QUANTUM GROUPS AS GLOBAL SYMMETRIES IN THE CONTINUUM

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Based on 2411.xxxxx with

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SYMMETRIES IN QFT

Great handle on QFTs given by symmetries

↳ predictions at strong coupling

↳ conserved quantities, selection sectors, ...

↳ predictions about IR phases of theories

Question: what is the most general internal symmetry a QFT can have?

Classic answer:

symmetries act on local operators, are described by groups

Group structure: $g^{-1} \cdot g = 1, \qquad g_1 \cdot g_2 = g_3$

On correlation fcts: $\langle g(\mathcal{O}_1 \dots \mathcal{O}_N) \rangle = \langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle$

On the Hamiltonian: $g H g^{-1} = H$

Can be discrete (e.g. $\mathbb{Z}_2, S_N, \dots$)
or continuous ($O(N), SU(N), \dots$)

New answer I: symmetries can be non-invertible

$g \rightsquigarrow \nexists g$

$g_1 \circ g_2 = g_3 + g_4 + \ldots$

See e.g. [Petkova, Zuber '00]
[Chang et al '18]

Better described by fusion categories.

Example: Kramers-Wanniers duality in 2d Ising CFT $\mathcal{N}$

$\mathcal{N} \cdot \mathcal{N} = \mathbb{1} + \eta$ → $\mathbb{Z}_2$ elements

$\mathcal{N} \cdot \sigma = 0$ ↳ spin field

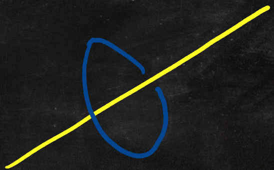
A lot known in $d=2$, also important in $d>2$ (e.g. ABJ anomaly)

New answer II. Symmetries can act on non-local operators

[Gaiotto, Kapustin, Seiberg, Willet '14]

p-form symmetry act on dimension p operators

- 0-form : acts on points
- 1-form : lines
- ⋮



E.g. • Polyakov center symmetry in $SU(N)$ pure Yang-Mills is $\mathbb{Z}_N^{(1)}$ symmetry

• Schwinger model (QED_{1+1}) w/ charge $q \to \mathbb{Z}_q^{(1)}$

THE QUESTION:

Is this all there is?

APPETIZER:

QUANTUM GROUPS ON THE LATTICE

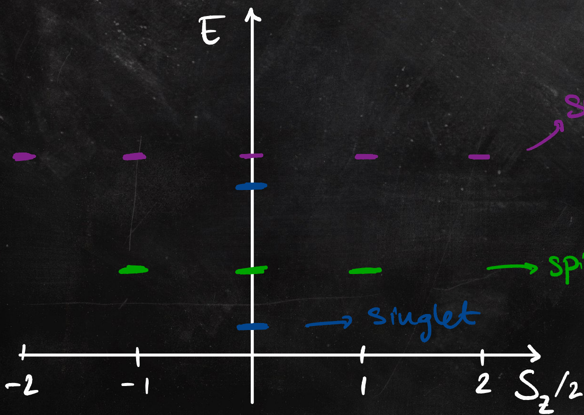
[Pasquier, Saleur '90]

HEISENBERG / XXX MODEL

Spin chain with spin ½ (with OBC)

$$H_{XXX} = \sum_{i=1}^{N-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z \right)$$

Spectrum at finite N



SU(2) symmetry

Generators:

$$S^z = \sum_{i=1}^{N} \sigma_i^z$$

$$S^\pm = \sum_{i=1}^{N} \sigma_i^\pm$$

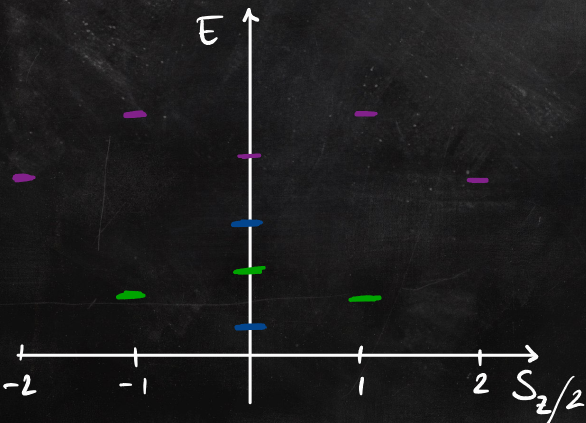
commute with H!

