{BH} its angular velocity.
- Because of static and cylindrical symmetry, one can Fourier transform fixing the frequency ω and the angular momentum m of the perturbation

$$\psi = \int d\omega \sum_{m,l} e^{i\phi m + it\omega} \psi_{\omega,m,l}(\theta, r)$$

- Substituting in the Teukolsky equation one finds that the θ and r variables dependence of ψ separates as

$$\psi_{\omega,m,l}(\theta, r) = S_{\omega,m,l}(cos(\theta)) R_{\omega,m,l}(r)$$

  where $S_{\omega,m,l}$ and $R_{\omega,m,l}$ satisfy two resulting linear second order ODEs in the corresponding variables.
- Both of them are (C)Heun equations [after the proper angular/radial dictionary]

  [Teukolsky, Batic & Schmid] (Petrov Type D)

Heun equation

- Linear second order ODEs with four regular singularities

$$\left(\frac{d^2}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-t}\right)\frac{d}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-t)}\right)w(z)=0\,,$$

where $\alpha+\beta+1=\gamma+\delta+\epsilon$. The auxiliary parameter q fixes the local expansion at $z\sim 0$ of the Heun function

$$\text{HeunG}\,(t,q,\alpha,\beta,\gamma,\delta,z)=1+\frac{q}{t\gamma}z+\mathcal{O}(z^2)\,.$$

- It reduces to the hypergeometric e.g. when $\epsilon=0$ and $q=\alpha\beta t$.
- The confluent Heun (HeunC) is obtained by $t\to\infty$ [$\epsilon\to -t\epsilon$, $q\to -tq$ and $\alpha\to\alpha/\epsilon$] It reads

$$\frac{d^2w}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\epsilon\right)\frac{dw}{dz}+\frac{\alpha z-q}{z(z-1)}w=0\,.$$

and is the relevant equation for the spheroidal harmonics (angular copy) and to Kerr BH (in the radial copy, confluence is $\Lambda\to 0$).

Heun equation

- HeunG is not exactly solved in math textbooks, but `Mathematica` knows it well numerically at arbitrary precision. Usually studied as expansion in hypergeometrics [**Mano-Suzuki-Takasaki**] or continued fraction [**Leaver**] methods.
- Heun functions can be identified with Virasoro classical (irregular) conformal blocks in CFT2. **[Polyakov]**
- **[Alday-Gaiotto-Tachikawa]** (AGT) correspondence gives well defined combinatorial formulas, as they get identified with equivariant Nekrasov partition functions in $\mathcal{N} = 2$ supersymmetric gauge theories in D=4 and $SU(2)$ gauge group.
- In short, the NS limit of equivariant multi-instanton counting solves Heun equations.
- We propose to study BHPT by using these exact analytical solution.
- To master the solutions of Heun equations for actual applications to BHPT, we need to exactly compute the connection matrices relating distant *local* solutions in the z-plane (explicit analytic continuation).

BPZ and Heun equation

- Reducible Virasoro representations are characterised by null states, which correspond to primary vertices $\chi_{r,s}$ with degenerate conformal dimension $\Delta_{r,s} = \frac{Q^2}{4} - \alpha_{r,s}^2$ (Liouville theory $c > 1$; $\alpha_{r,s} = -\frac{br}{2} - \frac{s}{2b}$, $Q = b + \frac{1}{b}$ and $c = 1 + 6Q^2$.).
- The lowest non trivial null operator is at level 2 and satisfies

$$(b^{-2}\partial_z^2 + T(z))\chi_{1,2}(z) = 0\,.$$

- Conformal Ward identities on correlation functions with a degenerate are the celebrated [BPZ] equations. Schematically, given a multi-vertex operator $\mathcal{O}_V(z_1, \ldots, z_N)$ with OPE

$$T(z)\mathcal{O}_V(z_1, \ldots, z_N) \sim V_i(z; z_i, \partial_{z_i})\mathcal{O}_V(z_1, \ldots, z_N) + \text{reg.} \quad \text{as} \quad z \sim z_i$$

one gets the linear PDE BPZ equation

$$[b^{-2}\partial_z^2 + \sum_i V_i(z; z_i, \partial_{z_i})]\Psi(z) = 0\,, \quad \Psi(z) = \langle \chi_{2,1}(z)\mathcal{O}_V(z_1, \ldots, z_N)\rangle$$

- To reduce this PDE to an ODE, one gets rid of the ∂_{z_i} in the V_i by a $b \to 0$ limit (semiclassical Liouville) properly rescaling the parameters of the multi-vertex operator $\mathcal{O}$. (e.g. for a primary, $\Delta \sim b^{-2}\hat{\Delta}$, a bit more complicated for irregular) [Polyakov 1986]

