Quantum corrections to the entropy of near extremal de Sitter black holes

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for “Black Holes, Holography and de Sitter spacetimes”

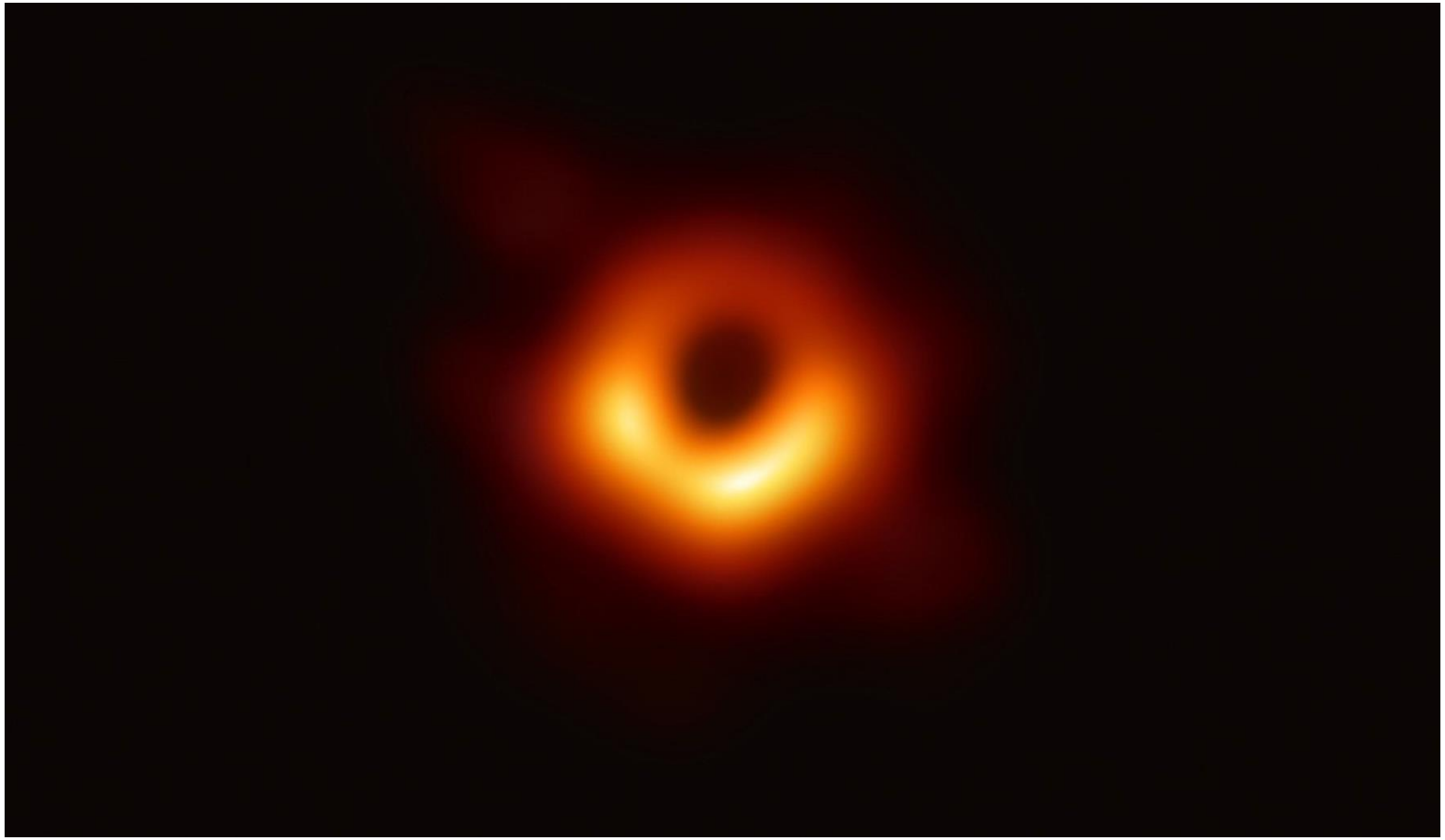
13/01/25

[2501.XXXXX] MJB, A. Castro, W. Sybesma, C. Toldo

Introduction

Why do we care about near-extremal black holes?

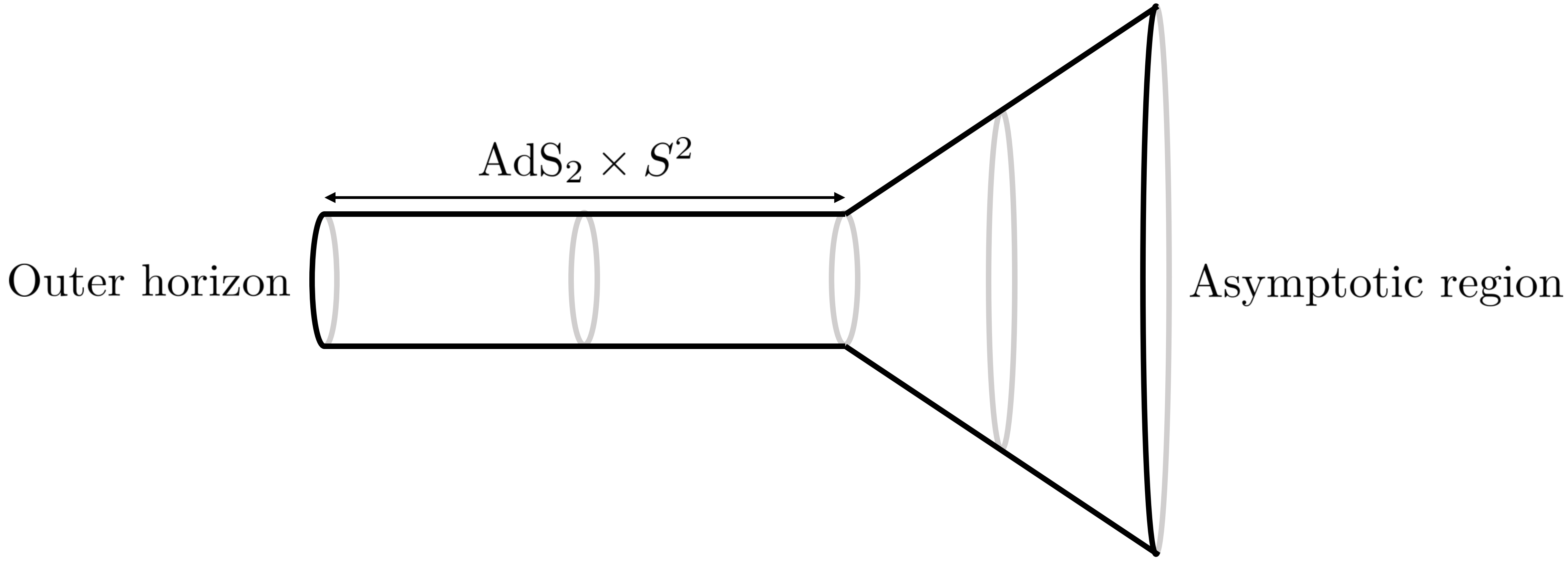
Black holes are thermal objects



[Event Horizon Telescope Collaboration]