XXZ MODEL

$$H_{XXZ} = \sum_{i=1}^{N-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q+q^{-1}}{2} \sigma_i^z \sigma_{i+1}^z \right)$$

$q = e^{i\theta}$, $\theta \in \mathbb{R}$

Spectrum ($q \neq \pm 1$)



Only $U(1)$ symmetry left for $q \neq \pm 1$

$[H, S^z] = 0$

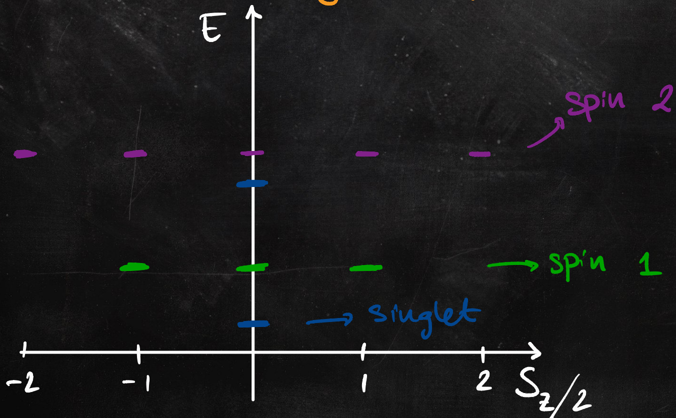
$[H, S^\pm] \neq 0$

Now add some imaginary surface terms

[Pasquier, Saleur '90]

$$H_{PS} = H_{XXZ} + \underbrace{\frac{q-q^{-1}}{2}}_{\hookrightarrow\, i\mathbb{R}} \left(\sigma_1^z - \sigma_N^z\right)$$

Spectrum (generic q)



But: no $SU(2)$

$$[H, S^{\pm}] \neq 0$$

only $U(1)$

$$[H, S^z] = 0$$

Degeneracies look like $su(2)$, but there is no $su(2)$!

Degeneracies explained by

$$E = \sum_i q^{\sigma_z/2} \otimes \dots \otimes q^{\sigma_z/2} \otimes \sigma^+ \otimes q^{-\sigma_z/2} \otimes \dots \otimes q^{-\sigma_z/2}$$

position i

$F =$ same with $\sigma^+ \to \sigma^-$

$$[H_{PS}, F] = [H_{PS}, E] = 0$$

E and F do not satisfy $su(2)$ algebra but together with the $u(1)$ generator, they form the quantum group $\mathcal{U}_q(sl_2)$

Quantum group symmetries on the lattice

* 1+1 D spin chains
* 2D stat mech models (e.g. loop models)

In the continuum they appear more indirectly

* crossing kernel of Virasoro blocks in minimal models $\to$ 6J symbols of $U_q(sl_2)$
* fusion rule for $SU(2)_k$ WZW models $\to$ fusion rules of $U_q(sl_2)$
* integrability: Yang-Baxter equation

GLOBAL SYMMETRIES IN THE CONTINUUM?

the previous continuum examples are not global symmetries!

But the QG symmetric spin chain is critical, it flows to a CFT!

Are QGs a new type of symmetry in QFT?

Q: can I build a CFT with $\mathcal{U}_q(sl_2)$ as a global symmetry?

Q: what does it look like? How do I find it?

Q: related to generalized symmetries?

TODAY'S PLAN

* ~~Motivation~~

* $U_q(sl_2)$ basics

* CFTs with Quantum Group symmetry

* An example: XXZ with non-local b.c.

* Connection with unitary theories

* Future directions

$U_q(sl_2)$ BASICS

Coproduct

Need to be able to act on multiple spins or operators. the coproduct Δ allows that

sl_2: $\Delta(X) = X \otimes \mathbb{1} + \mathbb{1} \otimes X \qquad X = E, F, H$

Needs to be compatible with the comm. rels.

$$\Delta([E, F]) = [\Delta(E), \Delta(F)]$$

$\mathcal{U}_q(sl_2)$: deformed comm. rels $\rightarrow$ deformed coproducts
Not a unique choice

$$\Delta(E) = E \otimes \mathbb{1} + q^{-H} \otimes E$$
$$\Delta(F) = F \otimes q^{H} + \mathbb{1} \otimes F$$
$$\Delta(H) = H \otimes \mathbb{1} + \mathbb{1} \otimes H \rightarrow \text{generates a "normal" } U(1).$$

NOT A GROUP!