BPZ and Heun equation

- Correlators in Liouville Theory holomorphically factorise in conformal blocks, that is (for example)

$$< \prod_{i=0}^{n} \Phi_{(p_i)}(z_i) >= \sum_{Q} C^{q^0}_{p_0 p_1} C^{q^1}_{q_0 p_2} \dots C^{q^{n-1}}_{q_{n-2} p_n} |\mathcal{F}\begin{pmatrix} p_0 p_1 & q_0 p_2 & \dots & q_{n-2} p_n \\ q^0 & q^1 & \dots & q^{n-1} \end{pmatrix}; z_0, \dots z_n) |^2$$

- Conformal blocks $\mathcal{F}$ with a degenerate vertex insertion are meromorphic solutions of the BPZ equation in a given pant decomposition of the punctured plane $\mathbb{C} \setminus \{z_i\}_{i=0,\dots n}$.
- Indeed, holomorphic factorisation is not unique, as it depends on the successive order of taking OPEs of the vertices: Crossing symmetry (reshuffling the order of the OPEs won't change the result) relates different local solutions of the very same differential equation.
- Therefore, local solutions of BPZ are related by the composition of linear operators (Crossing Kernel) determined by the recombination the three point functions C^r_{pq}.
- In Liouville theory, C^r_{pq} are given by the DOZZ **[Dorn-Otto-Zamolodchikov^2]** formula as a definite ratio of Γ_2-functions.

AGT and Conformal blocks

- The analytic structure of conformal blocks has been solved by the **[Alday-Gaiotto-Tachikawa]** (AGT) correspondence.
- AGT claims that BPS sectors of $\mathcal{N}=2$ supersymmetric gauge theories in D=4 can be characterised by two dimensional integrable QFTs
- In particular, S^4 partition functions of $\mathcal{N}=2$ supersymmetric quiver gauge theories with $SU(2)$ gauge nodes has been shown to coincide with Liouville multi-vertex correlation functions. For example for $SU(2)$ gauge theory and $N_f=4$, one

$$Z^{S^4}_{SU(2),N_f=4}(\{m_i\},q) = < \prod_{i=1}^{4} e^{\alpha_i \phi(z_i)} >^{\mathbb{P}^1}_{Liou}$$

  where $q=\frac{z_{12}z_{34}}{z_{13}z_{24}}$, $\{m_i\}\sim\{\alpha_i\}$ (up to lin.comb.) and $\epsilon_1/\epsilon_2 \sim b^{-2}$.
- To read the details of this formula one needs to separate the N/S patches contributions on S^4 and compare with the holomorphic factorised form of Liouville correlation functions.

AGT and Conformal blocks

- Using equivariant localisation, one has **[Pestun]**

$$Z^{S^4}_{SU(2),N_f=4}(\{m_i\},q) = \int da\,|Z^{\mathbb{R}^4}_{SU(2),N_f=4}(\{m_i\},q,a)|^2$$

  Here

$$Z^{\mathbb{R}^4}_{SU(2),N_f=4}(\{m_i\},q,a) = \mathcal{Z}_{1loop}(\{m_i\},a)\mathcal{Z}_{inst}(\{m_i\},q,a)$$

  where $\mathcal{Z}_{1loop}$ is a *given ratio of* Γ_2*-functions* and

$$\mathcal{Z}_{inst} = \sum_{Y_1,Y_2|\,\mathrm{Y.T.}} q^{|Y_1|+|Y_2|}\mathcal{R}(Y_1,Y_2;m_i,a)$$

  with $\mathcal{R}$ a *given rational function in its arguments.* **[Nekrasov]**
- In CFT

$$< \prod_{i=1}^{4} e^{\alpha_i\phi(z_i)} >^{\mathbb{P}^1}_{Liou} = \int d\alpha C^{\alpha}_{\alpha_1\alpha_2}C_{\alpha\alpha_3,\alpha_4}|\mathcal{F}(\alpha_1\alpha_2{}^{\alpha}\alpha_3\alpha_4|q)|^2$$

- Comparing, one consistently finds that

$$|\mathcal{Z}_{1loop}|^2 = CC \quad \text{and} \quad \mathcal{Z}_{inst} = \mathcal{F}.$$

- therefore Virasoro conformal blocks are calculable in terms of well defined combinatorial formulas.