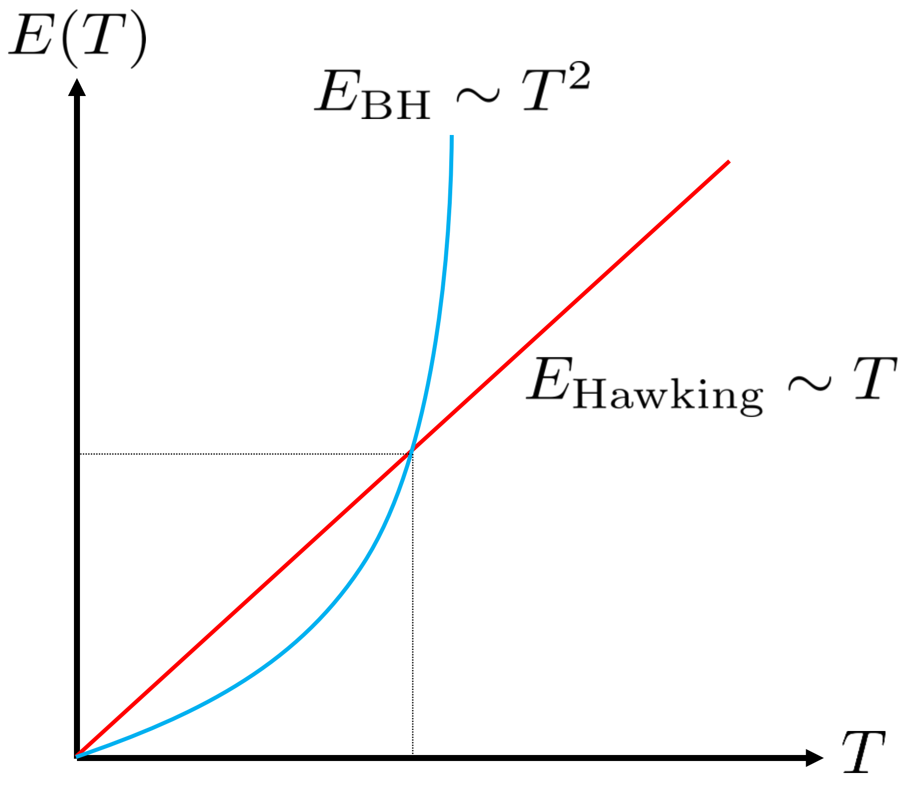
$$M = M(Q, J) \geq M_{\text{extremal}}$$

Near-extremal black holes



Two interesting puzzles

Puzzle 1



[Preskill, Schwarz, Shapere, Trivedi, Wilczek ’91]
[Maldacena, Michelson, Strominger ’98]

Puzzle 2

$$S_{\mathrm{extremal}} = k_B \frac{\pi\sqrt{Q^4 + 4c^2 J^2}}{c\hbar}$$
$$\simeq 10^{70} k_B \text{ for M87 ratio}$$

Big!

How do we resolve puzzles?

- Account for *light modes* near extremality
- Two options
  1. Schwarzian Theory ← we **DON'T** do this
  2. 4D Fluctuations ← we **DO** do this!
- Previous work:
  - Flat RN: [Banerjee, Saha '23], [Iliesiu, Murthy, Turiaci '23]
  - Kerr: [Kapec, Sheta, Strominger, Toldo '23]
- We asked: what if $\Lambda \neq 0$?

Outline

1. ~~Introduction~~
2. General Strategy
3. RN-dS_4 Black Holes
4. The Cold Limit
5. The Nariai Limit
6. Outlook

General Strategy

How do we compute the leading order corrections to the gravitational path integral?

Euclidean Path Integral

$$Z = \int [Dg]\,[DA]\,e^{-I[g,A]},$$

$$\text{where } I[g,A] = I_{\text{EM}} + I_{\text{boundary}} + I_{\text{gauge}}$$

$$I_{\text{EM}} = \tfrac{1}{16\pi G_N} \int d^4x\,\sqrt{-g}\Big(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}\Big)$$

$$g = \bar{g} + h \text{ and } A = \bar{A} + \tfrac{1}{2}a$$

$$Z \approx \exp\left(-I[\bar{g},\bar{A}]\right) \int [Dh][Da] \exp\Big[-\int d^4x\sqrt{\bar{g}}\,\big(h^* D[\bar{g},\bar{A}]h$$
$$+a^* P[\bar{g},\bar{A}]a + \big(h^* O_{\text{int}}[\bar{g},\bar{A}]a + \text{h.c}\big)\big)\Big]\ ,$$

Gaussian Approximation

$$S[\phi] \approx \int d^4x\sqrt{\bar{g}}\phi^*\Delta\phi, \text{ so } \int [D\phi] e^{-S[\phi]} \approx \frac{1}{\sqrt{\det(\Delta)}}.$$

$$Z \approx \exp\left(-I[\bar{g}, \bar{A}]\right) \frac{1}{\sqrt{\det(D)}} \frac{1}{\sqrt{\det(P)}}.$$

$$\delta \log Z = -\frac{1}{2}\sum_n \log\left(\delta\Lambda_n\right) \text{ where}$$

$$\delta\Lambda_n^{\text{graviton}} = \int d^4x\sqrt{\bar{g}} {h^n}^*_{\alpha\beta}\delta D^{\alpha\beta\mu\nu} {h^n}_{\mu\nu},$$

Extremal zero mode. *Why?*

$$\delta\Lambda_n^{\text{photon}} = \int d^4x\sqrt{\bar{g}} {a^n}^*_{\mu}\delta P^{\mu\nu} {a^n}_{\nu},$$

Why *extremal zero modes*?

Eigenvalue problem for $(h^* D[\bar{g}, \bar{A}]h$

$$Z \approx \exp\left(-I[\bar{g}, \bar{A}]\right) \int [Dh][Da] \exp\Big[-\int d^4x\sqrt{\bar{g}}\,\Big(h^* D[\bar{g}, \bar{A}]h$$
$$+a^* P[\bar{g}, \bar{A}]a + \left(h^* O_{\text{int}}[\bar{g}, \bar{A}]a + \text{h.c}\right)\Big)\Big]\ ,$$

Three options

1. Extremal non-zero modes $\lambda(T) = \lambda_0 + \mathcal{O}(T)$, $\delta\left(\log Z\right)$ polynomial in T
2. Extremal zero modes $\lambda(T) = \mathcal{O}(T)$, $\delta\left(\log Z\right)$ is $\log T$
3. Near-extremal zero modes $\lambda(T) = \mathcal{O}(T^2)$, $\delta\left(\log Z\right)$ polynomial in T

[Sen `08,`11]

What are our extremal zero modes?

