Can you hear the Planck mass?

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mainly based on 2406.00095 with
De Luca, De Ponti, Mondino

Milano, 14 January, 2025

Introduction

- In a compactification, gravitational potential: [e.g. $D = 4 + d \to 4$]

$$U \sim \frac{1}{m_{\mathrm{Pl},4}^2 r} \qquad \text{for } r \gg r_{\mathrm{KK}}$$

$$U \sim U_D \propto \frac{m_{\mathrm{Pl},D}^{2-D}}{r^{D-3}} \qquad \text{for } r \ll r_{\mathrm{KK}}$$

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$\Rightarrow$

Weyl law:

$$m_k \sim 2\pi \left(\frac{k}{\omega_d V(X)} \right)^{1/d}$$

$k \to \infty$

[Weyl '11,... Levitan '52]

$\omega_d = V(B_d)$
unit d-dim. ball

- but for warped compactification

$$e^{2A}(\mathrm{d}s_4^2 + \mathrm{d}s_d^2(X))$$

$$m_{\mathrm{Pl},4}^2 = m_{\mathrm{Pl},D}^{D-2} V_A(X)$$

$$\equiv$$

$$\int_X \mathrm{d}^d y \mathrm{e}^{(D-2)A} \sqrt{g}$$

and yet the Weyl law is still

$$m_k \sim 2\pi \left(\frac{k}{\omega_d V(X)}\right)^{1/d}$$

even in presence of singularities, if they are nice enough

[Hörmander '68, … Ambrosio, Honda, Tewodrose '17, Zhang, Zhu '17]

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$$V_A(X) \equiv \int_X \mathrm{d}^d y \, e^{(D-2)A} \sqrt{g}$$

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- What's going on?

The solution to the puzzle will involve the wavefunctions
and a property we call *weighted quantum ergodicity*

Plan

- Spin-two KK tower
- Weyl law from gravitational potential
- Quantum ergodicity
- The role of singularities

Spin-two KK tower

- KK masses: spectrum of internal diff. operators

famous review [Duff, Nilsson, Pope '85]

Example:
Freund–Rubin

Table 5
Mass operators from the Freund–Rubin ansatz

Spin	Mass operator
2^+	Δ_0
$(3/2)^{(1),(2)}$	$\not{D}_{1/2}+7m/2$
$1^{-(1),(2)}$	$\Delta_1+12m^2\pm 6m(\Delta_1+4m^2)^{1/2}$
1^+	Δ_2
$(1/2)^{(4),(1)}$	$\not{D}_{1/2}-9m/2$
$(1/2)^{(3),(2)}$	$3m/2-\not{D}_{3/2}$
$0^{+(1),(3)}$	$\Delta_0+44m^2\pm 12m(\Delta_0+9m^2)^{1/2}$
$0^{+(2)}$	Δ_L-4m^2
$0^{-(1),(2)}$	$Q^2+6mQ+8m^2$

- Model-dependent in general, but universal for spin-two

[Csaki, Erlich, Hollowood, Shirman'00;
Bachas, Estes '11]

'weighted Laplacian'

$$\Delta_f \psi \equiv -\mathrm{e}^{-f}\nabla^m(\mathrm{e}^f\nabla_m\psi)$$

$$\Delta_f\psi_k = m_k^2\psi_k$$

$$f=(D-2)A$$

$$\mathrm{e}^{2A}(\mathrm{d}s_4^2+\mathrm{d}s_d^2(X))$$

- Several universal bounds on m_k

[De Luca, De Ponti, Mondino, AT '21–23]

Usually one needs a local inequality of the type

$$R_{mn} - \nabla_m \nabla_n f + \tfrac{1}{n-N} \nabla_m f \nabla_n f \geqslant 0$$

$$f = (D-2)A$$
$$N = 2 - d < 0$$

and this indeed follows from internal Einstein, in string theory and more generally

