Branes in the Superconformal Index

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SISSA

Ongoing work [Aharony, Benini, OM]

NP-QFT

Motivation – Holography, microstate counting and beyond

How the macroscopic (semi-classical gravity) description arises from the microscopics (dual QFT)

- Example: SUSY black hole entropy from the dual CFT.
  [Benini, Hristov, Zaffaroni], [Benini & Milan], [Choi, Kim, Kim, Nahmgoong], [Cabo-Bizet, Cassani, Martelli, Murthy], ...
- Compute grand-canonical partition function.
  - CFT: count states, trace over Hilbert space.
  - Gravity: sum over Euclidean saddles.

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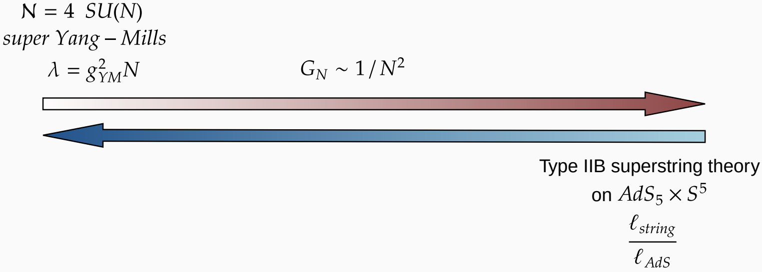
- Compute grand-canonical partition function.
  - CFT: count states, trace over Hilbert space.
  - Gravity: sum over Euclidean saddles.
- Many more saddles: black holes, orbifolds, wrapped branes. What's their origin? How are they related to one another?

Concentrate on a specific duality:



(but parts of the story apply to other theories, other dimensions, and other observables.)

$\mathcal{N} = 4$ $SU(N)$ Super Yang Mills

We will study the partition function of the theory on $S^1 \times S^3$.

Symmetries:

- $SO(6)_R$ R-symmetry. Charges: $R_{1,2,3}$ ($\Delta_{1,2,3}$).
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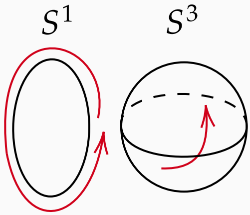

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If we tune the chemical potentials correctly,

$$\tau + \sigma - \sum_a \Delta_a = \pm 1$$

then the partition function is the superconformal index [Hosseini, Hristov, Zaffaroni], [Cabo-Bizet, Cassani, Martelli, Murthy], [Choi, Kim, Kim, Nahmgoong]

$$Z(\beta, \tau, \sigma, \Delta_{1,2,3}) = \mathcal{I}(\tau, \sigma, \Delta_1, \Delta_2)$$

How do we compute it? Gravity

On the gravity side, in the semi-classical limit $G_N \sim \frac{1}{N^2} \to 0$,

$$\mathcal{Z}_{grav} = \sum_{b \in bulks} e^{-N^2 I_b} \Bigg[a_0 + \frac{a_1}{N^2} + \cdots \\ + \sum_{Dp \in \text{branes}} e^{-N I_{Dp}} \left(b_0 + \frac{b_1}{N^2} + \cdots \right) \Bigg],$$

the sum is over all bulk geometries whose boundary conditions are the same as the background for the CFT, $\{\beta, \tau, \Delta_a\}$.

How do we compute it? QFT

Integrating out all the modes besides the holonomies of the gauge field, $u_i = \oint_{S^1} A^i$, [Kinney, Maldacena, Minwalla, Raju]

$$\mathcal{I} \propto \int_0^1 \left(\prod_{i=1}^{N-1} du_i \right) \frac{\prod_{a=1}^{3} \prod_{i \neq j}^{N} \widetilde{\Gamma}(\Delta_a + u_{ij}; \tau, \tau)}{\prod_{i \neq j}^{N} \widetilde{\Gamma}(u_{ij}; \tau, \tau)} \,.$$

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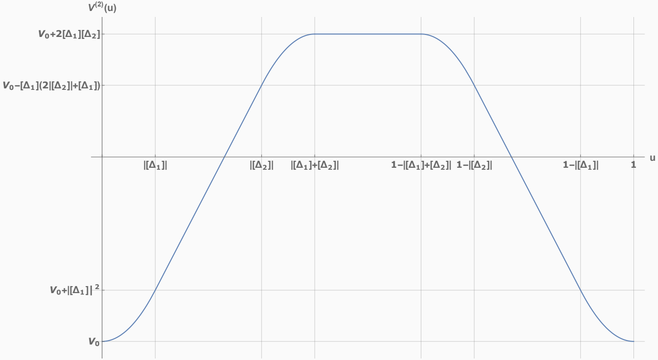
There are two ways to take a thermodynamic limit:

