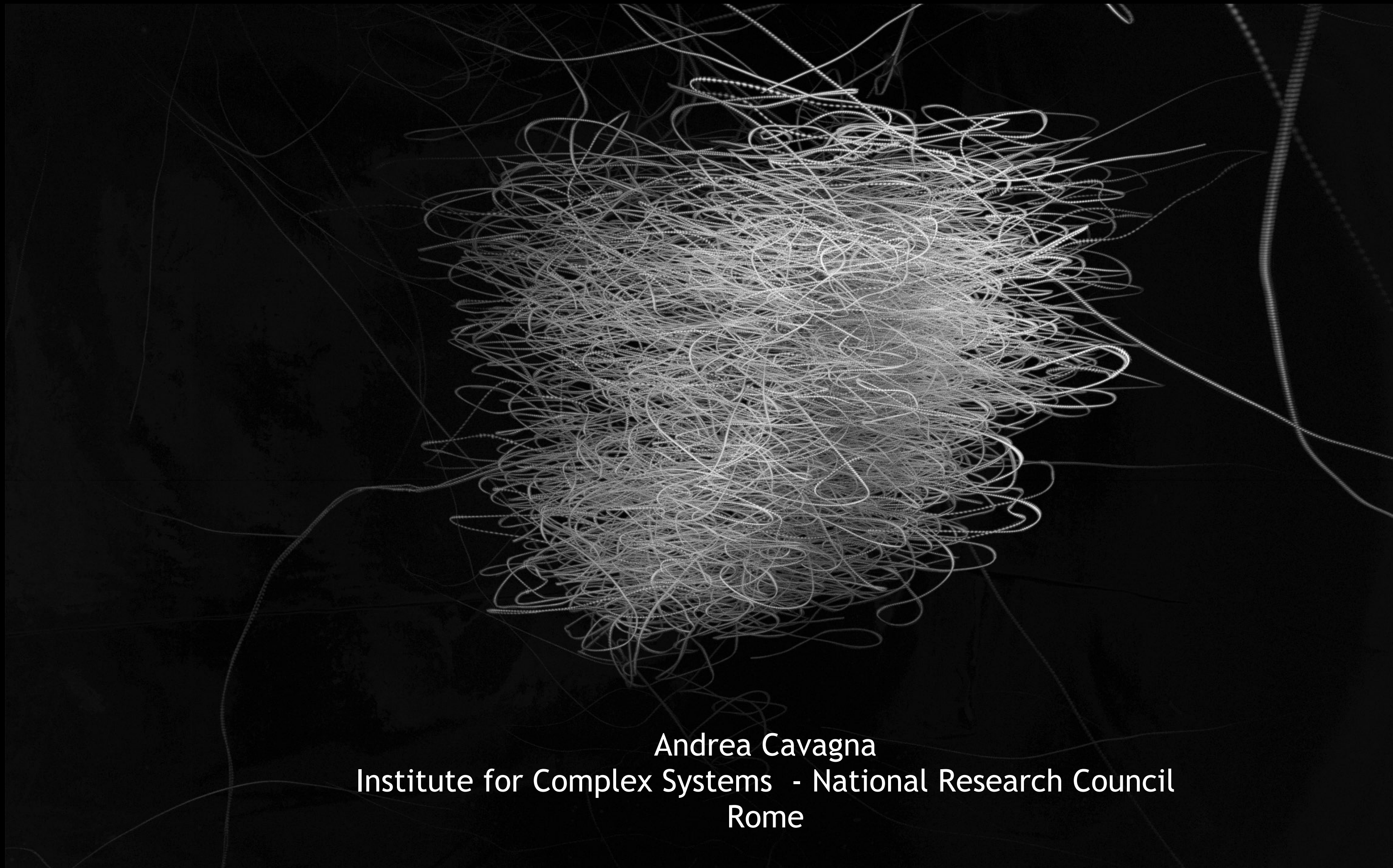
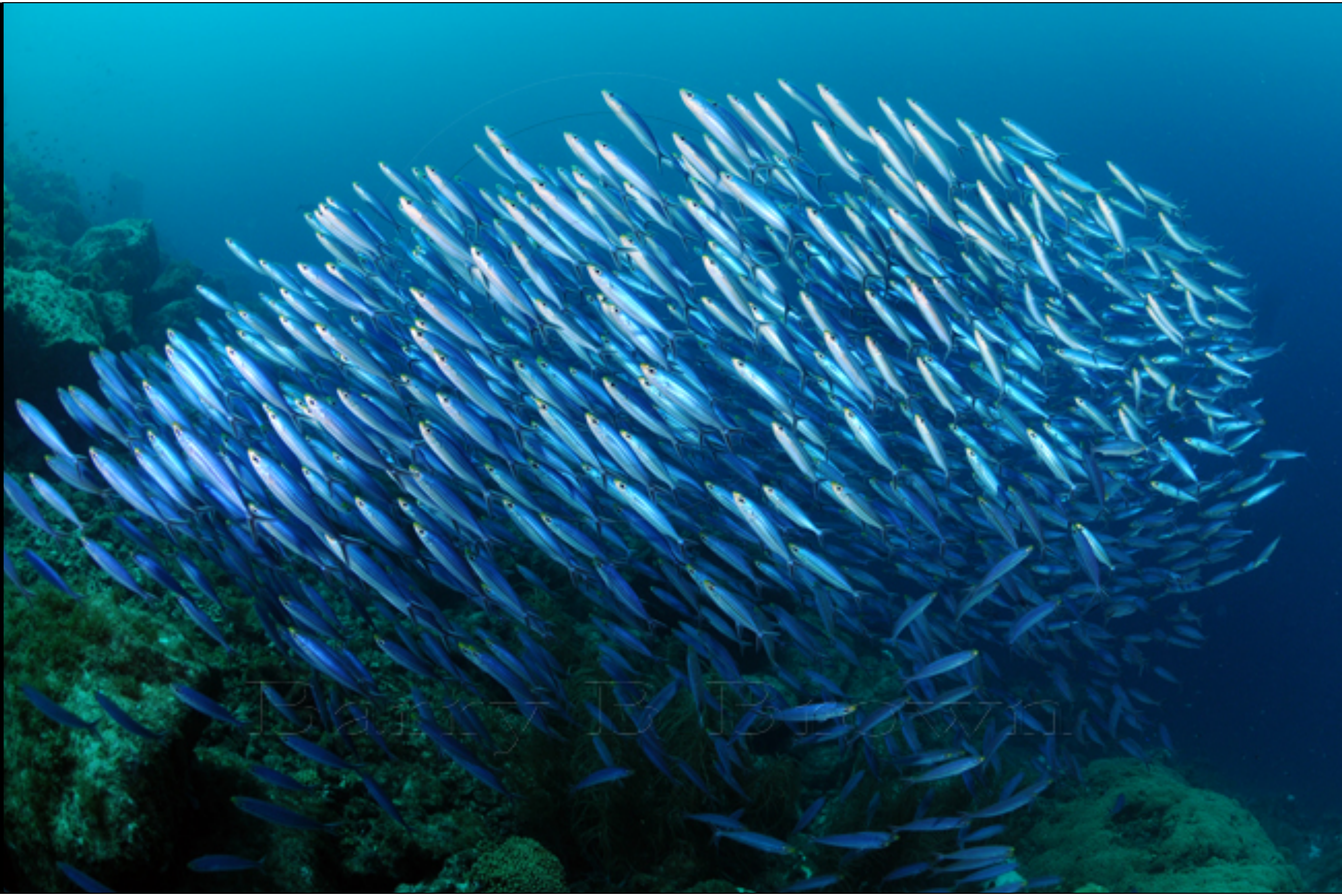
Natural Swarms in 3.99 Dimensions