Graviton zero modes

$$ds^2 \sim \mathcal{M} \times S^2$$

1. Tensor modes

$$x^\mu \rightarrow x^\mu + \zeta^\mu$$

$$\Rightarrow h_{\mu\nu} = \mathcal{L}_\zeta \bar{g}_{\mu\nu}$$

2. Vector modes

$$v_{\alpha\mu}^{(n,m)} = \underbrace{\epsilon_{\alpha\beta}\partial^\beta Y_{l=1,m}(\theta,\phi)}_{S^2} \times \underbrace{\partial_\mu \Phi_n(\tau,\eta)}_{\mathcal{M}}$$

Photon zero modes

$$(a_n)_\mu \propto \underbrace{\partial_\mu \Phi_n}_{\mathcal{M}}$$

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RN-dS_4 Black Holes

What happens to near-extremal black holes in a theory with a cosmological constant?

Cosmological EM Theory for $\Lambda > 0$

$$I_{\rm EM} = \frac{1}{16\pi G_N}\int d^4x\,\sqrt{-g}\Big(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}\Big)$$

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega^2\,, \qquad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

$$A = \frac{Q}{r}dt\,, \qquad F = -\frac{Q}{r^2}dr\wedge dt$$

$$V(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{r^2}{\ell_4^2}$$

$$= -\frac{1}{\ell_4^2 r^2}\left(r + r_+ + r_- + r_c\right)\left(r - r_-\right)\left(r - r_+\right)\left(r - r_c\right)$$

Three horizons!

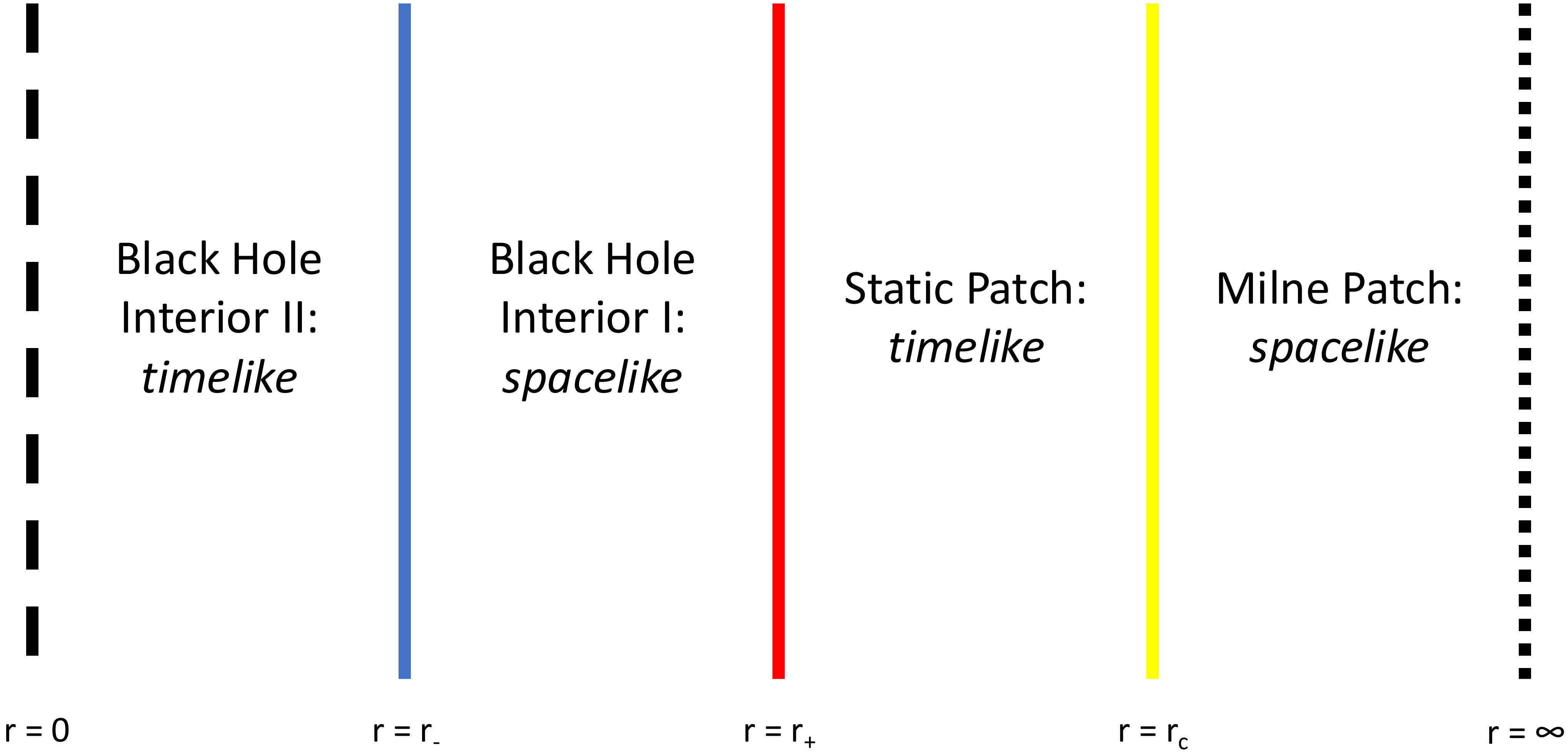
$$M = \frac{1}{2\ell_4^2}\left(r_+ + r_-\right)\left(\ell_4^2 - r_+^2 - r_-^2\right)\,,$$

$$Q^2 = \frac{r_+ r_-}{\ell_4^2}\left(\ell_4^2 - r_+^2 - r_-^2 - r_- r_+\right)\,,$$

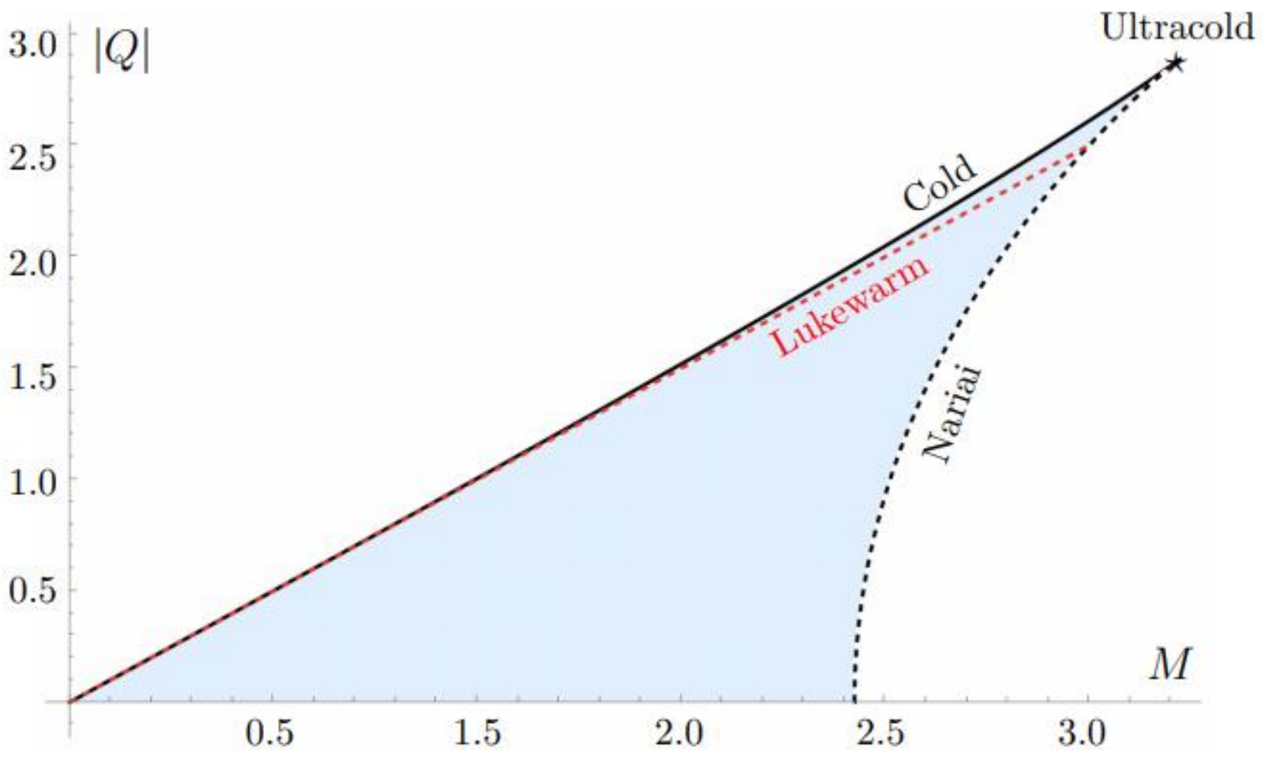
$$\ell_4^2 = r_c^2 + r_+^2 + r_-^2 + r_- r_+ + r_- r_c + r_c r_+\,.$$