CFT2 and Heun equation

- To apply to the computation of conformal blocks with degenerate insertions, one needs a multi-point generalisation of these prototypical formulas. It exists and it is proved.
- The insertion of degenerate primaries in gauge theory corresponds to insertions of surface operators. These can be obtained from quiver theories in a given ungauging limit.
- The classical limit in Liouville theory is known in the gauge theory as the Nekrasov-Shatashvili limit, where one considers the theory on a general Ω-background (equivariant parameters labeling $(\mathbb{C}^*)^2$ action on $\mathbb{C}^2$) and reduces the vev of the surface operator $\Psi(z)$

$$\Psi(z) \propto e^{-\frac{1}{\epsilon_1 \epsilon_2}\left(F^{\text{inst}}(\epsilon_1)+\epsilon_2 \mathcal{W}(z;\epsilon_1)+\mathcal{O}(\epsilon_2^2)\right)}, \quad (\epsilon_1, \epsilon_2) \rightarrow (\epsilon, 0)$$

- To obtain Heun functions as the classical limit of conformal blocks, one has to normalise and gets (four points at $(0, t, 1, \infty)$ and $0 < |z| < |t|$) the two linear independent solutions of Heun equation from the two blocks with shifted momenta (there are two channels in the OPE of the primaries with the degenerate)

$$\lim_{c\to\infty} \frac{\mathcal{F}(\{\alpha_i\}, \pm|z/t, t)}{\mathcal{F}(\{\alpha_i\}|t)} \propto \mathrm{HeunG}_{\pm}(z)$$

See **[B.-Iossa-Panea Lichtig-Tanzini]** (A whole class of confluences done explicitly)

CFT2 and Heun equation

- The last piece we need is the solution to the connection problem for Heun (and confluences)
- This is explicitly provided by the classical limit of the crossing relations of conformal blocks *after normalisation*.
- The crossing symmetry relations on degenerate conformal blocks are obtained by comparing the holomorphic factorization formulas before and after a global $\mathbb{P}SL(2,\mathbb{C})$ transformation γ.
- The resulting formulas come in the schematic form

$$\mathcal{F}^{(\gamma)}(\{\alpha_i\}|\gamma(w_i),\gamma(z)) = [Jac_\gamma](w_i,z)\sum_{\pm}\mathcal{C}_\pm(b\{\alpha_i\})\mathcal{F}(\{\alpha_i \pm e_i^\gamma b/2\}|w_i,z)$$

where $e_i^\gamma \in \{0,1\}$ label the momenta of the vertices crossed by the particular degenerate primary. $\mathcal{C}_\pm(b\{\alpha_i\})$ are shifted versions of the hypergeometric connection matrices

$$\mathcal{C}_{hyper} \sim \frac{\Gamma\Gamma}{\Gamma\Gamma}$$

arising from reshuffling the arguments in the Γ_2's of the DOZZ. The crossing kernels **do not compose as matrices (because of the shift in α_i) and this leaves crucial terms behind in the $b \to 0$ limit.**

CFT2 and Heun equation

- After normalisation and the classical limit one gets the connection matrices of Heun functions

$$\mathcal{C}_{\theta\theta'} = e^{-\theta\kappa_D}\mathcal{M}^{hyper}_{\theta\theta'}e^{-\theta'\kappa'_D} \quad \text{for simple crossings}$$

$$\mathcal{C}_{\theta\theta'} = \sum_{\theta"=\pm} e^{-\theta\kappa_D}\mathcal{M}^{hyper}_{\theta\theta"}e^{-\theta"\kappa_D"}\mathcal{M}^{hyper}_{\theta"\theta'}e^{-\theta'\kappa'_D}\ldots \quad \text{for multiple crossings}$$

  where κ_D is a given function of the CFT momenta and modulus.
- In gauge theory language κ_D is a combination of the quantum SW periods a and a_D, we refer to them as NS functions.
- Technically, these κ_D terms arise because in the $b \to 0$ limit the multiple shift in the crossed momenta of the normalised blocks exponentiates leaving them behind.
- The intertwining of the hypergeometric connection matrix with the exponential of the quantum periods gives the **explicit solution to the connection problem for Heun equations**.

BHPT and Heun functions

For Kerr BHs (**CHeun**), from the explicit form of the connection coefficients we compute **[B.,Panea-Lichtig,Iossa,Tanzini]**

- the **grey body factor** (that is the absorption coeff) $\sigma(\omega) \sim |\mathcal{C}^{in,out}_{h,\infty}/\mathcal{C}^{in,in}_{h,\infty}|^2$
- **exact QNMs quantization** $\mathcal{C}^{in,in}_{h,\infty} = 0$ (**previously conj. [Aminov-Grassi-Hatsuda]**)
- **Love Number coefficient** $k_L = \mathcal{C}^{in<}_{h-pw}/\mathcal{C}^{in>}_{h-pw}$
- **exact formula for the separation constant** $e^{ia\pi} = -1$ [angular dictionary]
- **Kerr BH Compton amplitude analysis** $A_{BH} = A_{point}(1 + A_{tidal})$ made unambiguous in terms of the two scattering channels, analytic properties [**Bautista-B-Iossa-Tanzini-Zhou]**