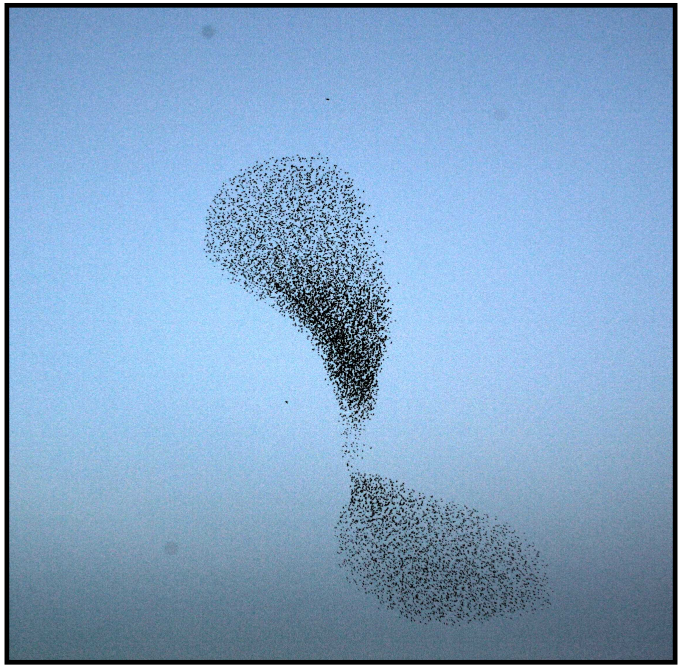
Andrea Cavagna
Institute for Complex Systems - National Research Council
Rome