1. Large N.
   - Bethe Ansatz (applies also at finite N) [Benini, Milan], [Closset, Kim, Willett], ...
   - Saddle point [Choi, Jeong, Kim, Lee]
2. "Cardy limit": $\tau, \sigma \to 0$, $\Delta_{1,2} \in \mathbb{R}$. Favors large charges.
   [Choi, Kim, Kim, Nahmgoong], [Murthy, Arabi Ardehali], [Komargodski, Cassani], ...

The Cardy limit

Use modular properties of the integrand to bring it to the form

$$\mathcal{I} = \frac{1}{N!} \int_0^{2\pi} \left(\prod_{i=1}^{N-1} du_i \right) \exp \left[-\frac{\pi i}{\tau^2} \sum_{i,j=1}^{N} V^{(2)} \left(u_{ij} \right) + O(e^{-1/\tau}) \right]$$



$V^{(2)}(u)$

The black hole saddle

One extrema happens when all the holonomies are at the same point.

- There are N such saddle points, for $u_i = \frac{\mathbb{Z}}{N}$.
- They give together [Honda], ..., [Cassani, Komargodski], [Murthy, Arabi Ardaheli]

$$N \exp\left[-\pi i (N^2-1)\frac{\Delta_1\Delta_2\Delta_3}{\tau^2} + O(e^{-1/\tau})\right] , \quad \Delta_3 = 2\tau - \Delta_1 - \Delta_2 - 1$$

Some Bethe Ansatz result

[Benini & Milan] , ... , [Aharony, Benini, OM, Milan], [OM]

In fact, the Bethe Ansatz gives a large N formula that keeps track of the $O(e^{-1/\tau})$ terms:

$$\mathcal{I} = N \frac{e^{\frac{\pi i}{12}}}{\tau \mathcal{I}_{U(1)}} e^{-\pi i N^2 \frac{\Delta_1 \Delta_2 \Delta_3}{\tau^2}} \prod_{k=1}^{\infty} \left[\frac{1 - \tilde{q}^k}{(1 - \tilde{y}_1^k)(1 - \tilde{y}_2^k)(1 - \tilde{y}_3^k)} \right]^2 \times \left(1 + O\left(e^{2\pi i N \frac{\Delta_a}{\tau}} \right) \right) ,$$

where $\tilde{q} = e^{-2\pi i/\tau}$, $\tilde{y}_a = e^{2\pi i \Delta_a/\tau}$.

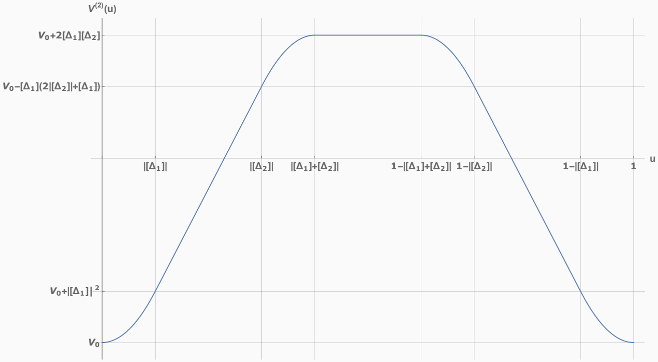
Gravity interpretation: on-shell action , 1-loop, wrapped branes, and the $\log(N)$ terms (coming from $B_2 \wedge dC_2 \wedge F_5$ toplogical term).

Branes?

But there are other saddles: pop out one eigenvalue to the plateau in the middle. Contributes additional

$$\exp\left[-2\pi i(N-1)\frac{\Delta_1\Delta_2}{\tau^2}\right]$$

Exciting a (different) brane!



Many branes?

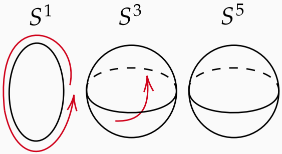
One can also move $N/2$ of the eigenvalues to the middle, contributing:

$$\frac{N}{2} \exp\left[-\pi i \frac{N^2}{2} \frac{(2\Delta_1)(2\Delta_2)(4\tau - 2\Delta_1 - 2\Delta_2 - 1)}{(2\tau)^2}\right]$$

A black hole

So what kinds of bulks are there?

