Holographic correlators with multi-particle states

Rodolfo Russo

January 14 2025

Queen Mary University of London

Introduction: the framework

I'll focus on the well-known holographic dualities — Maldacena 9711200

- Type IIB string theory on $\mathrm{AdS}_5 \times S^5 \leftrightarrow \mathcal{N} = 4$ SYM
- Type IIB string theory on[1] $\mathrm{AdS}_3 \times S^3 \times \mathcal{M}_4 \leftrightarrow$ D1D5 SCFT_2

When the CFT is strongly coupled ($\lambda \gg 1$) and the central charge is large ($c \gg 1$) the bulk physics is well approximated by supergravity

In this regime the sugra fluctuations around AdS are in one-to-one correspondence with the "single-particle" CFT states

Kim, Romans, van Nieuwenhuizen 1985, Deger, Kaya, Sezgin, Sundell 1998

However the spectrum is richer: it contains also multi-particle states obtained by taking the OPE of the single-particle ingredients

Example in $\mathcal{N} = 4$ SYM: $\underbrace{Tr[Z^2](x)}_{O}\,\underbrace{Tr[Z^2](y)}_{O} \sim \underbrace{(Tr[Z^2])^2(y)}_{O^2} + \dots$

[1] $\mathcal{M}_4 = T^4,\ K_3$

Introduction: the aim of this talk

I will use holography to calculate 4-point correlators in the supergravity regime of the class $\langle O^p O^p O O \rangle$

4-point correlators between single particle states have been thoroughly studied since the early days of the AdS/CFT

D'Hoker, Freedman, Mathur, Rastelli, Matusis; Arutyunov, Frolov; ...

Instead very little is known about correlators with multi-particle states

Ceplak, Giusto, Hughes, RR 2105.04670; Bissi, Fardelli, Manenti 2111.06857, 2412.19788 ; Ma, Zhou 2204.13419

Why are such correlators interesting?

- They contain new CFT data (couplings and anomalous dimensions)
- They provide a window on higher point correlators (but keeping the simpler 4-point kinematics)

I will present the first explicit results for the simplest 4-point correlators with two multi-particle insertions in both AdS_5 and AdS_3. I will discuss

- what type of functions appear in the configuration space result
- the structure of these correlators in Mellin space

References

Holographic correlators with BPS bound states in $\mathcal{N}=4$ SYM 2409.12911

Aprile, Giusto, RR and work in progress

Building on (for the AdS_3 case):

Holographic correlators with multi-particle states, 2105.04670

Ceplak, Giusto, Hughes, RR

Unitary 4-point correlators from classical geometries, 1710.06820

Bombini, Galliani, Giusto, Moscato, RR

Plan

I will first introduce the supergravity approach

- The holographic dictionary $\frac{1}{2}$-BPS geometries/heavy CFT states
- How to obtain the heavy-light 4-point correlators (HHLL) by studying the quadratic fluctuations around a given geometry

We then take the light limit of the HHLL correlators and obtain light 4-point correlators. We get

- the known results for correlators among single particle operators
- a new (compact) way of writing AdS_3 correlators with multi-particle and two single particle operators
- new results for correlators with two double-particle and two single particle operators in $\mathcal{N} = 4$ SYM

Microstate geometries

Consider a multi-particle state made of many copies of the same CPO

$$O_H^{(4)} \sim \Big(\mathrm{Tr}(Z^2)\Big)^p \qquad\qquad O_H^{(2)} \sim \left(\sum_r (\epsilon_{\dot A \dot B}\psi_r^{+\dot A}\tilde\psi_r^{+\dot B})\right)^p \equiv (O_{\frac12\frac12})^p$$

X, Y, Z are the three complex scalars in $\mathcal{N}=4$ SYM

$\psi_r^{\alpha\dot A}$ are $r=1,\ldots,N$ copies of free fermions and $\tilde\psi_r^{\dot\alpha\dot A}$ are the antiholomorphic partners

When $p \sim c$ these states are described by asymptotically AdS solutions:

- LLM geometries describe $\frac12$-BPS states in $\mathcal{N}=4$ SYM
  Lin, Lunin, Maldacena 2004
- 2-charge fuzzballs describe $\frac12$-BPS states in the D1D5 CFT
  Lunin, Mathur 2001; Lunin, Maldacena, Maoz 2002, Kanitscheider, Skenderis, Taylor 2007

The solutions above are labelled by continuous parameters: what is the relation with p? The dual CFT description is a coherent state. Example:
Kanitscheider, Skenderis, Taylor 2006

$$O_H^{(2*)} = \sum_{p=0}^{N} (1-\alpha^2)^{\frac{N-p}{2}}\,\alpha^p\,(O_{\frac12\frac12})_*^p \quad \text{peaked at } \bar p = N\alpha^2 \text{ for } N \gg 1$$