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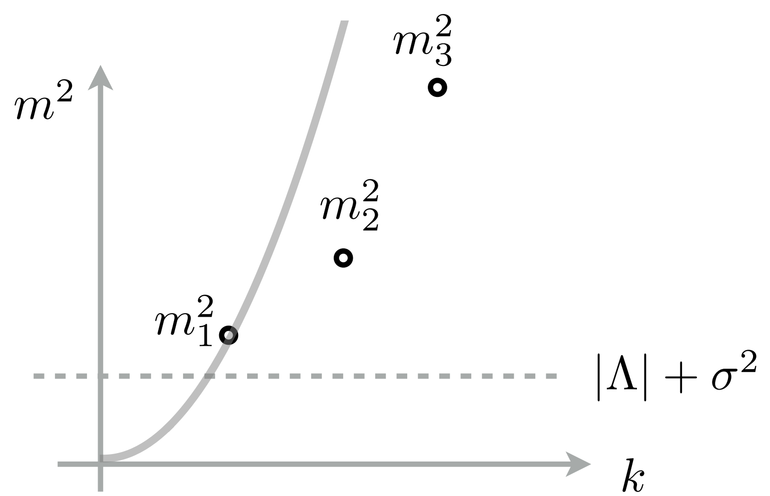
- A fun example:

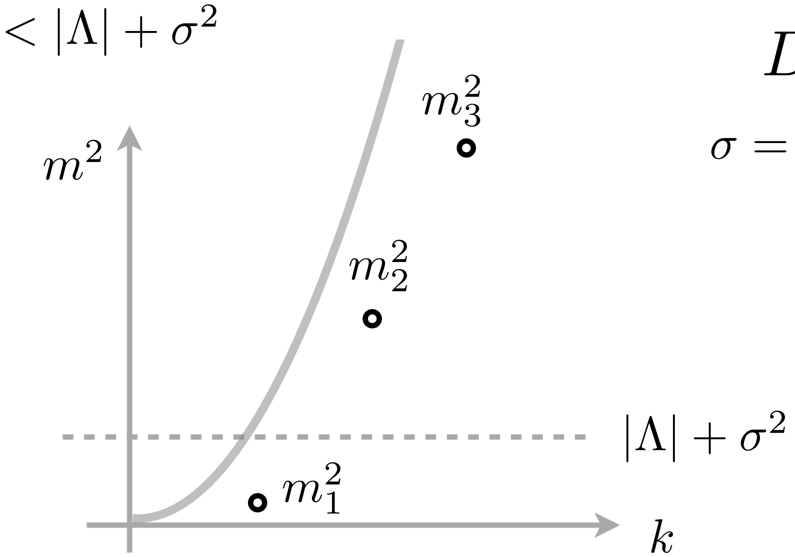
$$m_k^2 < 600 k^2 \max\left\{ m_1^2,\ |\Lambda| + \sigma^2 \right\}$$

$m_1^2 > |\Lambda| + \sigma^2$



higher masses constrained by m_1^2

$m_1^2 < |\Lambda| + \sigma^2$



higher masses constrained by $|\Lambda| + \sigma^2$

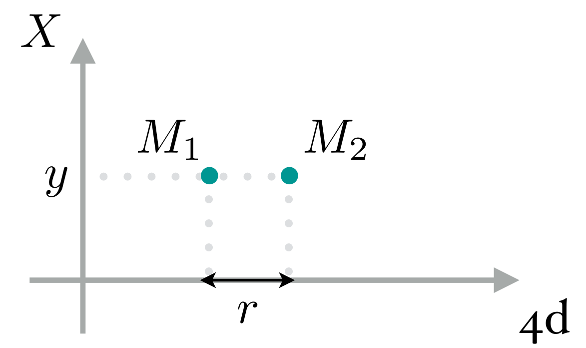
$$D = d + n$$
$$\sigma = \sqrt{D-2}\, \sup |\mathrm{d}A|$$

Weyl law from gravity

- $A = 0$ for now: unwarped

$$U \propto \frac{1}{m_{\mathrm{Pl},4}^2 r} \sum_k \psi_k^2(y) \mathrm{e}^{-m_k r} \quad \underset{r \to 0}{\overset{\text{expectation:}}{\sim}} \quad \frac{d!\,\omega_d}{(2\pi)^d} \frac{m_{\mathrm{Pl},D}^{2-D}}{r^{d+1}}$$

$$m_{\mathrm{Pl},4}^2 = m_{\mathrm{Pl},D}^{D-2} V(X)$$



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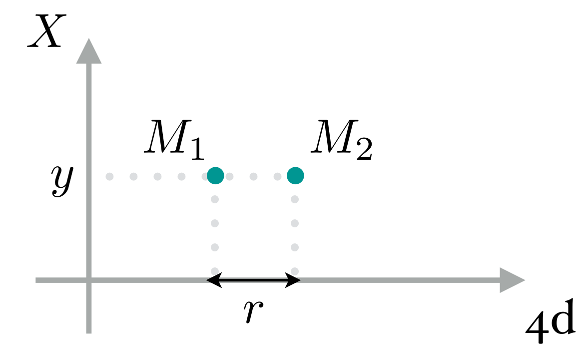