For (A)dS-Schwarzschild QNMs (**Heun**) **[Aminov-Arnaudo-B-Grassi-Tanzini]**

- **explicit power series of the QNMs in R_h for small BH**

  dS-Schwarzschild: purely imaginary, leading orders are negative. AdS-Schwarzschild w/Dirichlet bc: negative im-part, $Im[\omega_{n,\ell,s}(R_h)] \sim -c(n,s)R_h^{2\ell+2} + h.o. \quad \text{with} \quad c(n,s) > 0$ [exact BH stability - it's open for Kerr]
- **explicit power series of the QNMs in R_h^{-1} for large BH** [AdS-Schwarzschild w/Robin bc, improved precision wrt earlier literature.]

Effective actions in (A)dS-Kerr background

- One loop effective actions for spin s are obtained by integrating out fluctuation modes ψ leaving behind the standard correction

$$e^{-S_{eff,s}} \sim e^{-S^0_{eff,s}} \int D[\psi] e^{-\psi K_s \psi} \sim e^{-S^0_{eff,s}} \frac{1}{\sqrt{\mathrm{Det}' K_s}}$$

- In (A)dS-Kerr background the above determinants expand in the frequency and the spheroidal quantum numbers as the spectral problem separates for all spins.

$$\psi = \int d\omega \sum_{m,l} e^{i\phi m + it\omega} S_{s,\omega,m,l}(cos(\theta)) R_{s,\omega,m,l}(r)$$

so that

$$\frac{1}{\sqrt{\mathrm{Det}' K_s}} \sim \prod_{l,m,\omega} \frac{1}{\sqrt{\mathrm{Det}' \mathcal{D}^{\mathrm{Heun}}{}_{(s,l,m,\omega)}}}$$

where $\mathcal{D}^{\mathrm{Heun}}\psi = 0$ is the Heun equation, after the proper radial dictionary relating the abstract parameterisation to the BH data and the specific quantum numbers.

- Each partial contribution is computed by an auxiliary separated problem in the radial coordinate only. One can use exact quantization formulas (Gelfand-Yaglom theorem) for each partial wave contribution and then regularise the infinite product. See **[Dunne]**.

Effective actions in (A)dS-Kerr background

- Gelfand-Yaglom theorem is usually formulated for smooth potentials and boundary conditions at smooth points.

  (GY thm.: $\mathrm{Det}(\mathcal{D}^{\mathrm{II}}) \propto W_{ronskian}(\psi_1, \psi_2)$, with $\mathcal{D}^{\mathrm{II}}\psi_i = 0$ and appropriate b.c.)

- The differential operators $\mathcal{D}^{\mathrm{Heun}}$ entering the partial wave contributions for BH backgrounds are singular in normal form and the boundary conditions are placed at singularities (∞ or horizons).

- We developed a version of the Gelfand-Yaglom theorem for determinants of differential operators with regular (and irregular) singularities (and corresponding boundary conditions). We find

$$\mathrm{Det}(\mathcal{D}^{\mathrm{Heun}}) = \mathcal{C}^{\mathrm{Heun}}/\mathcal{N}_{\mathrm{Hyper}}$$

  where $\mathcal{C}^{\mathrm{Heun}}$ is a proper Heun connection coefficient and $\mathcal{N}_{\mathrm{Hyper}}$ a normalisation via a reference Hypergeometric problem (Rindler-like region subtraction). See also **[Law,Parmentier]**

Effective actions in (A)dS-Kerr background

- Substituting the partial waves determinant contributions in the one loop effective action, one gets (we did it for $s = 0, 1, 2$) **[Arnaudo-B-Tanzini2405]**

$$S_{1\mathrm{loop},s} = \frac{1}{2}\int d\omega \sum_{m,l} \log\left[\mathcal{C}^{\mathrm{Heun}}/\mathcal{N}_{\mathrm{Hyper}}\right]_{s,m,l,\omega}$$

  an explicit formula for the effective actions around (A)dS-Kerr (and (A)dS-Schwarzschild) BHs in terms of the NS function.