A quantum group is an algebra, not a group.

I cannot make group elements out of it!

Normally: X algebra generator $\to$ $\Delta(X) = X \otimes \mathbb{1} + \mathbb{1} \otimes X$

$g = e^{i\alpha X}$ group element $\to$ $\Delta(g) = g \otimes g$

$U_q(sl_2)$: cannot build g w/ a nice coproduct

REPRESENTATION THEORY

Two different cases

* q not root of unity: looks like $su(2)$
  spin - ℓ irreps, $2\ell+1$ dimensional

$$|\ell, m\rangle \quad , \quad m = -\ell, -\ell+1, \ldots, \ell$$

$$H|\ell, m\rangle = 2m|\ell, m\rangle$$

$$E|\ell, m\rangle = \# |\ell, m+1\rangle \qquad F|\ell, m\rangle = \# |\ell, m-1\rangle$$

$$E|\ell, \ell\rangle = 0 \qquad F|\ell, -\ell\rangle = 0$$

* q root of unity, $q^{p+1} = \pm 1$, $p \in \mathbb{Z}_{>0}$
  $E^p = F^p = 0$. More complicated

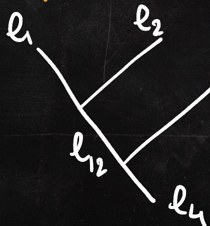
QUANTUM CLEBSCH - GORDON COEFFICIENTS

Some objects play the same role as in $su(2)$

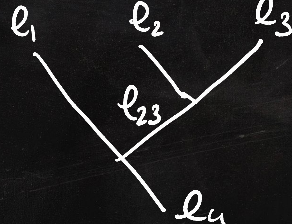
* quantum Clebsch - Gordon coefficients

$$|l_1, m_1\rangle\, |l_2, m_2\rangle = \sum_{\substack{l \\ m}} \begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}_q |l, m\rangle$$

* quantum 6J symbols



$$= \sum_{l_{23}} \begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l_4 & l_{23} \end{Bmatrix}_q$$



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CFTs with
QUANTUM GROUP
SYMMETRIES

OPERATORS IN A CFT

Work in 2D, generic $q = e^{i\theta}$, $\theta \in \mathbb{R}$

Internal symmetry:
- $\mathcal{O}$ transform under $\mathcal{U}_q(sl_2)$
- $[\mathcal{U}_q(sl_2), \text{Virasoro}] = 0$

$m \to u(1)$ charge, $|m| \le \ell$

$$\mathcal{O}_\ell^m(x) \qquad \text{weights } (h, \bar{h})$$

$\ell \to$ spin ℓ rep.

$$E\,\mathcal{O}_\ell^m = \#\ \mathcal{O}_\ell^{m+1}$$

$$F\,\mathcal{O}_\ell^m = \#\ \mathcal{O}_\ell^{m-1}$$

OPERATORS ARE NOT MUTUALLY LOCAL!

local operators : $\langle \mathcal{O}_1(x)\,\mathcal{O}_2(y)\rangle = \pm \langle \mathcal{O}_2(y)\,\mathcal{O}_1(x)\rangle$

↳ boson vs. fermions

Symmetry → Ward identity

$$\mathcal{O}_\pm \equiv \mathcal{O}^{\pm 1/2}_{1/2}$$

$$\langle \underbrace{\Delta(F)}_{F\otimes q^H + 1\otimes F} \; \mathcal{O}_+(x)\,\mathcal{O}_+(y)\rangle = 0$$

$$\rightsquigarrow \quad q\,\underbrace{\langle \mathcal{O}_-(x)\,\mathcal{O}_+(y)\rangle}_{(-1)^{2S}\langle \mathcal{O}_-(y)\,\mathcal{O}_+(x)\rangle \text{ from rotation}} + \langle \mathcal{O}_+(x)\,\mathcal{O}_-(y)\rangle = 0$$

$$\Rightarrow \quad \langle \mathcal{O}_+(x)\,\mathcal{O}_-(y)\rangle = -q\,(-1)^{2S}\,\langle \mathcal{O}_-(y)\,\mathcal{O}_+(x)\rangle$$