"One of the more conspicuous properties of nature
is the great diversity of size or length scales
in the structure of the world"

Kenneth G. Wilson

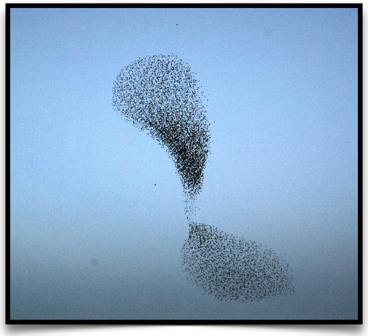

Barry B Brown

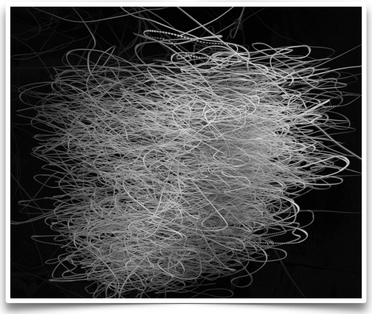
how do we tame such complexity?



by doing *outrageously* minimal models

like, all flocks and swarms of any species in three equations





$$\frac{d\boldsymbol{v}_i}{dt} = \boldsymbol{s}_i \times \boldsymbol{v}_i$$

$$\frac{d\boldsymbol{s}_i}{dt} = \boldsymbol{F}_i \times \boldsymbol{v}_i - \eta \boldsymbol{s}_i + \text{noise}$$

$$\frac{d\boldsymbol{r}_i}{dt} = \boldsymbol{v}_i$$

“microscopic details do not matter much

when large-scale structures emerge

in the system”

says who?

the renormalization group says so

THE JOURNAL OF CHEMICAL PHYSICS VOLUME 43, NUMBER 11 1 DECEMBER 1965

Equation of State in the Neighborhood of the Critical Point

B. WIDOM

Physics Vol. 2, No. 6, pp. 263-272, 1966.