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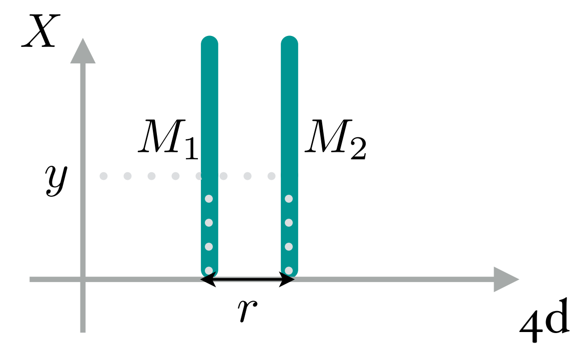
- smear particles on internal space: $\int_X \mathrm{d}^d y \sqrt{g}$ on both sides



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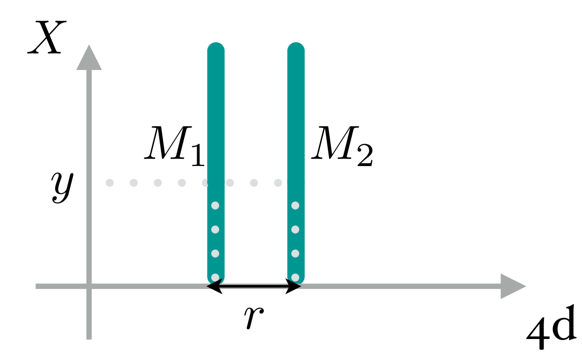

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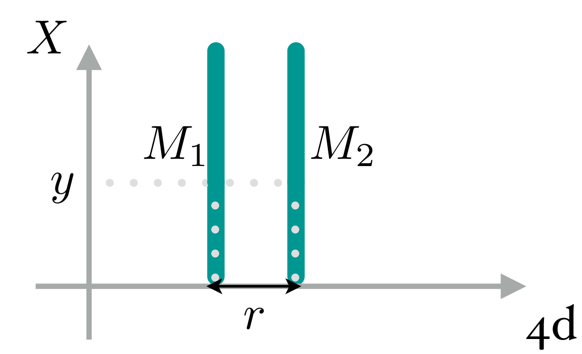
use 'Karamata's Tauberian theorem'

idea: $\sum_k \mathrm{e}^{-a k^{1/d} r} \sim \frac{1}{(ar)^d} \int_0^\infty \mathrm{d}k \mathrm{e}^{-k^{1/d}} = \frac{d!}{(ar)^d}$

$$\Rightarrow \quad m_k \sim 2\pi \left(\frac{k}{\omega_d V(X)} \right)^{1/d} \quad \checkmark$$

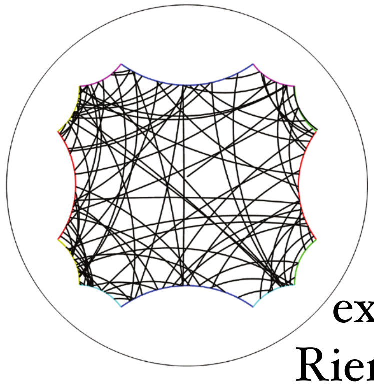
[similar to heat kernel proof]



Ergodicity

- classical ergodicity: for almost all initial conditions

  ⇨ trajectory dense in phase space



example on a
Riemann surface

pictures: [Dyatlov '21, '23]

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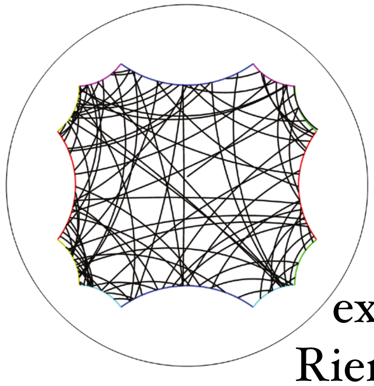

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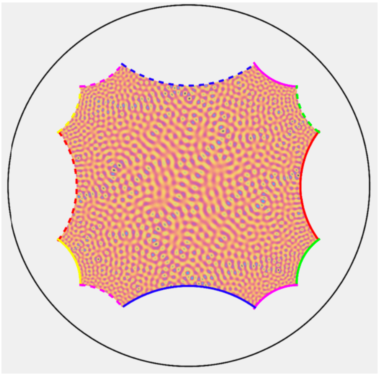
- classical ergodicity: for almost all initial conditions