There's a (complex) supersymmetric, Euclidean black hole solution in $AdS_5 \times S^5$. The boundary thermal cycle shrinks to zero at the horizon.



Its on-shell action is [Cabo-Bizet, Cassani, Martelli, Murthy]

$$I_b = \pi i N^2 \frac{\Delta_1 \Delta_2 \Delta_3}{\tau^2}$$

Exactly as the Cardy limit saddle!

What about D-branes?

[Aharony, Benini, OM, Milan]

The index is independent of $\lambda \propto \left(\frac{\ell_s}{\ell_{AdS}}\right)^4 \implies$ only D3-branes can appear.

So far, we had found two types of Euclidean SUSY D3-branes, located at the horizon and wrapping:

	AdS_5	S^5	Action I_{D3}
"Giant Graviton"-like	S^1	S^3	$2\pi i N \frac{\Delta_a}{\tau}$
"Dual Giant Graviton"-like	S^3	S^1	$-2\pi i N \frac{\Delta_a \Delta_b}{\tau^2}$

Exactly matching the exponential corrections, and the action for moving an eigenvalue!

Orbifolds

[Aharony, Benini, OM, Milan]

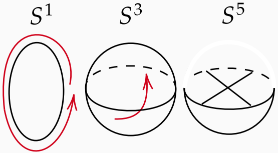
Start from a black hole with boundary conditions

$$\tilde\beta = m\beta, \quad \tilde\tau = m\tau, \quad \tilde\Delta_a = m\Delta_a + s_a\,, \quad s_a \in \mathbb{Z}, \quad 2\tilde\tau - \sum_a \tilde\Delta_a = 1$$

With $\phi_{1,2,3}$ angular coordinates on S^5, t_E on S^1_β, orbifold by

$$(t_E, \phi_1, \phi_2, \phi_3) \sim (t_E + \frac{\tilde\beta}{m}, \phi_1 + \frac{2\pi s_1}{m}, \phi_2 + \frac{2\pi s_2}{m}, \phi_3 + \frac{2\pi s_3}{m})$$

The boundary conditions are now $\{\beta, \tau, \Delta_a\}$.



Orbifolds II

Locally, the solution is the same as before the orbifolding. However, it reduced the volume of the S^5 by a factor of m, and the on-shell action becomes

$$I_b = \pi i \frac{N^2}{m} \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\tau}^2}\,, \qquad \tilde{\Delta}_3 = 2\tilde{\tau} - \tilde{\Delta}_1 - \tilde{\Delta}_2 - 1\,,$$

For $m = 2$: exactly the solution where we moved half the eigenvalues!

What's the dominant phase?

$$Z_{BH} \propto e^{-\pi i N^2 \frac{\Delta_1 \Delta_2 \Delta_3}{\tau^2}}, \qquad Z_{m=2} \propto e^{-\pi i \frac{N^2}{2} \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\tau}^2}}$$

Consider $-1 < \Delta_{1,2} < 0$.

For $\mathbb{R}\mathrm{e}(\tau) > 0$ the black hole dominates.

For $\mathbb{R}\mathrm{e}(\tau) < 0$ the orbifold dominates.

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For $\mathbb{R}\mathrm{e}(\tau) > 0$ the black hole dominates.

For $\mathbb{R}\mathrm{e}(\tau) < 0$ the orbifold dominates.

Moreover, the DGG branes on top of the BH gives

$$Z_{BH+DGG} = Z_{BH} e^{-2\pi i \frac{\Delta_a \Delta_b}{\tau^2}}$$

suppressed when $\mathbb{R}\mathrm{e}(\tau) > 0$, become unstable at $\mathbb{R}\mathrm{e}(\tau) < 0$! Suggesting the condensation of these branes end at the orbifold?

Quantization of charge

$$\mathcal{I}(\beta, \tau, \sigma, \Delta_{1,2}) = \mathrm{Tr}\left[e^{-\beta\{\mathcal{Q},\bar{\mathcal{Q}}\}+2\pi i\left(\tau J_1+\sigma J_2+\frac{1}{2}\sum_{a=1}^{3}\Delta_a R_a\right)}\right]$$

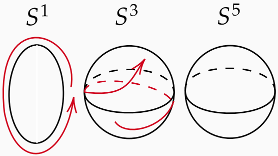
The charges are quantized, and so the index is periodic in τ, Δ. However, the black hole contributes as

$$\exp\left[\pi i N^2 \frac{\Delta_1\Delta_2\Delta_3}{\tau^2}\right]$$

which is not periodic. How does gravity restore the periodicity (and know about quantization of the charges?)