What does this look like??

The RN-dS_4 solution: <u>**not**</u> a Penrose diagram

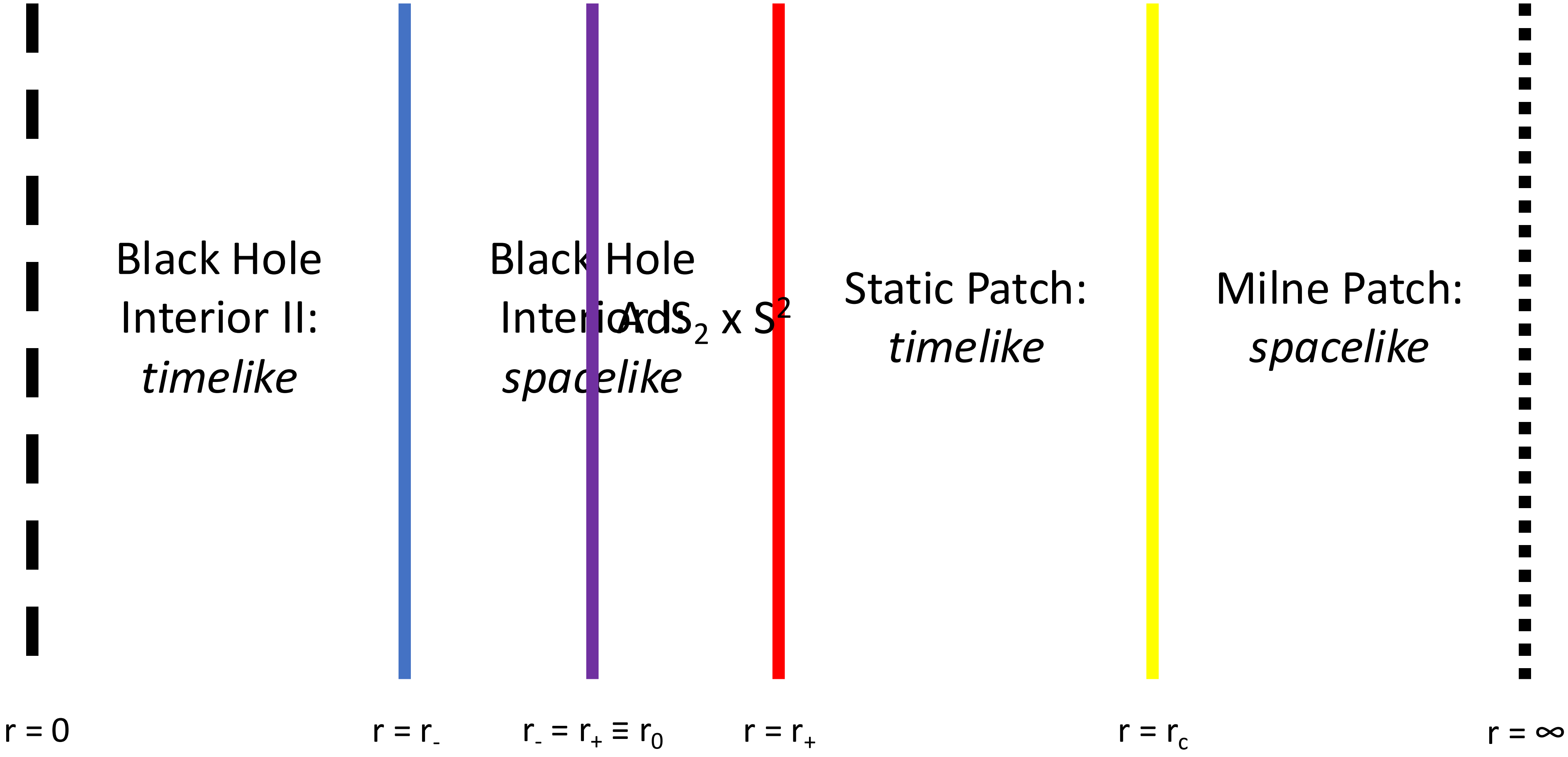


The RN-dS$_4$ solution: Sharkfin diagram

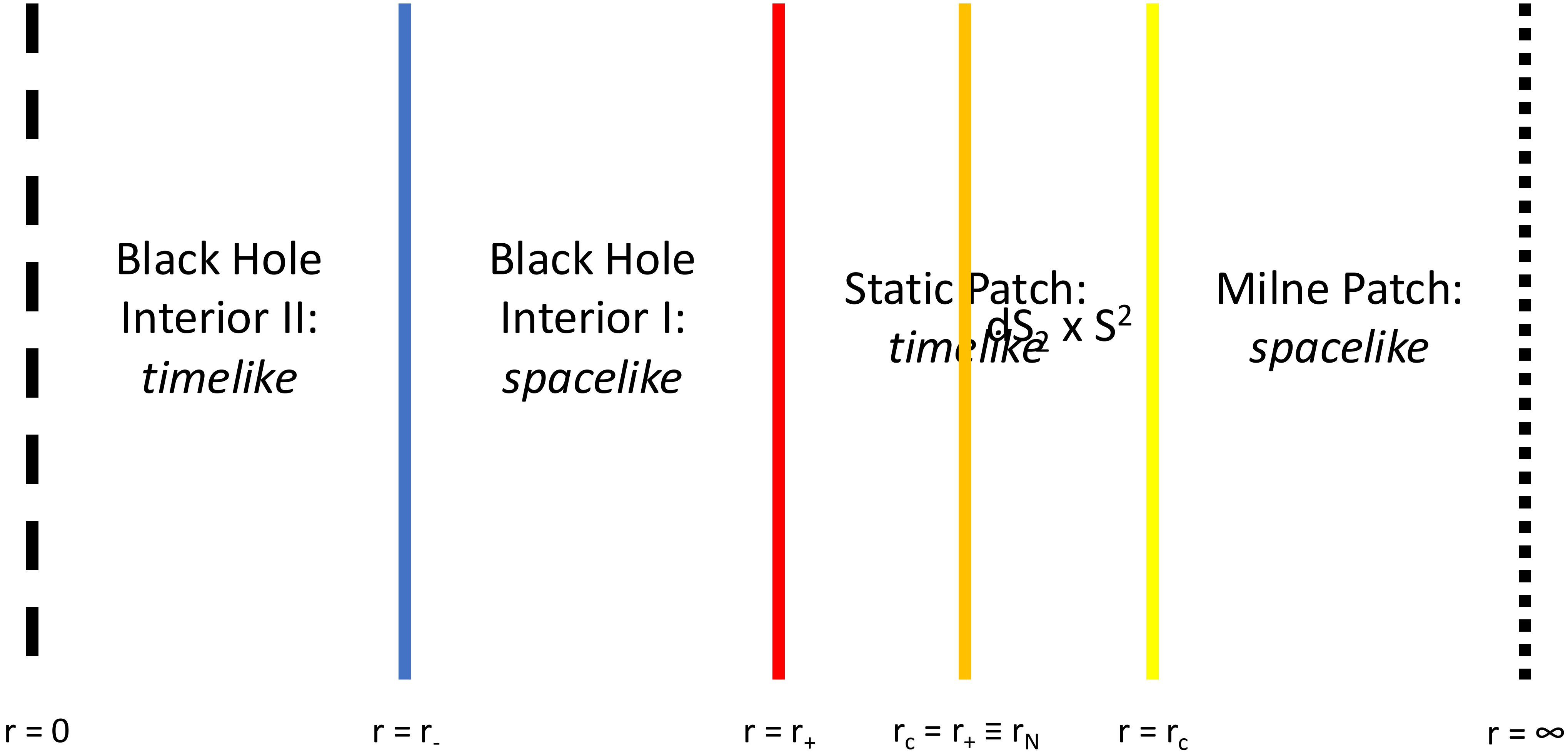


[Romans `92]
[Montero, Van Reit, Venken `20]

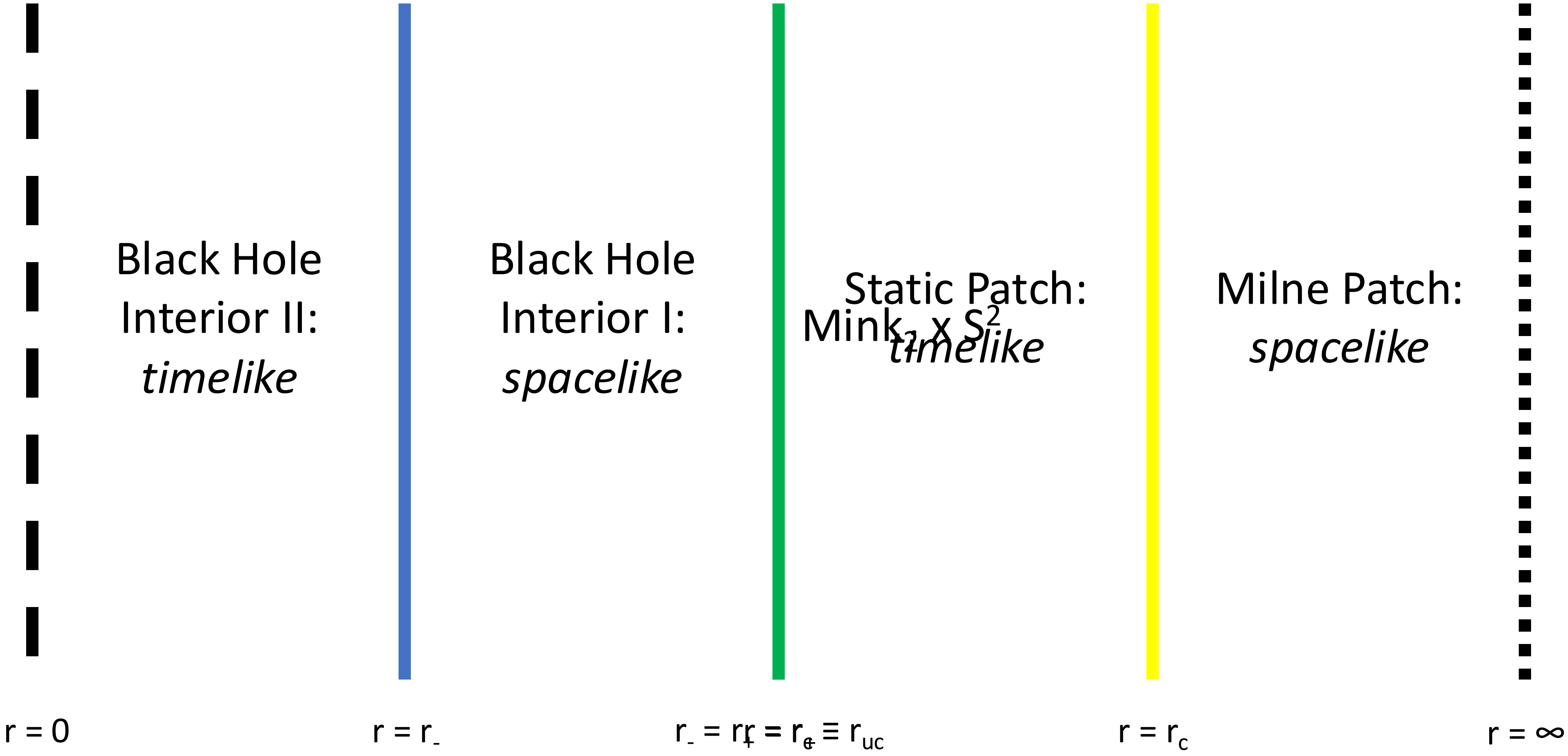
The Cold Limit



The Nariai Limit



The Ultracold Limit



Three Limits

1. Cold limit: $AdS_2 \times S^2$
2. Nariai limit: $dS_2 \times S^2$
3. Ultracold limit: $Mink_2 \times S^2$

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The Cold limit

What happens to the $AdS_2 \times S^2$ story when we have a cosmological constant?