SCALING LAWS FOR ISING MODELS NEAR T_c*

LEO P. KADANOFF†

PHYSICAL REVIEW VOLUME 177, NUMBER 2 10 JANUARY 1969

Scaling Laws for Dynamic Critical Phenomena

B. I. HALPERIN AND P. C. HOHENBERG

Volume 29A, number 6 PHYSICS LETTERS 2 June 1969

ON THE MICROSCOPIC FOUNDATION OF SCALING LAWS

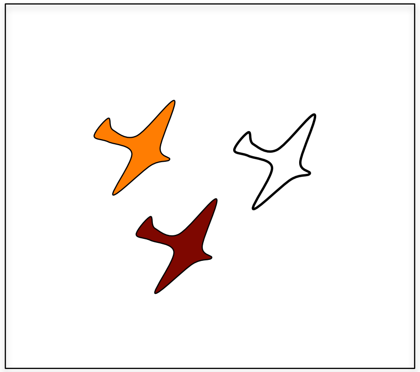
C. DI CASTRO and G. JONA-LASINIO

VOLUME 28, NUMBER 4 PHYSICAL REVIEW LETTERS 24 JANUARY 1972

Critical Exponents in 3.99 Dimensions*

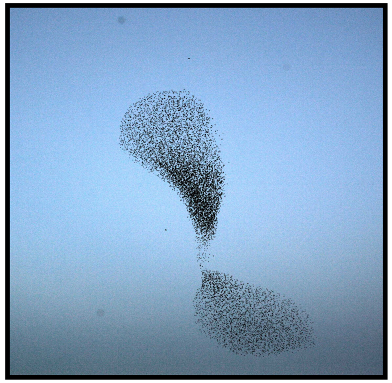
Kenneth G. Wilson and Michael E. Fisher

"more is different"



one (microscopic) scale

scaling up…



many scales

- if different scales do not interact with each other: **easy** case
- if different scales interact with each other: **tough** case

Renormalization Group
for pedestrians

short scale

Bangkok

London

large scale

Bangkok

London

large scale

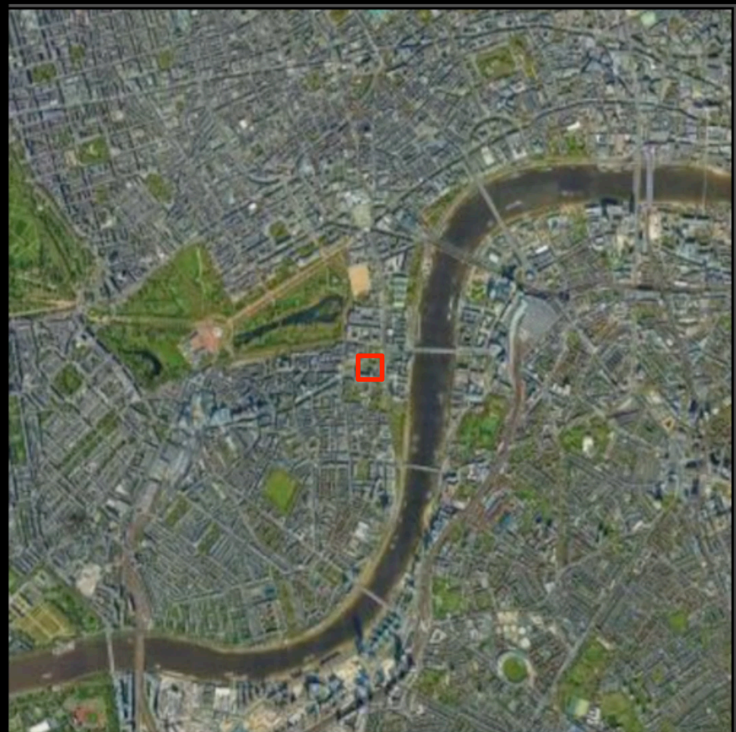
Bangkok

London

“microscopic details do not matter much when we are interested in the large-scale behaviour of the system”

but… wait a minute!

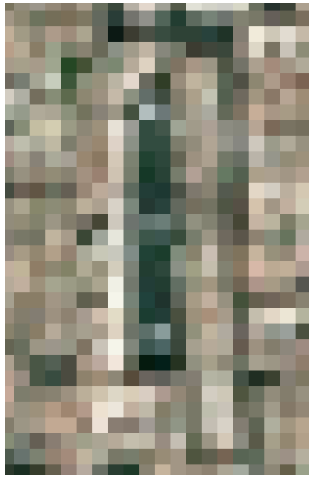
Westminster Abbey is still there, isn't it?



in fact, Westminster Abbey is gone!