- There are also analog formulas for Kerr and for extremal (A)dS-Kerr (both described by CHeun in the radial sector)

$$\mathcal{C}^{\mathrm{Heun}} \to \mathcal{C}^{\mathrm{CHeun}} \quad \text{and} \quad \mathcal{N}_{\mathrm{Hyper}} \to \mathcal{N}_{\mathrm{Wittaker}}$$

Euclidean continuation & improved DHS

- We can apply these formulas to the analysis of quantum correction of BH entropy after Euclidean rotation.
- The Euclidean rotation implies the usual thermal S^1 compactification.

$$\int d\omega \quad \to \quad \sum_{\omega \in \text{thermal}}$$

The thermal Matsubara frequencies are computed by requiring the good analytic continuation of the corresponding wave functions.

[Castro, Keeler and Szepietowski]

- For rotating BHs, in the QNMs sector ($\overline{QNM}$s are in $1-1$ because of PT symmetry) one gets

$$\omega_k^{(M)} = m\Omega_H + 2\pi i T_H (k+s)$$

where m is the angular q.nr. and $k \geq 0$ the thermal q.nr. while the temperature and the angular velocity at the event horizon read

$$T_H = \frac{R_h - R_i}{8\pi M R_h}, \qquad \Omega_H = \frac{a_{\text{BH}}}{2MR_h}. \tag{1}$$

Euclidean continuation & improved DHS

- Substituting all the variables, one gets an explicit (improved) Denef-Hartnoll-Sachdev formula for (A)dS-Kerr. **[Arnaudo-B-Tanzini2412]**
- $$Z^{(s)}_{1\,loop} = \left| \prod_{k\geq 0} \left[\prod_{l\geq 0} \prod_{-l\leq m\leq l} \left[\mathcal{C}^{\text{Heun}} / \mathcal{N}_{\text{Hyper}} \right]^{-1}_{s,m,l,\omega} \right]_{\omega=\omega_k^{(M)}} \right|$$

  keeping into account also $\overline{QNM}$s.
- see the paper for the fully explicit formulas.

Euclidean version and its scaling properties at NEK

- To study the (leading) scaling properties in T_H of the euclidean one-loop effective action, one has to identify the QNMs which are parametrically small in the near extremal limit.
- QNMs are the resonant modes of the BH. The resonance condition is the vanishing of the very same connection coefficient appearing in the determinant

$$\left[\mathcal{C}^{\text{Heun}}\right]_{s,m,l,\omega} = 0$$

- The near extremal limit is the confluent limit of Heun and the CHeun connection is related to the Heun one by

$$\mathcal{C}^{\text{Heun}} \propto 1/\Gamma(B)\left(\mathcal{C}^{\text{CHeun}} + \mathcal{O}(\delta)\right)$$

  where B is a function of the parameter δ (that is $\propto T_H$) which measures the distance between the singularities (horizons).
- The QNMs we look for are therefore among the solutions of the condition

$$B = -n\,, \text{with } n \in \mathbb{N}_*$$

  which determines the set of zeros $\omega_\star$ in the frequencies.

Euclidean version and its scaling properties at NEK

- After substituting the gravitational dictionary one finds

$$\omega_\star = \frac{m}{2a_{BH}} + \mathcal{O}(T_H)$$

  which is real at the leading order for any overtone n. Therefore the whole set of modes which are decoupling in the NEK regime are Zero Damping Modes (ZDM) (vanishing imaginary part) and we have

$$det(Kerr)_{l,m,s} = det(ZDM)_{l,m,s}\left[det(EKerr)_{l,m,s} + \mathcal{O}(T_H)\right]$$

- The factor to be analysed to compute the scaling of the one loop effective action is therefore the contribution from the ZDM sector to the QNMs

$$\prod_{k\geq 0}\prod_{l\geq 0}\prod_{-l\leq m\leq l}\Gamma\left(B|_{s,m,l,\omega=\omega_k^{(M)}}\right)$$

- By direct inspection, one finds that only the $m=0$ modes (Hod's modes) can give a contribution to the leading scaling term, which finally reads (for $s>0$!!! for $s=0$ one should improve to next order.)

$$\prod_{k\geq 0}\prod_{l\geq 0}[T_h(k+s)]^{-1} \sim T_H^{\frac{s}{2}-\frac{1}{4}}$$

Scaling properties of gravitational perturbation at NEK

- The gravitational perturbations are described by the Teukolsky equation with $s = \pm 2$ which correspond to the two helicity states of the on-shell graviton.
- $s \to -s$ swaps QNMs and anti-QNMs in the analysis of Matsubara frequencies
- We conclude that

$$Z^{\text{grav,NEK}}_{\text{1-loop}} = Z^{\text{s=2,NEK}}_{\text{1-loop}} Z^{\text{s=-2,NEK}}_{\text{1-loop}} \sim T_H^{\frac{3}{2}} \, Z^{\text{EK}}_{\text{1-loop}},$$

  where $Z^{\text{EK}}_{\text{1-loop}}$ is the one-loop partition function of the extremal Kerr geometry.
- It's nature is universal! It depends only on the ZDM contribution which is generated only by the confluence procedure (e.g. it is independent on Λ).