Precision holography

The definition of multi-particle states is subtle. Example:

$$(|O_{\frac{1}{2}\frac{1}{2}}\rangle)^p_* = \sum_{\substack{r_1,r_2,\dots,r_p \\ r_1\neq r_2\neq\dots r_p}} (\epsilon_{\dot{A}\dot{B}}\psi^{+\dot{A}}_{r_1}\tilde{\psi}^{+\dot{B}}_{r_1})\dots(\epsilon_{\dot{A}\dot{B}}\psi^{+\dot{A}}_{r_p}\tilde{\psi}^{+\dot{B}}_{r_p}) \neq \Big(\sum_r \epsilon_{\dot{A}\dot{B}}\psi^{+\dot{A}}_{r}\tilde{\psi}^{+\dot{B}}_{r}\Big)^p = (|O_{\frac{1}{2}\frac{1}{2}}\rangle)^p$$

For $p = 2$ the difference is a single-particle state with $h = \bar{h} = 1$

$$(O_{\frac{1}{2}\frac{1}{2}})^2 - (O_{\frac{1}{2}\frac{1}{2}})^2_* \sim O_{11}$$

Similarly in $\mathrm{AdS}_5/\mathcal{N} = 4$ SYM the expansion of graviton gas states is

$$O^{\mathbf{A}}_H \simeq 1 + \alpha\,\mathrm{Tr}(Z^2) + \alpha^2\Big[\big(\mathrm{Tr}(Z^2)\big)^2 + \mathbf{A}\,\mathrm{Tr}(Z^4)\Big] + \mathcal{O}(\alpha^4)$$

LLM plane $r = \big(1 - \frac{\alpha}{2}\cos(k\phi)\big)_{k=2}$

$\Downarrow$

$\mathbf{A} \neq 0$

Tyukov, Turton to appear

The geometry dual to the $\mathbf{A} = 0$ case is known

Liu, Lu, Pope, Vazquez-Poritz 0703184

Giusto, Rosso 2401.01254 (CFT interpretation)

Derive the correlator for generic α from the quadratic fluctuations around the appropriate geometry. Then take $\alpha \to 0$ and extract the correlators with $p = 1, 2\dots$ Exchange order of limits? OK in the BPS case (I think)

Holographic 4-point functions generalities

Technically, the HHLL correlator is the regular, non-normalisable solution to the (appropriate) wave equation that at the boundary ($\rho \to \infty$) scales as

$$z = e^{i(\tau+\sigma)} = e^{(\tau_e + i\sigma)} \qquad \text{vev of } O_L$$

$$\phi_\Delta(\rho; z, \bar{z}) \stackrel{\rho\to\infty}{\longrightarrow} \delta^2(z-1)\,\rho^{\Delta-d} + C_\alpha(z,\bar{z})\,\rho^{-\Delta}$$

$$\text{source for } \bar{O}_L$$

This the HHLL correlator $C_\alpha(z,\bar{z}) = \langle O_H(x_1)\bar{O}_H(x_2)\bar{O}_L(x_3)O_L(x_4)\rangle_{GF}$, where GF stays for gauged fixed: $x_1 = 0$, $x_2 \to \infty$, $x_{34}^2 = (1-z)(1-\bar{z})$

In the small α limit, we have (schematically)

$$C_\alpha \sim \sum_p C_p \alpha^{2p}\ , \text{ with } C_p \sim \langle O^p O^p O\, O\rangle_{\text{tree-con}}$$

We capture the "tree-level connected" Witten diagrams, see below

The bulk computation

Instead of using a perturbation dual to a CPO, we consider a minimally coupled scalar in the full geometry (dual to a superdescendant)

Additionally we focus on the lowest Kaluza-Klein mode on the sphere

Then for O_L with $\Delta = 2$ in AdS_5 ($\Delta = 1$ in AdS_3) we need to study the massless Klein-Gordon equation in a KK-reduced 5D (3D) metric. In the AdS_5

$$ds_5^2 = \Omega_0^2 \left[\frac{d\xi^2}{(1-\xi^2)^2} + \frac{\xi^2}{1-\xi^2} \, d\Omega_3^2 \right] - \frac{\Omega_1^2}{1-\xi^2} \, dt^2 \qquad \text{Dual to } O_H = e^{\frac{1}{2}\tanh\left(\frac{\epsilon}{2}\right) \mathrm{Tr} Z^2}$$