$\Rightarrow$ trajectory dense in phase space

$\Downarrow$ [Schnirelman '74, Colin de Verdière '85, Zelditch '87]

- quantum ergodicity: almost all ψ_k oscillate around constant



$$\lim_{\substack{k\to\infty \\ k\notin e}} \frac{\int_B \sqrt{g}\psi_k^2}{\int_X \sqrt{g}\psi_k^2} = \frac{V(B)}{V(X)} \qquad \forall B \subset X$$

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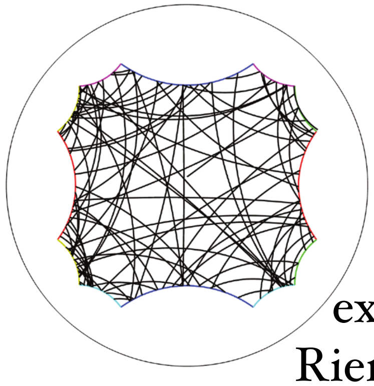

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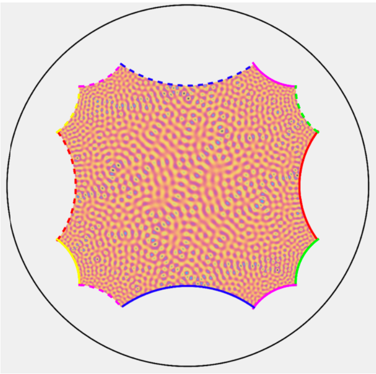

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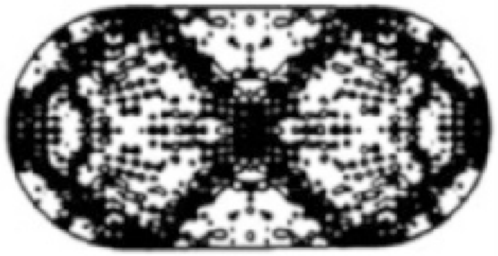
- occasional eigenfunction can be 'scarred':
  peaked around classical closed trajectory

example on
'Bunimovich stadium'
picture: [Reichl '92]



if there are no scars,
quantum unique ergodicity

- QE expected to be common. If it holds: **Still $A = 0$!**

$$U \propto \frac{1}{m_{\mathrm{Pl},4}^2 r} \sum_k \psi_k^2(y) \mathrm{e}^{-m_k r} \quad \underset{r \to 0}{\overset{\text{expectation:}}{\sim}} \quad \frac{d! \omega_d}{(2\pi)^d} \frac{m_{\mathrm{Pl},D}^{2-D}}{r^{d+1}}$$

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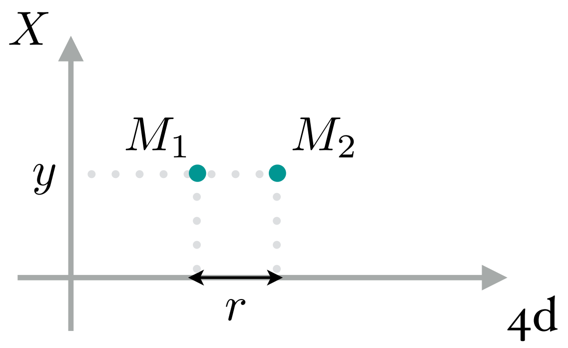

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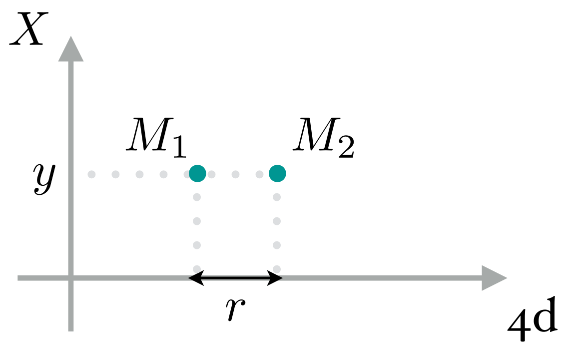
$$\Rightarrow \qquad \sum_k \frac{1}{r} \mathrm{e}^{-m_k r} \quad \underset{r \to 0}{\overset{\text{expectation:}}{\sim}} \quad \frac{d!\omega_d}{(2\pi)^d} \frac{V(X)}{r^{d+1}}$$

and conclude Weyl law as before.