$\rightarrow$ spacetime spin

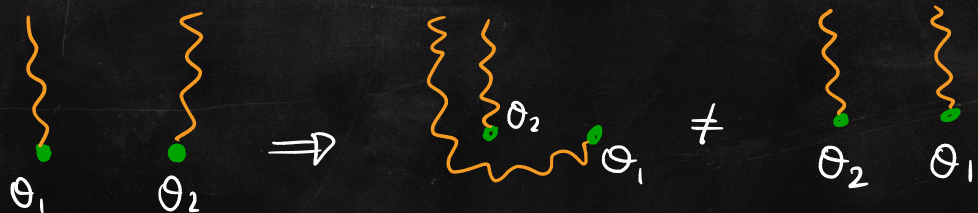
$$\langle \Theta_+(x)\,\Theta_-(y)\rangle = -q\,(-1)^{2s}\,\langle \Theta_-(y)\,\Theta_+(x)\rangle$$

↳ either $s \neq \mathbb{Z}/2$

↳ or $[\Theta_+, \Theta_-] \neq 0$

In both cases, operator not mutually local.

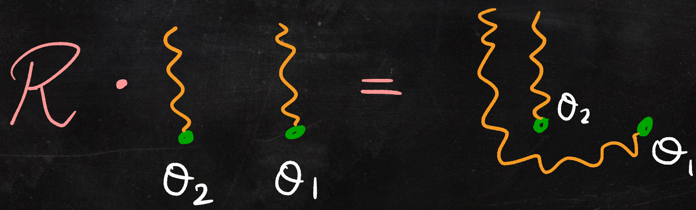
Simplest modification: operators are attached to lines!



Lines are topological:

can be moved freely as long as they do not cross other operators.

$U_q(sl_2)$ object: R-matrix, swaps two operators



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* ~~CFTs with Quantum Group symmetry~~

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AN EXAMPLE:

NON-LOCAL B.C. in XXZ

Simpler to study spin chains with translational invariance

but: I don't know any $\mathcal{U}_q(sl_2)$ system w/ periodic boundary conditions

Closest thing: XXZ_q. Deformation of XXZ spin chain w/ non-local interactions.

[Grosse, Pallua, Prester, Raschhofer '94]

....

Not unitary, not a problem in Euclidean space.

Critical and integrable: CFT spectrum known.

Parametrize $q = \exp\left(i\pi \frac{\mu}{\mu+1}\right)$ $\mu \in \mathbb{R}_+$

central charge $c = 1 - \frac{6}{\mu(\mu+1)}$

Operators $W^{m}_{r,s}$ $m \to u(1)$ label; $r,s \to$ kac labels

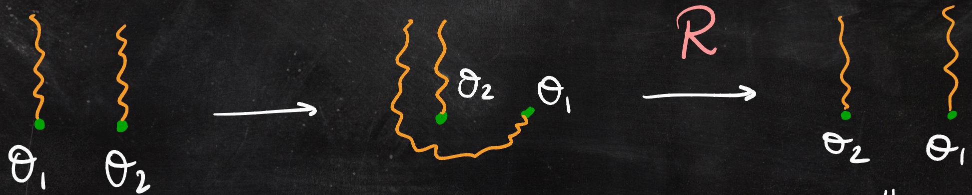
↳ Virasoro weights: $(h_{r,s}, h_{r,1}) \to$ spin not integer

↳ $U_q(sl_2)$ quantum numbers $\ell = \frac{s-1}{2}$, m.

Not known: OPE coefficients / $n \geq 3$ point functions

EVERYTHING MAKES SENSE!

- check I :
  existence of lines and $U_q(sl_2)$ properties imply relation between spacetime spin $s = h - \bar{h}$ and $U_q(sl_2)$ spin ℓ



$$\langle \theta_1 \theta_2 \rangle \qquad \qquad \langle \theta_2 \theta_1 \rangle$$

✓ Satisfied by all operators

- Check II:

  is the theory crossing symmetric?

Need to find the OPE coefficients of the theory $\mathcal{O}_1 \cdot \mathcal{O}_2 = \sum_k \lambda_{12k}\, \mathcal{O}_k$

Crossing symmetry:

$$\langle \mathcal{O}_1(0)\, \mathcal{O}_2(z)\, \mathcal{O}_3(1)\, \mathcal{O}_4(\infty)\rangle = \sum_J \lambda_{12J}\, \lambda_{34J}\, \mathcal{F}_J^{(s)}(z)\, \mathcal{F}_J^{(s)}(\bar{z})$$

$$= \sum_k \lambda_{23k}\, \lambda_{14k}\, \mathcal{F}_k^{(t)}(z)\, \mathcal{F}_k^{(t)}(\bar{z})$$