$$\Omega_0 = 1 - \frac{\epsilon^4}{504} (1-\xi^2)^2 (6+\xi^2) - \frac{\epsilon^6}{99792} (1-\xi^2)^2 (-360 + 332\,\xi^2 + 196\,\xi^4 + 63\,\xi^6) + O(\epsilon^8)$$

$$\Omega_1 = 1 - \frac{\epsilon^2}{6} (1-\xi^2)^2 + \frac{\epsilon^4}{504} (1-\xi^2)^2 (21 - 20\,\xi^2 - 15\,\xi^4) - \frac{\epsilon^6}{498960} (1-\xi^2)^2 (6391 - 11150\,\xi^2 - 1090\,\xi^6 + 2960\,\xi^6 + 4275\,\xi^8) + O(\epsilon^8)$$

The correlator for the CPO is obtained by integrating a Ward Identity

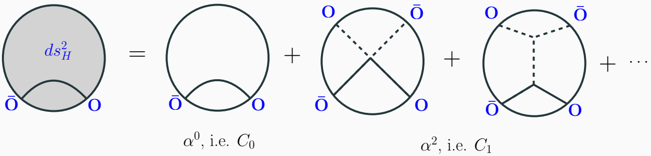
The dictionary between the geometry and the CFT state is fixed by matching protected 3-point functions

precision holography: Skenderis, Taylor 2006 ...

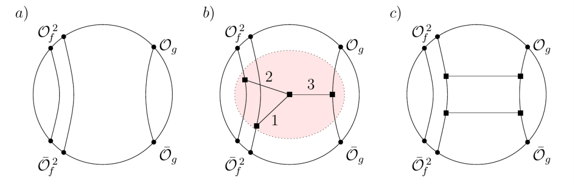
Diagrammatic interpretation

In pictures for C_1 we have the following interpretation in terms of Witten diagrams (the dashed propagators are encoded in the geometry)

see Turton, Tyukov 2408.16834 for more details in AdS_5



At the next order C_2 contains b), but not the disconnected diagrams a) and c) (again our approach avoids the use of bulk-to-bulk propagators)



Rewriting holographic 4-point functions (I)

The supergravity results for C_n are written in terms Li_k with $k \leq n$ multiplied by rational functions of z, $\bar{z}$

Of course C_1, we can be written in terms of the 4D box integral

$$D_{1111} = \int \frac{d^4x}{i\pi^2} \frac{1}{(x-x_1)^2 (x-x_2)^2 (x-x_3)^2 (x-x_4)^2}$$

$$D_{\Delta_1+1\Delta_2+1\Delta_3\Delta_4} = \frac{\partial}{\partial x_{12}^2} D_{\Delta_1\Delta_2\Delta_3\Delta_4} \quad \text{and similar for the other } \tfrac{\partial}{\partial x_{ij}^2}$$

$$\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(z,\bar{z}) = \left[\frac{2\prod_{i=1}^4 \Gamma(\Delta_i)}{\pi^{d/2}\Gamma\left(\frac{\hat{\Delta}-d}{2}\right)} \frac{|x_{13}|^{\hat{\Delta}-2\Delta_4} |x_{24}|^{2\Delta_2}}{|x_{14}|^{\hat{\Delta}-2\Delta_1-2\Delta_4} |x_{34}|^{\hat{\Delta}-2\Delta_3-2\Delta_4}} D_{\Delta_1\Delta_2\Delta_3\Delta_4} \right]_{GF}$$

An example: $C_1^{AdS_3} = \left[\langle O^f_{\frac{1}{2}\frac{1}{2}} \bar{O}^f_{\frac{1}{2}\frac{1}{2}} \bar{O}^g_{\frac{1}{2}\frac{1}{2}} O^g_{\frac{1}{2}\frac{1}{2}} \rangle \right]_{GF}$ (with different flavours $f \neq g$)

$$C_1^{AdS_3} = -\frac{1}{N} \frac{1}{|1-z|^2} \left[1 + |z|^2 \bar{D}_{1122} \right]$$

Giusto, RR, Wen 2018 and generalised in Rastelli, Zhou 2019, Giusto, RR, Tyukov, Wen 2019, 2020

The same pattern continues also for $n > 1$ (!)