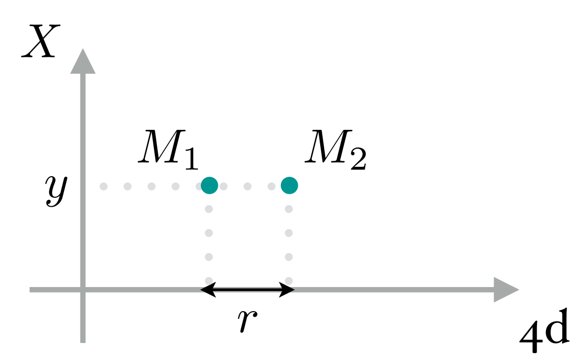


- Now warped case: $e^{2A}(ds_4^2 + ds_d^2(X))$

$$m^2_{\mathrm{Pl},4} = m^{D-2}_{\mathrm{Pl},D} V_A(X)$$

why doesn't V_A appear in the Weyl law?

$$\int_X d^d y\, e^{(D-2)A}\sqrt{g}$$



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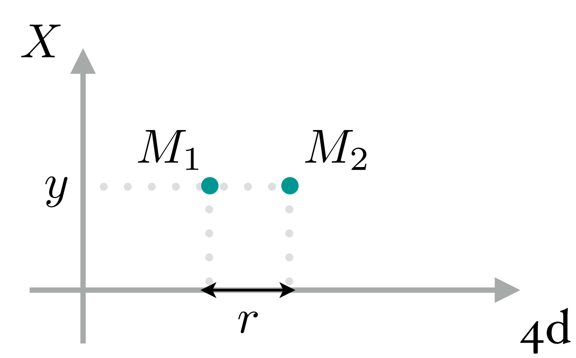
[De Luca, De Ponti, Mondino, AT '24]

$$e^{(D-2)A}\sum_{k=0}^{\infty}\frac{1}{r}e^{-m_k r}\psi_k^2 \quad \underset{r\to 0}{\sim} \quad \frac{d!\,\omega_d V_A(X)}{(2\pi r)^{d+1}}$$



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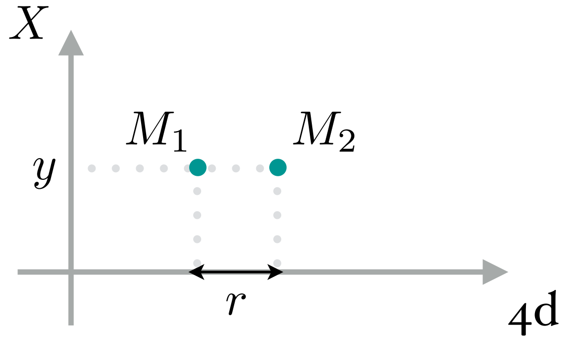
- Quantum ergodicity can't be right! LHS would depend on point, RHS doesn't



- Weighted quantum ergodicity: almost all ψ_k oscillate around e^{-f}

[De Luca, De Ponti, Mondino, AT '24]

$$\lim_{\substack{k\to\infty \\ k\notin e}} \frac{\int_B \sqrt{g}\,\mathrm{e}^{f}\,\psi_k^2}{\int_X \sqrt{g}\,\mathrm{e}^{f}\,\psi_k^2} \;=\; \frac{V(B)}{V(X)} \qquad \forall B\subset X \qquad\qquad f=(D-2)A$$

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our norm. $\frac{\int_B \sqrt{g}\mathrm{e}^{f}\psi_k^2}{V_A(X)}$

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$$||$$

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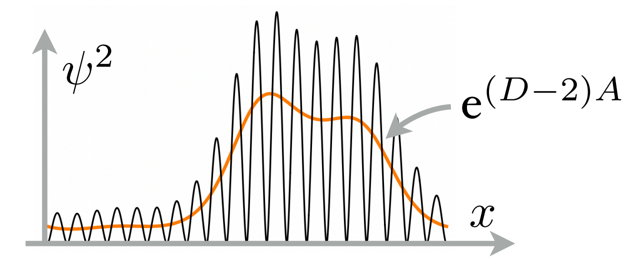
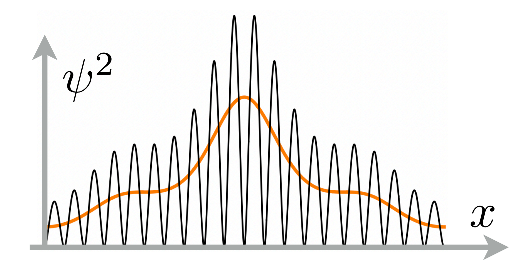
- also supported by
  - some numerical and analytic models
  - 'analogue Schrödinger' approach:

$$\Delta_f = e^{-f}\,(\Delta_0 - \underbrace{e^{-f/2}\Delta_0\, e^{f/2}}_{\text{'potential'}})\,e^f$$