Other black holes

Some other combinations of the thermal cycle and the hopf fiber of the S^3 can close in the bulk [Aharony, Benini, OM, Milan]



with on-shell actions are

$$I_b = \pi i N^2 \frac{\Delta_1 \Delta_2 \Delta_3}{(\tau + r)^2}\,, \qquad r \in \mathbb{Z}$$

From the bulk perspective, this restores the periodicity in τ. These correspond (in 1-to-1 fashion) to contributions from the Bethe Ansatz.

Summary

- There are many gravitational saddles.
- Summing over saddles is needed to reproduce discreteness of charges.
- We can find their origin as saddles of the $\mathcal{I}$.
- Moving one holonomy – exciting a brane.
- When these branes become unstable, they ”nucleate” into another gravitational solution.

Things I glossed over

1. Periodicity in Δ comes from stability of 1-loop, other branes.
2. Different regimes of the chemical potentials.
3. Branes in the Bethe ansatz – moving one eigenvalue from Hong-Liu configurations. Tractable at large N. Part of a continuum.
4. Moving xN holonomies – gravity interpretation?

Thank you!

Beyond the action: a topological contribution

Type IIB SUGRA contains a topological term,

$$\int B_2 \wedge dC_2 \wedge F_5 = N \int_{\mathrm{AdS}_5} B_2 \wedge dC_2$$

since $\int_{S^5} F_5 = N$. In the black hole topology, there are N distinct flat B_2 solutions with the same on-shell action [Witten], [Aharony, Witten]. The contribution to Z is actually

$$Ne^{-I_b}$$

This is the bulk manifestation of the breaking of the center $\mathbb{Z}_N^{(1)}$ symmetry.

Orbifolds and center symmetry

In the orbifold the volume of S^5 had shrunk by a factor of m.

In order for the flux $\int_{S^5} F_5 \in \mathbb{Z}$, we need to have $m|N$.

Topological action is $\frac{N}{m} \int B_2 \wedge dC_2 \implies$ center symmetry breaking $\mathbb{Z}_N^{(1)} \to \mathbb{Z}_m^{(1)}$.

Some comments

- Similar results from Bethe Ansatz (large N).
  - DGG branes come from continuum solutions at large N. Can argue for their action.
  - Continuum behaves like the flat direction of Cardy limit.
- Similar saddles appear in other $\mathcal{N} = 1$ SCFTs.
- The orbifolds can be constructed for more general bulks of the form $AdS_5 \times X_5$, for some Sasaki-Einstein manifold X_5 which is a $U(1)$ fibration.
- Should apply other observables, such as expectation values for defects [Chen, Heydeman, Wang, Zhang].

Summary

In the Cardy limit (or at large N) many different gravitational saddles emerge from the QFT (black holes, orbifolds).

Giant gravitons – exponential corrections around eigenvalue configurations.

Dual giant gravitons – moving single eigenvalue. Nucleation leads to other gravitational saddles.

Phase diagram might have regions with partial breaking of center symmetry (orbifolds)!

Bulks for moving arbitrary fraction of eigenvalues?

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- $\mathbb{Z}_N^{(1)}$ center symmetry (1-form symmetry).

The partition function is:

$$Z(\beta, \Omega_i, \Phi_a) = \mathrm{Tr}\left(e^{-\beta H + \sum_i \beta\,\Omega_i J_i + \frac{1}{2}\sum_a \beta\,\Phi_a R_a}\right)$$

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Charges are quantized! Periodic in $\tau, \sigma, \Delta_{1,2,3}$.

The Cardy limit

The integrand is invariant under integer shifts of the Δ's. When all the Δ's are in case 1 (or 2), we can use modular properties of the integrand to bring it to the form

$$Z = \frac{1}{N!} \int_0^{2\pi} \left(\prod_{i=1}^{N-1} du_i \right) e^{-\frac{\pi i}{2\tau^2} \sum_{i,j=1}^{N} V^{(2)}(u_{ij}) - \frac{\pi i}{2\tau} \sum_{i,j=1}^{N} V^{(1)}(u_{ij}) + O(e^{-1/\tau})}$$

If we wouldn't have shifted the Δ's, the $O(e^{-1/\tau})$ corrections inside the exponent would not have been suppressed!