Recap

- We have used equivariant multi-instanton calculus in D=4 $\mathcal{N} = 2$ supersymmetric gauge theories to concretely solve spinning BH perturbation theory.
- This is done by translating the relevant differential equations in CFT2 and characterise their solutions via AGT correspondence in terms of the NS function.
- This gives a new exact expressions of one loop effective actions in *(A)dS-Kerr* BHs backgrounds.
- We have analysed the scaling behaviour in the temperature in the near extremal regime of the euclidean semiclassical gravitational partition function finding its leading scaling behaviour.

Open problems

- Further computations in BH thermodynamics: $s = 0$, fermions, susy cases, NLO in T_H. Universality (confluences vs different extremal limits).
- More application on GR: e.g. QQNMs and 2^{nd} order BHPT, more on gravitational amplitudes, applications to primordial GWs.
- The method extends to general Fuchsian equations and their confluences, explicit expressions for the connection matrices. [The solution corresponds to the partition functions in a linear quiver $\mathcal{N} = 2$ gauge theories with $SU(2)$ nodes (class $\mathcal{S}$ at genus 0) with a surface operator insertion.]
- Higher rank confluences are harder. [Already a basic CFT2 problem, irregular states]
- Math open problem: obtain Nekrasov combinatorics from the analysis of the relevant differential equations

Remarks

- Here we DO NOT claim that there is any sort of duality between the gravitational problem and the gauge theory, but just that there are some quantities on the two sides which obey the same 2^{nd} order linear ODE.
- Integrable systems perspective (NS function is the YY function, ODE/IM techniques) **[Fioravanti]**
- The detailed analysis of gravitational wave emitted by BHs or by different compact objects via exact techniques is potentially a rich method to obtain a fine structure spectrum to distinguish different objects from far away and test possible higher order or quantum corrections to classical GR.

  See e.g. **[Bianchi-Consoli-Grillo-Morales, Consoli-Fucito-Morales-Poghossian]**
- The method works because of convergence of the power series. [Convergence of the Nekrasov partition function.] **[Arnaudo-B-Tanzini].**
- Many other problems (e.g. in QM, GR, QFT) reduce to Heun equation (or higher Fuchsian equations) or one of its confluences/reductions **[Fitziev, Hortacsu].** Our results and techniques can be exported to other kinds of problems to obtain interesting new results.

Thank you!

Technical slides:

CFT - CB		$SU(2)$ Gauge Theory	Heun
$\mathfrak{F}$	Regular	$N_f = 4$	HeunG
${}_1\mathfrak{F}$	Confluent	$N_f = 3$	HeunC
${}_{\frac{1}{2}}\mathfrak{F}$	Reduced Confluent	$N_f = 2$ asymmetric	HeunRC
${}_1\mathfrak{D}_1$	Doubly Confluent	$N_f = 2$ symmetric	HeunDC
${}_1\mathfrak{E}_{\frac{1}{2}}$	Reduced Doubly Confluent	$N_f = 1$	HeunRDC
${}_{\frac{1}{2}}\mathfrak{E}_{\frac{1}{2}}$	Doubly Reduced Doubly Confluent	$N_f = 0$	HeunDRDC

$$
\begin{array}{ccccccc}
 & & & {}_{\frac{1}{2}}\mathfrak{F} & & & \\
 & & \nearrow & & \searrow & & \\
\mathfrak{F} \rightarrow & {}_1\mathfrak{F} & & & & {}_1\mathfrak{E}_{\frac{1}{2}} & \rightarrow {}_{\frac{1}{2}}\mathfrak{E}_{\frac{1}{2}} \\
 & & \searrow & & \nearrow & & \\
 & & & {}_1\mathfrak{D}_1 & & &
\end{array}
$$

Connection formulas: example - the regular block before $c \to \infty$

Expand the following at $0 < |z| < |t| < 1$

$$\langle \Delta_\infty | V_1(1) V_t(t) \Phi(z) | \Delta_0 \rangle = \sum_{\theta=\pm} \int d\alpha \, C^{\alpha_{0\theta}}_{\alpha_{2,1}\alpha_0} C^{\alpha}_{\alpha_t \alpha_{0\theta}} C_{\alpha_\infty \alpha_1 \alpha} \left| \mathcal{F}\left({}^{\alpha_\infty}_{\alpha_1} \, \alpha \, {}^{\alpha_t} \, \alpha_{0\theta} \, {}^{\alpha_{2,1}}_{\alpha_0} | t, \frac{z}{t}\right) \right|^2$$