Perturbing away from the extremal limit

- Separate the coincident horizons

$$r_- = r_0 - 2\pi\ell_{\rm AdS}^2 T_+, \qquad r_+ = r_0 + 2\pi\ell_{\rm AdS}^2 T_+,$$

$$\text{where } \ell_{\rm AdS}^2 = \frac{\ell_4^2 r_0^2}{\ell_4^2 - 6r_0^2} \geq r_0^2$$

- Zoom in on the outer horizon

$$r = r_+ + 4\pi T_+ \ell_{\rm AdS}^2 \sinh^2\frac{\eta}{2}$$

- Wick rotate

$$t = \frac{1}{2\pi T_+}(-i\tau)$$

Perturbing away from the extremal limit

- Heat up geometry $g_{ab} = \bar{g}_{ab} + \delta g_{ab} T_+$

$$\bar{g}_{ab} dx^a dx^b = \ell_{\text{AdS}}^2 \left(\sinh^2 \eta d\tau^2 + d\eta^2\right) + r_0^2 d\Omega_2^2,$$

$$\frac{\delta g_{ab} dx^a dx^b}{4\pi \ell_{\text{AdS}}^2} = \frac{\ell_{\text{AdS}}^2 \left(\ell_{\text{AdS}}^2 + 2r_0^2\right) (2 + \cosh \eta) \tanh^2 \frac{\eta}{2}}{3r_0^3} \left(-\sinh^2 \eta d\tau^2 + d\eta^2\right)$$

$$+ r_0 \cosh \eta d\Omega_2^2$$

- Perturb gauge field

$$A_\tau = i \frac{\ell_{\text{AdS}} \sqrt{\ell_{\text{AdS}}^2 + r_0^2}}{\sqrt{2} r_0} (\cosh \eta - 1) - i T_+ \frac{\sqrt{2} \pi \ell_{\text{AdS}}^3 \sqrt{\ell_{\text{AdS}}^2 + r_0^2}}{r_0^2} \sinh^2 \eta$$

Graviton (tensor) modes

$$h_{\mu\nu}^{(n)} dx^\mu dx^\nu = \frac{i e^{in\tau} \ell_{\mathrm{AdS}} \sqrt{|n|(n^2-1)}}{\sqrt{8}\pi r_0} \tanh^n\left(\frac{\eta}{2}\right) \left(\frac{d\eta^2}{\sinh^2\eta} + \frac{2i d\eta d\tau}{\sinh\eta} - d\tau^2\right),$$

$$\delta\Lambda_n = \delta\left(\int d^4x \sqrt{g}\, h_{\alpha\beta}^{(n)\,*} D^{\alpha\beta\mu\nu} h_{\mu\nu}^{(n)}\right) = \frac{|n| T_+}{16 r_0} + \mathcal{O}\left(T_+^2\right).$$

$$(\delta \log Z)_{\mathrm{tensor}} = 2\,(-1/2) \log \prod_{n\geq 2} \delta\Lambda_n = \frac{3}{2} \log \frac{T_+}{r_0} - \log\left(64\sqrt{2\pi}\right),$$

Graviton (vector) modes

$$\Phi_n = \frac{1}{\sqrt{2\pi^3|n|r_0^2}} \left(\frac{\sinh \eta}{1+\cosh \eta} \right)^n e^{in\tau} .$$

$$\delta\Lambda_n = \delta \left(\int d^4x \sqrt{g}\, v_{\alpha\beta}^{(m,n)^*} D^{\alpha\beta\mu\nu} v_{\mu\nu}^{(m,n)} \right) = \frac{\left(\ell_{\mathrm{AdS}}^2 + 2r_0^2\right)|n|T_+}{48r_0^3} + \mathcal{O}\left(T_+^2\right) .$$

$$(\delta \log Z)_{\mathrm{vector}} = \frac{1}{2} \log \frac{\left(\ell_{\mathrm{AdS}}^2 + 2r_0^2\right) T_+}{r_0^3} - \log\left(4\sqrt{6\pi}\right) ,$$

Gauge (photon) modes

$$a_\mu^{(n)} = \frac{\sqrt{\pi} r_0}{2} \partial_\mu \Phi_n (\tau, \eta)$$

$$\delta \Lambda_n = \delta \left(\int d^4 x \sqrt{g} a_\mu^{(n)} P^{\mu\nu} a_\nu^{(n)} \right) = \frac{\left(\ell_{\mathrm{AdS}}^2 - r_0^2 \right) |n| T_+}{24 r_0^3} + \mathcal{O}\left(T_+^2 \right)$$

$$(\delta \log Z)_{\mathrm{photon}} = \frac{1}{2} \log \frac{\left(\ell_{\mathrm{AdS}}^2 - r_0^2 \right) T_+}{r_0^3} - \log \left(4\sqrt{3\pi} \right) , ,$$

Summing

$$\delta \log Z \simeq \frac{3}{2} \log \frac{T_+}{r_0} + \frac{3}{2} \log \frac{\left(\ell_{\mathrm{AdS}}^2 + 2 r_0^2\right) T_+}{r_0^3} + \frac{1}{2} \log \frac{\left(\ell_{\mathrm{AdS}}^2 - r_0^2\right) T_+}{r_0^3},$$

$$\log Z \sim \frac{7}{2} \log T_+ .$$

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The Nariai limit

What story can we tell in $dS_2 \times S^2$?

Perturbing away from the extremal limit

- Separate the coincident horizons

$$r_+ = r_N - 2\pi\ell_{\mathrm{dS}}^2 T_c, \qquad r_c = r_N + 2\pi\ell_{\mathrm{dS}}^2 T_c$$

$$\text{where } \ell_{\mathrm{dS}}^2 = \frac{\ell_4^2 r_N^2}{6r_N^2 - \ell_4^2}$$

- Zoom in on the outer horizon

$$r = r_c + 4\pi T_c \ell_{\mathrm{dS}}^2 \sinh^2\frac{\eta}{2}$$

- Wick rotate

$$t = \frac{1}{2\pi T_c}(-i\tau)$$

Perturbing away from the extremal limit

- Heat up geometry $g_{ab} = \bar{g}_{ab} + \delta g_{ab} T_c$

$$\bar{g}_{ab} dx^a dx^b = -\ell_{\rm dS}^2 \left(\sinh^2 \eta d\tau^2 + d\eta^2\right) + r_N^2 d\Omega_2^2,$$

$$\frac{\delta g_{ab} dx^a dx^b}{4\pi \ell_{\rm dS}^2} = \frac{\ell_{\rm dS}^2 \left(\ell_{\rm dS}^2 - 2r_N^2\right)(2 + \cosh\eta)\tanh^2 \frac{\eta}{2}}{3r_N^3} \left(-\sinh^2 \eta d\tau^2 + d\eta^2\right) + r_N \cosh \eta d\Omega_2^2.$$

- Perturb gauge field

$$A_\tau = i \frac{\ell_{\rm dS}\sqrt{\ell_{\rm dS}^2 - r_N^2}}{\sqrt{2} r_N} (\cosh\eta - 1) - i T_c \frac{\sqrt{2}\pi \ell_{\rm dS}^2 \sqrt{\ell_{\rm dS}^2 - r_N^2}}{r_N^2} \sinh^2 \eta$$

Graviton (tensor) modes

$$h_{\mu\nu}^{(n)} dx^\mu dx^\nu = \frac{e^{in\tau} \ell_{\mathrm{dS}} \sqrt{|n|(n^2-1)}}{\sqrt{8}\pi r_N} \tanh^n \frac{\eta}{2} \left(\frac{d\eta^2}{\sinh^2 \eta} + \frac{2i d\eta d\tau}{\sinh \eta} - d\tau^2 \right),$$

$$\delta\Lambda_n = \delta \left(\int d^4x \sqrt{g} h_{\alpha\beta}^{(n)*} D^{\alpha\beta\mu\nu} h_{\mu\nu}^{(n)} \right) = -\frac{|n| T_c}{16 r_N} + \mathcal{O}\left(T_c^2\right).$$

Divergent integral!