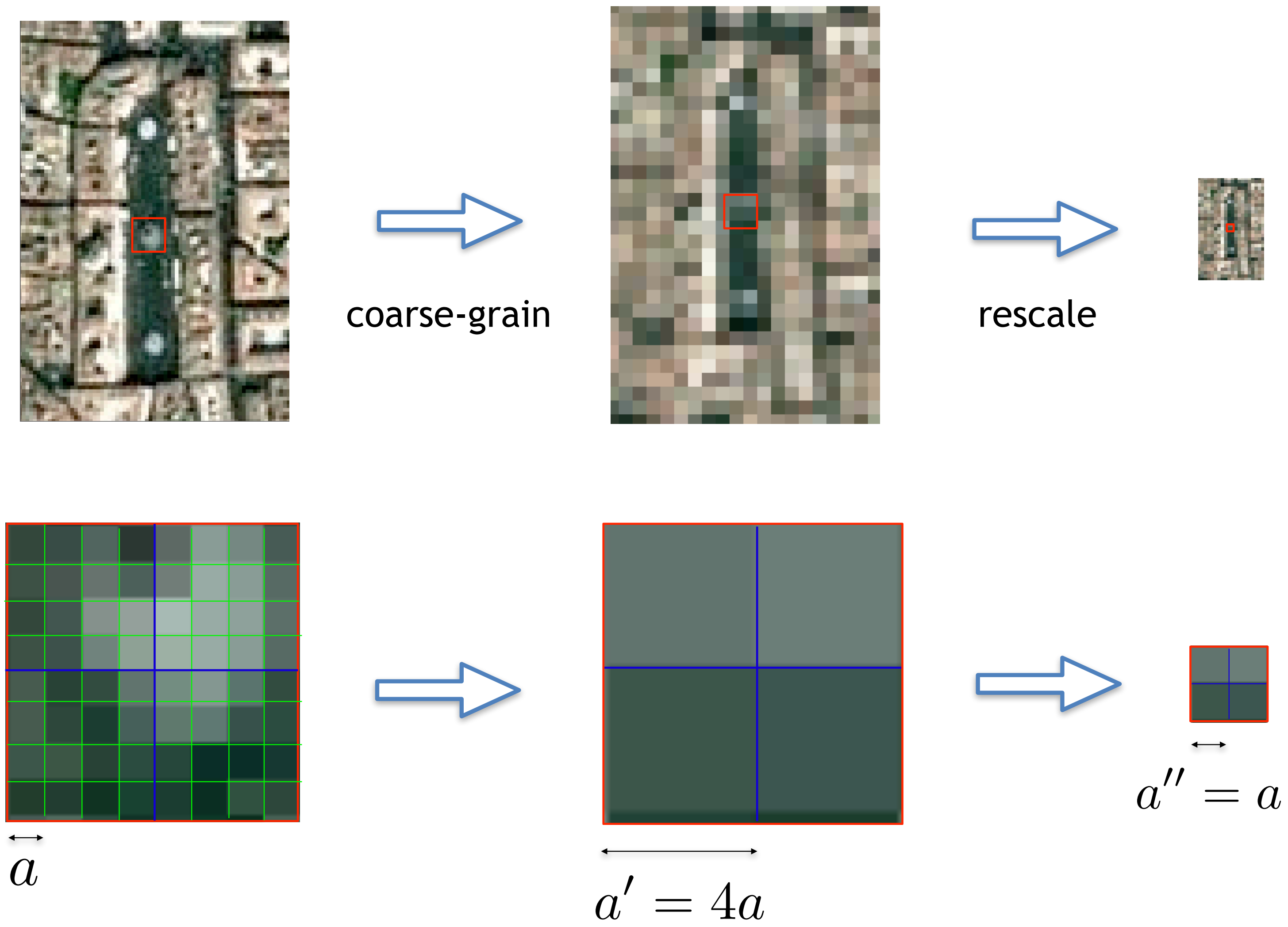
RG zooming out is actually composed of *two* steps