Rewriting holographic 4-point functions (II)

By defining

Usyukina, Davydychev 1993 and and Isaev 2003

$$D^{(2)} = \int \frac{d^4x_a}{i\pi^2}\frac{d^4x_b}{i\pi^2}\frac{1}{x_{a1}^2x_{a2}^2x_{a3}^2x_{ab}^2x_{1b}^2x_{2b}^2x_{b4}^2} = \frac{(1-z)(1-\bar{z})}{z-\bar{z}}\,\frac{\mathcal{P}_2}{x_{12}^4\,x_{34}^2}$$

In general the n-ladder integral is $D^{(n)} = \frac{(1-z)(1-\bar{z})}{z-\bar{z}}\,\frac{\mathcal{P}_n}{x_{12}^{2n}\,x_{34}^2}$ with

$$\mathcal{P}_n = \sum_{r=0}^{n}\frac{(-1)^r(2n-r)!}{n!(n-r)!\,r!}\log^r|z|^2\,\left(\mathrm{Li}_{2n-r}(z) - \mathrm{Li}_{2n-r}(\bar{z})\right)$$

we can rewrite C_2 in terms of derivatives of $D^{(2)}$. An AdS$_3$ example:

PolyLog expression in Ceplak, Giusto, Hughes, RR 2105.04670

$$C_2^{AdS_3} = -\frac{2\,|z|^2}{N^2}\left[\frac{\partial D^{(2)}}{\partial x_{34}^2} - (z+\bar{z})\,\frac{\partial^2 D^{(2)}}{\partial (x_{34}^2)^2} + \frac{\partial D^{(1)}}{\partial x_{34}^2}\right]_{GF}$$

We checked that this structure (involving $D^{(n)}$) holds also for $C_n^{AdS_3}$ with $n > 2$ and also for $C_2^{AdS_5}$ and $C_3^{AdS_5}$ (it's likely a general property)

Mellin space formulation

We can rewrite C_2 in Mellin space as done for C_1. We use Aprile, Vieira 2007.09176 see also Allendes et al 1205.6257

$$\left[D^{(2)}\right]_{GF} = \int \frac{ds}{2\pi i}\frac{dt}{2\pi i}\Gamma^2(-s)\Gamma^2(-t)\Gamma^2(-u)K(u,t)U^sV^t$$

$$\text{with } s+t+u=-1$$

$$K(u_1,t) = -\pi^2-(\psi^{(0)}(-t))^2+\psi^{(1)}(-t)-(\psi^{(0)}(-u))^2+\psi^{(1)}(-u)+2\psi^{(0)}(-t)\,\psi^{(0)}(-u)$$

Let's see how it works for AdS_3. We first extract the dynamical part of the correlator $\mathcal{H}_2 = C_2\frac{U^2}{V^2}$. Then we get ($\Delta = 1$ in AdS_3)

$$\mathcal{H}_2 = \int \frac{ds}{2\pi i}\frac{dt}{2\pi i}\,\Gamma^2(-s)\Gamma^2(-t)\Gamma^2(-u)\,U^{s+\Delta}V^t \left\{K(u,t)\,A + \left[\psi^{(0)}(-t)-\psi^{(0)}(-u)\right]B + C\right\} \qquad s+t+u=\Delta+\tfrac{d}{2}$$

$$A = \frac{1}{s+1}\left(1-\frac{2tu}{s+2}\right), \quad B = \frac{2(t-u)}{(1+s)(2+s)}, \quad C = \frac{2s}{(1+s)(2+s)}$$

The AdS$_5$ case

In the $\mathcal{N} = 4$ SYM case $\mathcal{H}_2$ has again the same structure

$$A = \frac{32}{(s+1)(s+2)(s+3)}\left[(-16-3s+3tu) + \frac{18tu}{(s+4)} - \frac{18t^2u^2}{(s+4)(s+5)}\right]$$

$$B = \frac{32(t-u)}{(s+1)(s+2)}\left[\frac{18(\,(s+5)-tu)}{(s+4)(s+5)} - \frac{1}{(t+1)(u+1)}\right]$$

$$C = \frac{16}{(s+1)}\left[\frac{3(s^2+17s+64-12tu)}{(s+4)(s+5)} - \frac{1}{(t+1)(u+1)}\right]$$

The apparent poles at $s = -3, -4, -5$ cancel!

The corresponding configuration space result (derived from sugra) can again be written in terms of derivatives of $D^{(2)}$ times rational functions

$\mathcal{H}_3$ follows a similar pattern with apparent poles up to $s = -7$. In configuration space it can be written in terms of $D^{(3)}$ and its derivatives (contrary to the 2-loop result for $\langle OOOO \rangle$!)