Graviton (vector) modes

$$\Phi_n = \frac{1}{\sqrt{2\pi^3|n|r_N^2}}\left(\frac{\sinh\eta}{1+\cosh\eta}\right)^n e^{in\tau}$$

$$\int d^4x\sqrt{\bar{g}}v_{\mu\nu}^{(m,n)*}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}v_{\alpha\beta}^{(m,n)} = -1$$ ← Uh oh!

$$\delta\left(\int d^4x\sqrt{g}v_{\alpha\beta}^{(m,n)*}D^{\alpha\beta\mu\nu}v_{\mu\nu}^{(m,n)}\right) = \frac{\left(2r_N^2 - \ell_{\mathrm{dS}}^2\right)|n|T_c}{48r_N^3} + \mathcal{O}\left(T_c^2\right)$$

↑ Sometimes negative!

Gauge (photon) modes

$$a_\mu^{(n)} = \frac{\sqrt{\pi} r_N}{2} \partial_\mu \Phi_n \left(\tau, \eta\right)$$

$$\int d^4x \sqrt{\bar{g}} a_\mu^{(n)*} \bar{g}^{\mu\nu} a_\nu^{(n)} = -1$$ ← Uh oh!

$$\delta \left(\int d^4x \sqrt{g} a_\mu^{(n)} P^{\mu\nu} a_\nu^{(n)} \right) = -\frac{\left(r_N^2 + \ell_{\mathrm{dS}}^2\right) |n| T_c}{24 r_N^3} + \mathcal{O}\left(T_c^2\right)$$

↑ Divergent integral!

So what went wrong

$$ds^2 = (-EAdS_2) \times S^2$$

Mode	Norm	$\delta\Lambda_n$
Graviton (tensor)	+1	-1
Graviton (vector)	-1	+/-
Photon (gauge)	-1	-1

A possible solution

$$r \to i\rho \;\; (r_N \to i\rho_N)$$

$$T_c = -i\tilde{T}_c$$

$$ds^2 = -(EAdS_2 \times S^2)$$

Graviton (tensor) modes: good!

$$\int d^4x\sqrt{\bar{g}}h_{\mu\nu}^{(n)^*}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}h_{\alpha\beta}^{(n)} = 1.$$

$$\delta\Lambda_n = \delta\left(\int d^4x\sqrt{g}h_{\alpha\beta}^{(n)^*}D^{\alpha\beta\mu\nu}h_{\mu\nu}^{(n)}\right) = \frac{|n|\tilde{T}_c}{16\rho_N} + \mathcal{O}\left(\tilde{T}_c^2\right)$$

Safe sign!

Graviton (vector) modes

$$\int d^4x \sqrt{\bar{g}} v_{\mu\nu}^{(m,n)*} \bar{g}^{\mu\alpha} \bar{g}^{\nu\beta} v_{\alpha\beta}^{(m,n)} = 1$$

← Safe sign!

$$\delta\Lambda_n = \delta\left(\int d^4x \sqrt{g} v_{\alpha\beta}^{(m,n)*} D^{\alpha\beta\mu\nu} v_{\mu\nu}^{(m,n)}\right) = \frac{\left(2\rho_N^2 + \ell_{\mathrm{dS}}^2\right)}{48\rho_N^3} |n| \tilde{T}_c + \mathcal{O}\left(\tilde{T}_c^2\right)$$

$$= \frac{4\rho_N^2 + \ell_4^2}{6\rho_N^2 + \ell_4^2} \frac{|n|\tilde{T}_c}{16\rho_N} + \mathcal{O}\left(\tilde{T}_c^2\right)$$

↑ Safe sign!

Gauge (photon) modes

$$\int d^4x\sqrt{\bar{g}}a_\mu^{(n)\,*}\bar{g}^{\mu\nu}a_\nu^{(n)} = -1$$ ← Uh oh!

$$\delta\left(\int d^4x\sqrt{g}a_\mu^{(n)}P^{\mu\nu}a_\nu^{(n)}\right) = \frac{(\rho_N^2 - \ell_{\rm dS}^2)}{24\rho_N}|n|\tilde{T}_c + \mathcal{O}\left(\tilde{T}_c^2\right)$$

$$= \frac{4\rho_N^2}{6\rho_N^2 + \ell_4^2}\frac{|n|\tilde{T}_c}{16\rho_N} + \mathcal{O}\left(\tilde{T}_c^2\right).$$

↑ Safe sign!

Recap of signs

$$ds^2 = (-EAdS_2) \times S^2$$

Mode	Norm	$\delta\Lambda_n$
Graviton (tensor)	+1	-1
Graviton (vector)	-1	+/-
Photon (gauge)	-1	-1

$$ds^2 = -(EAdS_2 \times S^2)$$

Mode	Norm	$\delta\Lambda_n$
Graviton (tensor)	+1	+1
Graviton (vector)	+1	+1
Photon (gauge)	-1	+1

Summing

$$\delta \log Z \simeq \frac{3}{2} \log \frac{\tilde{T}_c}{\rho_N} + \frac{3}{2} \log \frac{\left(2\rho_N^2 + \ell_{\text{dS}}^2\right) \tilde{T}_c}{\rho_N^3} + \frac{1}{2} \log \frac{\left(\rho_N^2 - \ell_{\text{dS}}^2\right) \tilde{T}_c}{\rho_N^3},$$

$$\Rightarrow Z \propto \left(\tilde{T}_c\right)^{7/2} \frac{\left(2\rho_N^2 + \ell_{\text{dS}}^2\right)^{3/2} \sqrt{\rho_N^2 - \ell_{\text{dS}}^2}}{\rho_N^{15/2}} Z_0,$$

$$\Psi \propto \left(\tilde{T}_c\right)^{7/2} \frac{\left(2\rho_N^2 + \ell_{\text{dS}}^2\right)^{3} \sqrt{\rho_N^2 - \ell_{\text{dS}}^2}}{\rho_N^{15/2}} \Psi_0$$

$$= (-iT_c)^{7/2} \frac{\left(2\rho_N^2 + \ell_{\text{dS}}^2\right)^{3} \sqrt{\rho_N^2 - \ell_{\text{dS}}^2}}{\rho_N^{15/2}} \Psi_0,$$

Compared to [Maldacena, Turiaci, Yang `19]
[Cotler, Jensen, Maloney `19]

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Conclusions

What have we learnt, what don't we understand, and where to next?

What have we learnt?

- Gaussian path integral can resolve puzzles in near-extremal black holes
- The cosmological constant *matters*
- Things are messier in de Sitter

Where to next?