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$$e^f \sum_{k=0}^{\infty} \frac{1}{r} e^{-m_k r} \psi_k^2 \quad \underset{r\to 0}{\sim} \quad \frac{d!\,\omega_d V_A(X)}{(2\pi r)^{d+1}}$$

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- $\int_B \sqrt{g}$ on both sides:

$$V_A(X) \frac{V(B)}{V(X)} \sum_{k=0}^{\infty} \frac{1}{r} \mathrm{e}^{-m_k r} \underset{r\to 0}{\sim} V(B) \frac{d!\omega_d V_A(X)}{(2\pi r)^{d+1}}$$

- $\color{red}{\text{Weighted}}$ quantum ergodicity: almost all ψ_k oscillate around $\color{red}{e^{-f}}$

[De Luca, De Ponti, Mondino, AT '24]

$$\lim_{\substack{k\to\infty \\ k\notin e}} \frac{\int_B \sqrt{g}\, e^f \psi_k^2}{\int_X \sqrt{g}\, e^f \psi_k^2} = \frac{V(B)}{V(X)} \qquad \forall B \subset X \qquad f = (D-2)A$$

unwarped!

$$\text{our norm.} \quad \frac{\int_B \sqrt{g}\, e^f \psi_k^2}{V_A(X)}$$

- The expectation $U_4 \underset{r\to 0}{\sim} U_D$ now reads

[De Luca, De Ponti, Mondino, AT '24]

$$e^f \sum_{k=0}^{\infty} \frac{1}{r} e^{-m_k r} \psi_k^2 \underset{r\to 0}{\sim} \frac{d!\,\omega_d V_A(X)}{(2\pi r)^{d+1}}$$

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⇨ warping disappears ⇨ $m_k \sim 2\pi \left(\dfrac{k}{\omega_d V(X)}\right)^{1/d}$

'unwarped' Weyl law.

- possible spin-off application of WQE: gravity localization

$$\begin{array}{c}|||\\ \text{models where } U_4 \underset{r\gg r_0}{\sim} 1/r \text{ even when } X \text{ noncompact}\end{array}$$

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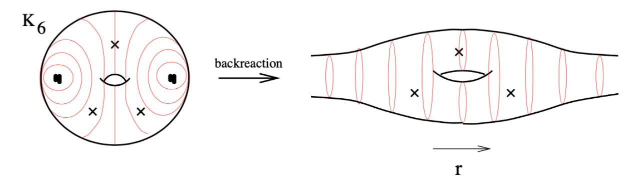
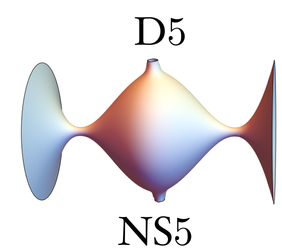
  - RSII: D3-branes



[Verlinde '99, Chan, Paul, Verlinde '00]

  - KR: hol. duals of defects in $\mathcal{N} = 4$ super-YM



[Bachas, Lavdas '17, '18] based on [D'Hoker, Estes, Gutperle '07] [Assel, Bachas, Estes, Gomis '11]

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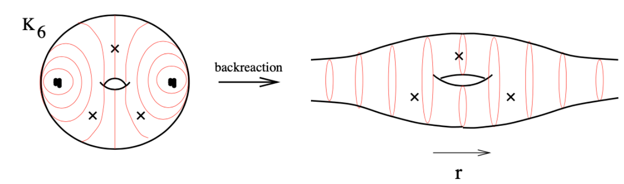

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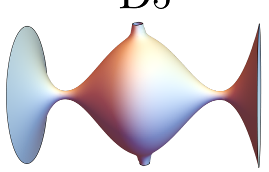

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here $m_0 \ll \sqrt{|\Lambda|} \ll m_1$

easy

problematic, because:

$$m_k^2 < 150 k^2 \max\left\{ m_0^2,\ |\Lambda| + \sigma^2 \right\}$$

[De Luca, De Ponti, Mondino, AT '23]

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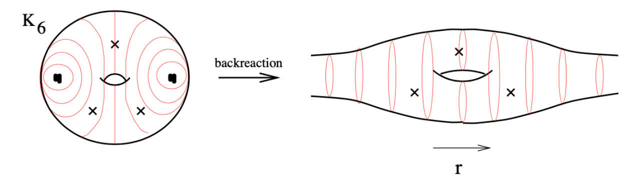