The full correlator

The relation between the full and the connected AdS_5 correlators reads

$$
\begin{aligned}
&\langle O^2(1)O^2(2)O(3)O(4)\rangle_{\text{conn}} = \langle O^2(1)O^2(2)O(3)O(4)\rangle - \mathbf{4\langle O(1)O(2)\rangle\langle O(1)O(2)O(3)O(4)\rangle} \\
&-2\Big(\langle O(1)O(3)\rangle\langle O(1)O^2(2)O(4)\rangle + \langle O(1)O(4)\rangle\langle O(1)O^2(2)O(3)\rangle \\
&\quad + \langle O(2)O(3)\rangle\langle O^2(1)O(2)O(4)\rangle + \langle O(2)O(4)\rangle\langle O^2(1)O(2)O(3)\rangle\Big) \\
&-4\langle O(1)O(2)O(3)\rangle\langle O(1)O(2)O(4)\rangle - \langle O(1)^2O(2)^2\rangle\langle O(3)O(4)\rangle + 4\langle O(1)O(2)\rangle^2\langle O(3)O(4)\rangle \\
&+8\langle O(1)O(2)\rangle\left(\langle O(1)O(3)\rangle\langle O(2)O(4)\rangle + \langle O(1)O(4)\rangle\langle O(2)O(3)\rangle\right)
\end{aligned}
$$

The terms of order N^0 and N^{-2} on the rhs cancel and the first contribution scales as N^{-4}

The N^{-4} contribution of the term in bold can be obtained by using known 1-loop results for the AdS_5 single-particle correlators

several interesting papers by Aharony, Alday, Aprile, Bissi, Drummond, Heslop, Paul, Zhou (in various collaborations)

Thus combining the 1-loop result with the connected contribution on the previous slide, one obtains the full correlator $\langle O^2 O^2 O O\rangle$

Checks

The light expansion of the HHLL correlator has been checked in the Regge limit up to $\mathcal{O}(\epsilon^4)$ where it does reproduce the expected eikonal exponentiation

for the AdS_3 case see Ceplak and Hughes 2102.09549

We checked in an AdS_3 example that A, B do not depend on the value of **A** (see the "precision holography" section) as expected from the CFT interpretation (non-trivial from the sugra point of view)

The Euclidean correlator is free of unwanted singularities (the apparent singularities at $\bar{z} \to z$ all cancel!)

The OPE analysis of C_2 is consistent with known protected CFT data (such as known BPS 3-point couplings)

To be done: use our $n > 1$ correlators to derive the integrated correlator. One should reproduce the results obtained from localisation

Binder, Chester, Pufu, Wang 1902.06263; ...

More checks: (super)block decomposition

In $\mathcal{N} = 4$ there is a powerful non-renormalisation theorem

$$\mathcal{C}_{(p,q)} \equiv \langle O_2^p(x_1,y_1) O_2^p(x_2,y_2) O_q(x_3,y_3) O_q(x_4,y_4)\rangle$$

$$\hat{\mathcal{C}}_{(p,q)} \equiv \frac{\mathcal{C}_{(p,q)}}{\langle O_2^p O_2^p\rangle\langle O_q O_q\rangle} = \mathcal{G}^{\text{free}}_{(p,q)} + \mathcal{I}\,\mathcal{H}_{(p,q)}$$

$$\mathcal{I} = V + \sigma V(V-1-U) + \tau(1-U-V) + \sigma\tau U(U-1-V) + \sigma^2 UV + \tau^2 U$$

We can decompose in superblocks $\mathcal{G}^{\text{free}}$ and $\mathcal{H}$: accidentally BPS (semi-short) multiplets in $\mathcal{G}^{\text{free}}$ can recombine with contributions from $\mathcal{H}$

In the crossed channel $\langle O O^2 O O^2\rangle$ there should be protected twist-4 semishort multiplets in the $[020]_\ell$ $SU(4)$ irrep (with spin ℓ)

These couplings were derived from $\langle OOOO\rangle$ and $\langle OOO^2O\rangle$ and match perfectly the results obtained by decomposing our $\mathcal{H}_2$

Doobary Heslop 1508.03611; Aprile Drummond Paul Heslop 1912.01047

More CFT checks are possible (in progress)

Conclusions

A detailed study of holographic correlators with multi-particle states is possible. I think that the results presented here are just a first step in a more general story. Some immediate questions:

- Extract new CFT data coupling with 3 double-particles (in progress)
- Extend the results to generic KK modes of the single-particle operators (is there a conformal symmetry à la Caron-Huot-Trinh?). A recursion relation connecting correlators with different values of p?
- Are there any novelties when considering lower susy cases, such as $\frac{1}{4}$-BPS operators (which are always multi-particle states)?

It's of course interesting to look at other AdS/CFT pairs: does the same pattern persist? More general questions include:

- Can we use this information to reconstruct a full holographic 6-point correlator among single particle states $\langle OOOOOO \rangle$?
- What about more general correlators such as $\langle O^2 O^2 O^2 O^2 \rangle$?