coarse-grain

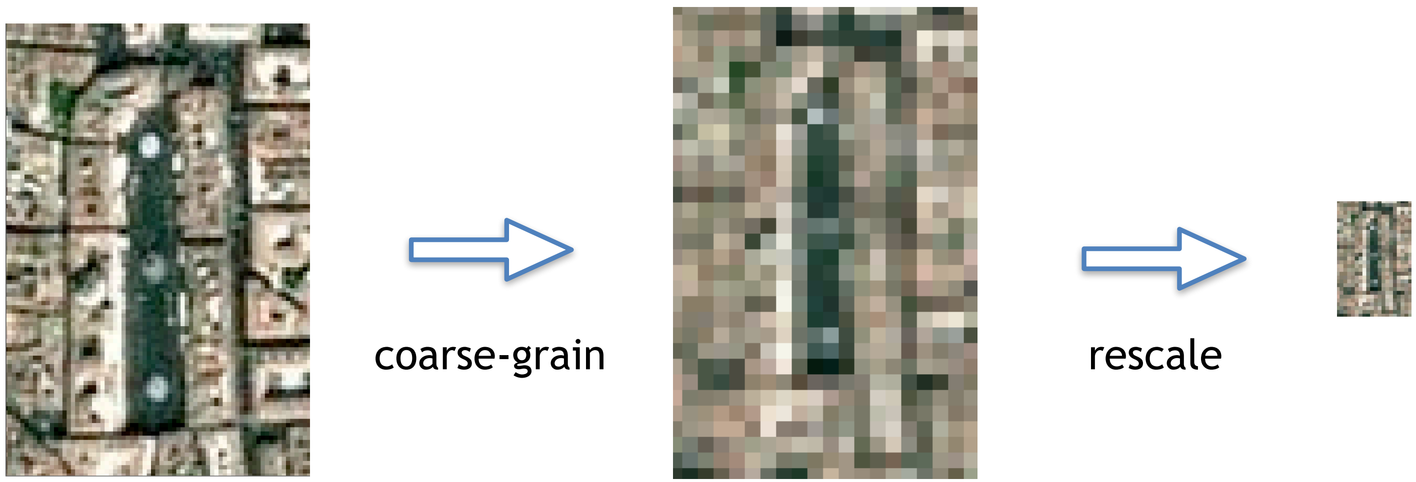


rescale

zooming out is actually composed of *two* steps



what is good in losing information on the short scales?



smaller features are blurred but the larger ones are unaffected

the new systems has the *same* large-scale properties as the original one
but with all finer scales fluctuations eliminated