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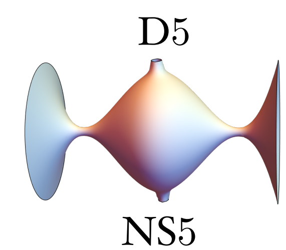

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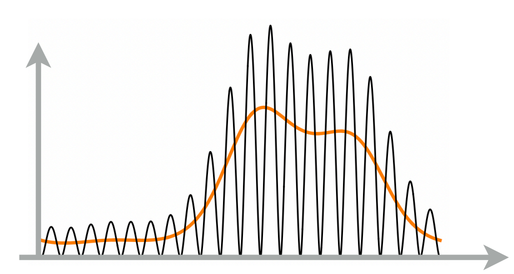
'localization' only up to cosmological scale?

- wave-function suppression could help pushing localization scale down.



Singularities

- We expect $U_4 \underset{r\to 0}{\sim} U_D$ also with physical singularities

[D-branes, O-planes]

- WQE argument: $\int_B$, far from singularities ✓

on the other hand, not fully rigorous [limits vs. integrals…]

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- Nasty enough singularities can break Weyl:

∃ 2d examples with
- $m_k \sim ck^\alpha \quad \forall \alpha < 1/2$
- $m_k^2 \log m_k \sim 2\pi k$

[Dai, Honda, Pan, Wei '22]

- Weyl law proven for 'RCD' singularities under a certain condition on geodesic balls

[Ambrosio, Honda, Tewodrose '17; Zhang, Zhu '17]

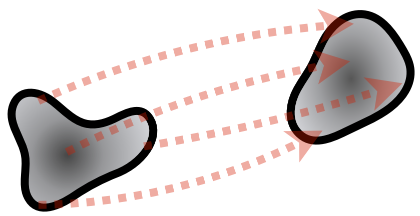
- RCD spaces: replace inequality

$$R_{mn} - \nabla_m \nabla_n f + \frac{1}{n-N} \nabla_m f \nabla_n f \geqslant 0$$

with an integrated version:

a bound on $\partial_t^2 S$ for geodesic motion of probability distributions.

$$S = N \left(1 - \int_M \sqrt{g}\, \mathrm{e}^f \rho^{\frac{N-1}{N}}\right)$$



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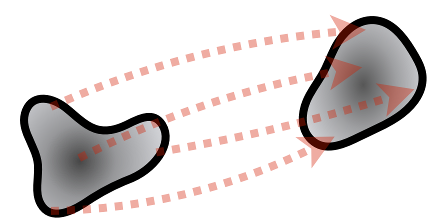
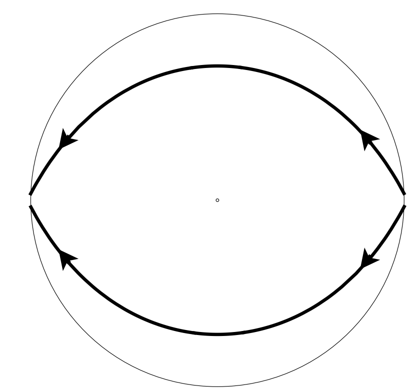


- D-brane singularities are indeed RCD

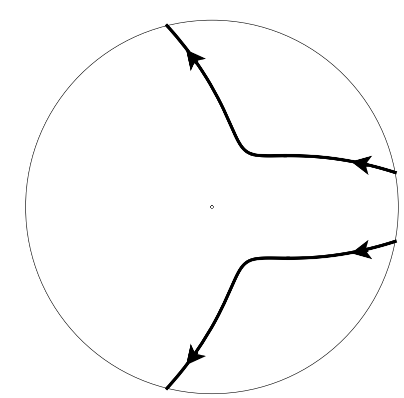
[De Luca, De Ponti, Mondino, AT '22]

geodesics attracted by a D-brane...



- O-planes are not.

... but repelled by an O-plane



- The additional necessary condition for Weyl:

$$\lim_{r\to 0^+}\int_X \frac{r^n}{\mathfrak{m}(B_r(x))}\mathrm{d}\mathfrak{m} = \int_X \lim_{r\to 0^+}\frac{r^n}{\mathfrak{m}(B_r(x))}\mathrm{d}\mathfrak{m} < \infty$$

weighted measure of geodesic balls.