The same correlator can be expanded for $z \sim t$ and $t << 1$ after the transformation $\gamma(x) = \frac{x-t}{1-t}$

$$\langle \Delta_\infty | V_1(1) V_t(t) \Phi(z) | \Delta_0 \rangle = \left| (1-t)^{\Delta_\infty - \Delta_1 - \Delta_t - \Delta_{2,1} - \Delta_0} \right|^2 \langle \Delta_\infty | V_1(1) V_0\left(\frac{t}{t-1}\right) \Phi\left(\frac{z-t}{1-t}\right) | \Delta_t \rangle =$$

$$= \sum_{\theta=\pm} \int d\alpha \, C^{\alpha_{t\theta}}_{\alpha_{2,1}\alpha_t} C^{\alpha}_{\alpha_0 \alpha_{t\theta}} C_{\alpha_\infty \alpha_1 \alpha} \left| (1-t)^{\Delta_\infty - \Delta_1 - \Delta_t - \Delta_{2,1} - \Delta_0} \mathcal{F}\left({}^{\alpha_\infty}_{\alpha_1} \, \alpha \, {}^{\alpha_0} \, \alpha_{t\theta} \, {}^{\alpha_{2,1}}_{\alpha_t} | \frac{t}{t-1}, \frac{t-z}{t}\right) \right|^2 .$$

Comparing the two: eliminate a common $C_{\alpha_\infty \alpha_1 \alpha}$; express the resulting $(CC)^\theta_{LHs} = |\mathcal{M}_{\theta\theta'}|^2 (CC)^{\theta'}_{LHs}$ using DOZZ, split in holomorphic and anti-holomorphic getting

$$\mathcal{F}\left({}^{\alpha_\infty}_{\alpha_1} \, \alpha \, {}^{\alpha_t} \, \alpha_{0\theta} \, {}^{\alpha_{2,1}}_{\alpha_0} | t, \frac{z}{t}\right)$$

$$= \sum_{\theta'=\pm} \mathcal{M}_{\theta\theta'}(b\alpha_0, b\alpha_t; b\alpha) e^{i\pi(\Delta - \Delta_0 - \Delta_{2,1} - \Delta_t)} (1-t)^{\Delta_\infty - \Delta_1 - \Delta_t - \Delta_{2,1} - \Delta_0} \mathcal{F}\left({}^{\alpha_\infty}_{\alpha_1} \, \alpha \, {}^{\alpha_0} \, \alpha_{t\theta'} \, {}^{\alpha_{2,1}}_{\alpha_t} | \frac{t}{t-1}, \frac{t-z}{t}\right)$$

where $\mathcal{M}_{\theta\theta'}$ are the hypergeometric connection coefficients.

Kerr:Teukolski-CHeun dictionary

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dt\, d\phi$$

where

$$\Sigma = r^2 + a^2\cos^2\theta\,, \quad \Delta = r^2 - 2Mr + a^2 = (r-r_-)(r-r_+)$$

$$\Phi_s = e^{im\phi - i\omega t} S_{\lambda,s}(\theta, a\omega) R_s(r)$$

$$\Delta\frac{d^2R}{dr^2} + (s+1)\frac{d\Delta}{dr}\frac{dR}{dr} + \left(\frac{K^2 - 2is(r-M)K}{\Delta} - \Lambda_{\lambda,s} + 4is\omega r\right)R = 0\,,$$

$$\partial_x(1-x^2)\partial_x S_\lambda + \left[(cx)^2 + \lambda + s - \frac{(m+sx)^2}{1-x^2} - 2csx\right]S_\lambda = 0$$

where $x = \cos\theta$, $c = a\omega$, $K = (r^2+a^2)\omega - am$ and $\Lambda_{\lambda,s} = \lambda + a^2\omega^2 - 2am\omega$. We determine the separation constant λ as the eigenvalue of the angular eq. by imposing regularity at $\theta = 0, \pi$. Both the above equation reduce to CHeun. For example the radial, by setting $z = \frac{r-r_-}{r_+-r_-}$ and $\psi(z) = \Delta(r)^{\frac{s+1}{2}} R(r)$ becomes

$$\frac{d^2\psi(z)}{dz^2} + V_r(z)\psi(z) = 0 \quad where \quad V_r(z) = \frac{1}{z^2(z-1)^2}\sum_{i=0}^{4}\hat{A}^r_i z^i$$