- Ultracold black hole
- Working in static patch vs Milne patch [MJB, Hartnoll `23]
- Wheeler DeWitt interpretations [Maldacena, Turiaci, Yang `19]
- Going away from near horizion region see e.g. [Kapec, Law, Toldo ‘24] [Kolanowski, Marolf, Rakic, Rangamani, Turiaci ‘24]

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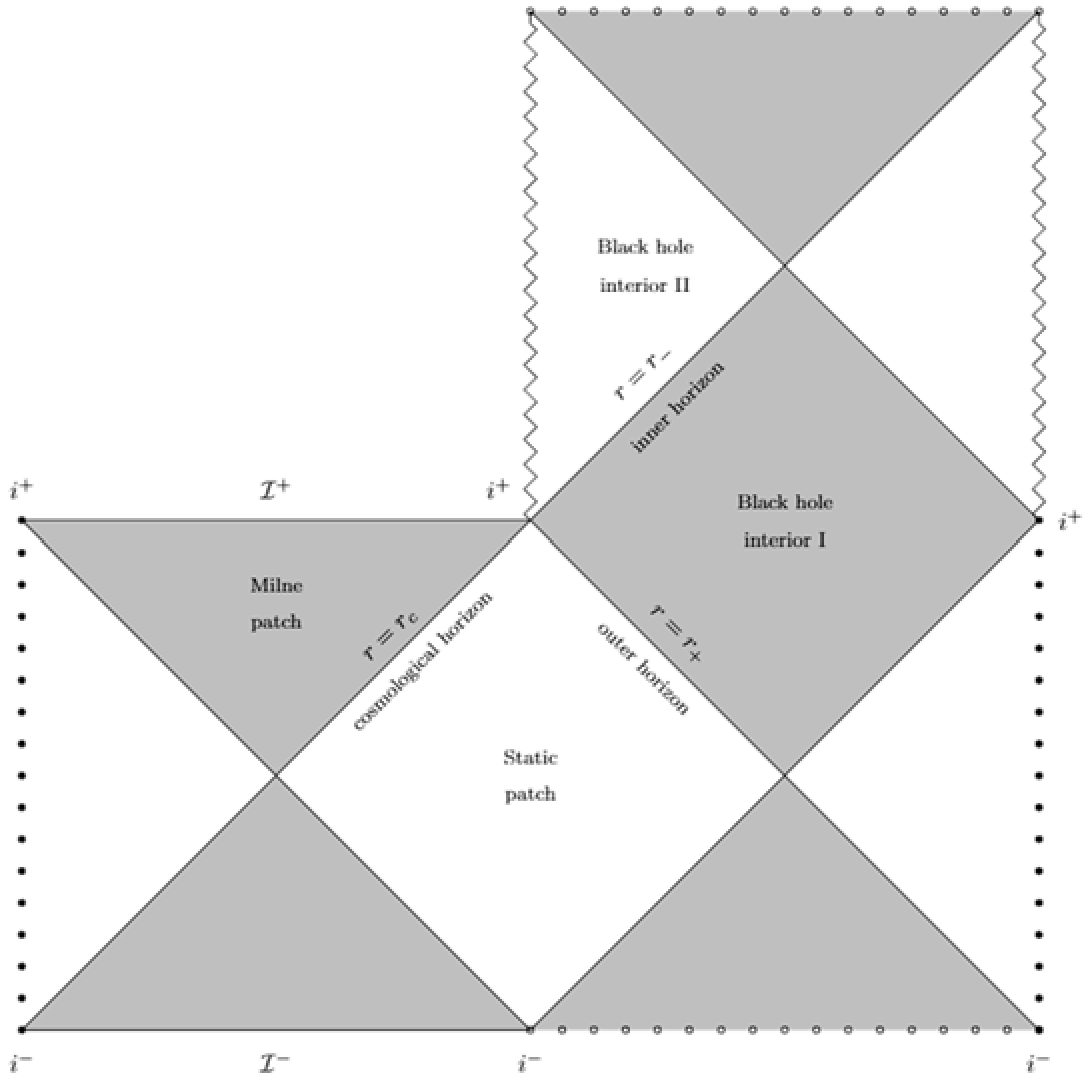
Thank you!

Details

For inevitable questions

The RN-dS$_4$ solution: Penrose diagram



General Strategy: Gauges and Ghosts

$$\mathcal{L}_{\text{diffeo}} = \frac{1}{32\pi}\bar{g}_{\mu\nu}\left(\bar{\nabla}_{\alpha}h^{\alpha\mu} - \frac{1}{2}\bar{\nabla}^{\mu}h^{\alpha}_{\ \alpha}\right)\left(\bar{\nabla}_{\beta}h^{\beta\nu} - \frac{1}{2}\bar{\nabla}^{\nu}h^{\beta}_{\ \beta}\right),$$

$$\mathcal{L}_{U(1)} = \frac{1}{32\pi}\left(\bar{\nabla}_{\alpha}a^{\alpha}\right)^{2}.$$

$$\mathcal{L}_{\text{Ghost}} = -\frac{1}{16\pi}\left(b_{\mu}\left(\bar{g}^{\mu\nu}\bar{\Box} + \bar{R}^{\mu\nu}\right)c_{\nu} + b^{(i)}\bar{\Box}c^{(i)} - 2b^{(i)}\bar{F}^{\mu\nu}\bar{\nabla}_{\mu}c_{\nu}\right),$$

General Strategy: Operators

$$h^*_{\alpha\beta} D^{\alpha\beta,\mu\nu}[\bar{g}] h_{\mu\nu} = -\frac{1}{16\pi} h^*_{\alpha\beta} \Big(\frac{1}{4} \bar{g}^{\alpha\mu} \bar{g}^{\beta\nu} \bar{\Box} - \frac{1}{8} \bar{g}^{\alpha\beta} \bar{g}^{\mu\nu} \bar{\Box} + \frac{1}{2} \bar{R}^{\alpha\mu\beta\nu} + \frac{1}{2} \bar{R}^{\alpha\mu} \bar{g}^{\beta\nu}$$
$$- \frac{1}{2} \bar{R}^{\alpha\beta} \bar{g}^{\mu\nu} - \frac{1}{4} \bar{R} \bar{g}^{\alpha\mu} \bar{g}^{\beta\nu} + \frac{1}{8} \bar{R} \bar{g}^{\alpha\beta} \bar{g}^{\mu\nu} + \frac{1}{8} F^2 \left(2 g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu} \right)$$
$$- F^{\alpha\mu} F^{\beta\nu} - 2 F^{\alpha\gamma} F^{\mu}{}_{\gamma} g^{\beta\nu} + F^{\alpha\gamma} F^{\beta}{}_{\gamma} g^{\mu\nu}$$
$$- \frac{\Lambda}{2} g^{\alpha\mu} g^{\beta\nu} + \frac{\Lambda}{4} g^{\alpha\beta} g^{\mu\nu} \Big) h_{\mu\nu} \ .$$

$$a^*_\mu P^{\mu\nu} a_\nu = -\frac{1}{32\pi} a^*_\mu \left(\bar{g}^{\mu\nu} \Box - \bar{R}^{\mu\nu} \right) a_\nu ,$$

$$h^*_{\alpha\beta} O^{\alpha\beta\mu}_{\text{int}} a_\mu = \frac{1}{16\pi} h^*_{\alpha\beta} \left(4 \bar{g}^{\alpha[\mu} \bar{F}^{\nu]\beta} + F^{\mu\nu} \bar{g}^{\mu\nu} \right) \nabla_\mu a_\nu .$$