- We proved that this holds for D6, D7, D8
  [± brute force]

[De Luca, De Ponti, Mondino, AT '24]

[the spectrum is continuous in the presence of D3, D4]

Conclusions

- Gravity compactifications give a perspective on Weyl law
- Physical argument particularly clean if ergodicity holds
- For warped compactifications, a new weighted ergodicity appears
- Rigorous version available also with D-brane singularities

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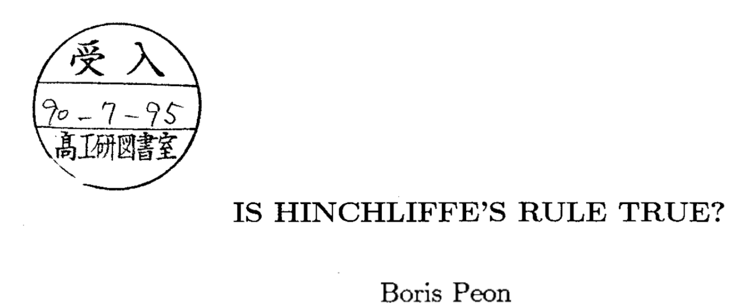
- No, one can't hear the Planck mass.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IS HINCHLIFFE'S RULE TRUE?

Boris Peon

Backup Slides

Bounds.

[De Luca, AT '20;
De Luca, De Ponti,
Mondino, AT '21,'22]

	weighted volume [or 4d Planck mass]	Cheeger	diameter
upper	m_k [smooth; warp.]	m_1 [D-branes; warp.] $\frac{m_k}{m_1^2}$ [O-planes]	m_k [smooth; warp.]
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- [smooth] $\subset$ [D-branes] $\subset$ [O-planes]
  spaces with D-brane sing. — spaces with O-plane sing.

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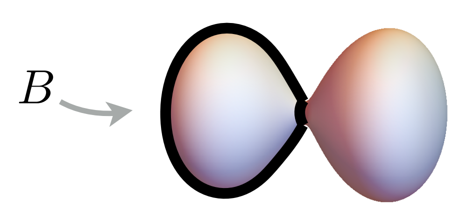
- Cheeger constant h_1: small when space has small 'neck'

$$h_1 = \min_B \frac{\mathrm{vol}_A(\partial B)}{\mathrm{vol}_A(B)}$$

'min. of $\frac{\text{perimeter}}{\text{area}}$'



$$\mathrm{vol}_A(B) \equiv \int_B \sqrt{g}\, e^{(D-2)A}$$

Some examples.

- $m_k^2 \leqslant \max\Big\{(D-2)\sigma^2,\ \frac{1}{n-1}(|\Lambda|+\sigma^2)\Big\} + \beta\left(k\,\underbrace{\frac{\sup(\mathrm{e}^{(D-2)A})}{\int \mathrm{d}^n y\sqrt{\bar{g}_n}\,\mathrm{e}^{(D-2)A}}}_{\text{weighted volume} \sim \text{Planck mass}}\right)^{2/n}$

  easy to make second term large:

  e.g. $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_p,\ p \to \infty$

[M_n smooth]

total dimension:

$$D = d + n$$

$\sigma = \sqrt{D-2}\,\sup|\mathrm{d}A|$

[$\alpha, \beta, \gamma \sim 10^4$]

[De Luca, AT '21]
using [Hassannezhad '12]

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weighted volume ~ Planck mass

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[De Luca, AT '21] using [Hassannezhad '12]

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- this issue is eliminated working with the diameter:

[De Luca, AT '21] using [Setti '98]
[De Luca, De Ponti, Mondino, AT '25]

$$m_k^2 \leqslant \left(|\Lambda| + (D-1)\sigma^2\right) + \gamma\frac{k^2}{\mathrm{diam}^2}$$

but now problem is 'nonlocal': how large is diam?

for sphere quotients:
[Greenwald '00,
Gorodski, Lange, Lytchak, Mendes '19,
Collins, Jafferis, Vafa, Xu, Yau '22]

- two bounds in terms of the Cheeger constant:

[De Ponti, Mondino '19;
De Luca, De Ponti, Mondino, AT '21,'22]

$$m_1^2 < \max\left\{\tfrac{21}{10}\sqrt{|\Lambda| + \sigma^2}\,h_1,\, \tfrac{22}{5}h_1^2\right\} \qquad\qquad m_k \leqslant 8\sqrt{2}k\,\frac{m_1^2}{h_1}$$

- combining them:

$$m_k^2 < 600k^2\max\left\{m_1^2,\, |\Lambda| + \sigma^2\right\}$$