Two methods of fixing OPE coeffs

1. Degeneracy of operators [Belavin, Polyakov, Zamolodchikov '84]

$$W_{r,s} \rightsquigarrow (h_{r,s}, h_{r1}) \quad \text{with} \quad r,s \in \mathbb{N}$$

$$\rightsquigarrow \langle D_{r,s}\, W_{r,s} \,\ldots\ldots\ldots \rangle = 0$$

↳ differential operator of degree $r \cdot s$

Solve BPZ diff. eq for 4 point function;

get Virasoro blocks;

impose crossing symmetry → fix OPE coeffs

B. Coulomb Gas approach

Get correlation functions from larger auxiliary theory

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_{c<1} = \int \prod_{i=1}^{m} dw_i \, \langle \mathcal{O}_1 \ldots \mathcal{O}_n V_1(w_1) \ldots V_m(w_m) \rangle_{c=1}$$

Originally from [Dotsenko, Fateev '84]

modified to include quantum groups.

In both approaches, find that theory is well defined (i.e. crossing symmetric). ✓

CONNECTION WITH UNITARY THEORIES

$U_q(s\ell_2)$ appears in many 2d CFTs,

but not as a global symmetry.

E.g. Minimal models

$$\langle \mathcal{O}_1(0)\,\mathcal{O}_2(z)\,\mathcal{O}_3(1)\,\mathcal{O}_4(\infty)\rangle = \sum_J{}' \lambda_{12J}\,\lambda_{34J}\,\mathcal{F}^{(s)}_J(z)\,\mathcal{F}^{(s)}_J(\bar{z})$$

$$= \sum_k{}' \lambda_{23k}\,\lambda_{14k}\,\mathcal{F}^{(t)}_k(z)\,\mathcal{F}^{(t)}_k(\bar{z})$$

$$\mathcal{F}^{(t)}_k(z) = \sum_J{}' \mathbb{F}_{kJ}\,\mathcal{F}^{(s)}_J(z)$$

↳ crossing/fusion kernel

known that [Furlan, Ganchev, Petkova '90]

$$\mathbb{F}_{jk} = \begin{pmatrix} \text{ratio of OPE} \\ \text{coefficients} \end{pmatrix} \cdot \ \mathcal{U}_q(sl_2) \ 6J \text{ symbols}$$

Why do $\mathcal{U}_q(sl_2)$ 6J symbols appear?

XXZ_q gives an explanation!

Crossing kernel explained

→ Crossing symmetry of XXZ_q can be written exactly as the crossing kernel for (some) operators

$$\mathcal{F}^{(t)}_k = \sum_J (\ldots) \underbrace{\{ \ldots \}_q}_{6J \text{ symbol}} \mathcal{F}^{(s)}_J$$

→ Virasoro blocks only care about scaling dimensions, same for XXZ_q and minimal model

→ $\mathcal{U}_q(sl_2)$ appears in minimal models because the XXZ_q CFT exists

→ Coulomb gas explains the same for all other operators.

$U_q(sl_2)$ as hidden symmetry in minimal models

$$q = \exp\left(i\pi \frac{\mu}{\mu+1}\right) \Rightarrow c = 1 - \frac{6}{\mu(\mu+1)}$$

For $\mu \in \mathbb{N}$, recover c of minimal models $M_{\mu,\mu+1}$

Here XXZ_q develops a subsector which is a (twist of) minimal model

E.g. $\mu = 3 \rightsquigarrow Z_{XXZ_q} \supset Z_{\substack{\text{fermionic} \\ \text{Ising}}}$

known for the lattice [Grosse et al '94]

checked that the same happens for the CFT

FUTURE DIRECTIONS

To sum up

CFTs with $U_q(sl_2)$ symmetry exist

* $U_q(sl_2)$ mixes local and non-local (defect ending) operators

* One such example arises from the XXZ spin chain with non-local boundary conditions

* this theory is non-unitary

* contains unitary and local subsectors in some cases
explains appearence of $U_q(sl_2)$ in unitary theory

OPEN QUESTIONS

- Any relations to non-invertible symmetries? Claims in the literature [Saleur, Read '07]

  ↳ can I build a group-like element $\Delta(g) = g \otimes g$ in the CFT?

- Unitary theory?

- Generalizations to $U_q(s\ell_{N>2})$?

- Higher dimensions: line operators in $d=3$ in non-topological theories