$$\hat{A}^r_0 = \frac{a^2(1-m^2) - M^2 + 4amM\omega(M - \sqrt{M^2-a^2}) + 4M^2\omega^2(a^2 - 2M^2) + 8M^3\sqrt{M^2-a^2}\,\omega^2}{4(a^2-M^2)} + (is)\frac{am\sqrt{M^2-a^2} - 2a^2M\omega + 2M^2\omega(M - \sqrt{M^2-a^2})}{2(a^2-M^2)} - \frac{s^2}{4}$$

$$\hat{A}^r_1 = \frac{4a^2\lambda - 4M^2\lambda + (8amM\omega + 16a^2M\omega^2 - 32M^3\omega^2)\sqrt{M^2-a^2} + 4a^4\omega^2 - 36a^2M^2\omega^2 + 32M^4\omega^2}{4(a^2-M^2)} + (is)\left(-i + \frac{(2a^2\omega - am)\sqrt{M^2-a^2}}{a^2-M^2}\right) + s^2$$

$$\hat{A}^r_2 = -\lambda - 5a^2\omega^2 + 12M^2\omega^2 - 12M\omega^2\sqrt{M^2-a^2} + (is)(i - 6\omega\sqrt{M^2-a^2}) - s^2$$

$$\hat{A}^r_3 = 8a^2\omega^2 - 8M^2\omega^2 + 8M\omega^2\sqrt{M^2-a^2} + (is)4\omega\sqrt{M^2-a^2}$$

$$\hat{A}^r_4 = 4(M^2-a^2)\omega^2\,,$$

Explicit conformal blocks (regular case)

$$\mathcal{F}\left(\begin{matrix}\alpha_1 & & \alpha_t\\ \alpha_\infty & \alpha & \alpha_0\end{matrix};t\right) = t^{\Delta-\Delta_t-\Delta_0}(1-t)^{-2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}+\alpha_t)}\times$$

$$\times\sum_{\vec{Y}} t^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right)\prod_{\theta=\pm} z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_t+\theta\alpha_0\right) z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_1+\theta\alpha_\infty\right)$$

$$z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\mu\right) = \prod_{k=1,2}\prod_{(i,j)\in Y_k}\left(\alpha_k+\mu+b^{-1}\left(i-\frac{1}{2}\right)+b\left(j-\frac{1}{2}\right)\right)$$

$$z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right) = \prod_{k,l=1,2}\prod_{(i,j)\in Y_k} E\left(\alpha_k-\alpha_l,Y_k,Y_l,(i,j)\right)\prod_{(i',j')\in Y_l}\left(Q-E\left(\alpha_l-\alpha_k,Y_l,Y_k,(i',j')\right)\right)$$

$$E\left(\alpha,Y_1,Y_2,(i,j)\right) = \alpha - b^{-1}L_{Y_2}((i,j)) + b\left(A_{Y_1}((i,j))+1\right) .$$

$$\mathcal{F}\left(\begin{matrix}\alpha_1 & & & \alpha_{2,1}\\ \alpha_\infty & \alpha & {}^{\alpha_t}\ \alpha_{0\theta} & \alpha_0\end{matrix};t\frac{z}{t}\right) =$$

$$= t^{\Delta-\Delta_t-\Delta_{0\theta}} z^{\frac{bQ}{2}+\theta b\alpha_0}(1-t)^{-2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}-\alpha_t)}\left(1-\frac{z}{t}\right)^{-2(\frac{Q}{2}+\alpha_t)(\frac{Q}{2}+\alpha_{2,1})}(1-z)^{-2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}+\alpha_{2,1})}\times$$

$$\times\sum_{\vec{Y},\vec{W}} t^{|\vec{Y}|}\left(\frac{z}{t}\right)^{|\vec{W}|} z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right) z_{\text{vec}}\left(\vec{\alpha_{0\theta}},\vec{W}\right) z_{\text{bifund}}\left(\vec{\alpha},\vec{Y},\vec{\alpha_{0\theta}},\vec{W};\alpha_t\right)\times$$

$$\times\prod_{\sigma=\pm} z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_1+\sigma\alpha_\infty\right) z_{\text{hyp}}\left(\vec{\alpha_{0\theta}},\vec{W},\alpha_{2,1}+\sigma\alpha_0\right)$$

$$z_{\text{bifund}}\left(\vec{\alpha},\vec{Y},\vec{\beta},\vec{W};\alpha_t\right) = \prod_{k,l=1,2}$$

$$\times\prod_{(i,j)\in Y_k}\left[E\left(\alpha_k-\beta_l,Y_k,W_l,(i,j)\right)-\left(\frac{Q}{2}+\alpha_t\right)\right]\prod_{(i',j')\in W_l}\left[Q-E\left(\beta_l-\alpha_k,W_l,Y_k,(i',j')\right)-\left(\frac{Q}{2}+\alpha_t\right)\right]$$