the new blurred system has new interactions

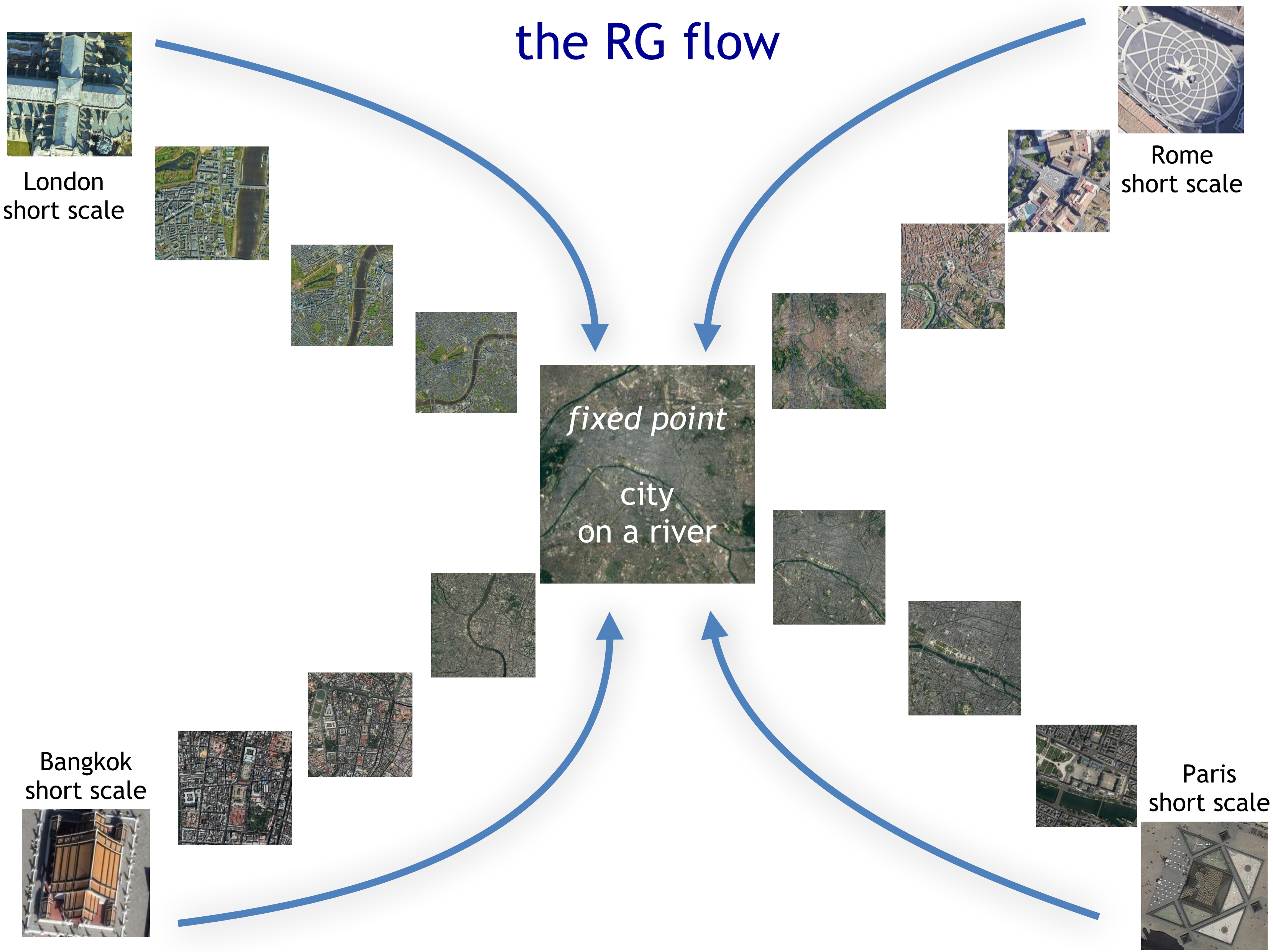
there are **couplings** between the new coarse-grained variables
which determine the **probability** of the new system

RG

- if different scales are *not* coupled: the interactions are the same
- if different scales *are* coupled: the interactions are changed!

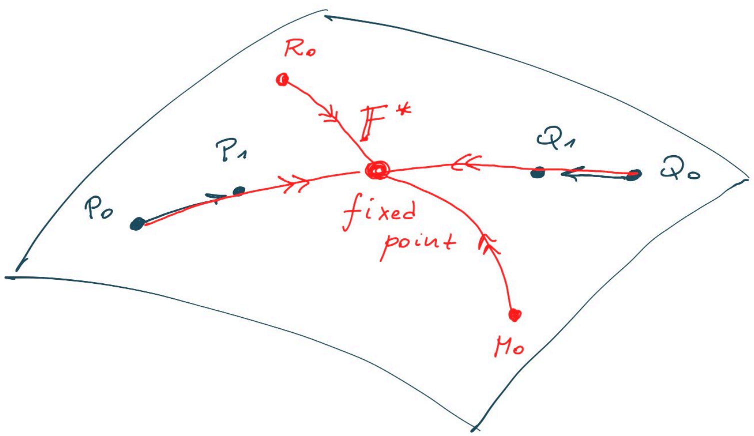
flow in the space of systems!

the RG flow


London
short scale
Rome
short scale
fixed point
city
on a river
Bangkok
short scale
Paris
short scale

the same RG fixed point rules the **large scale** behaviour of systems with different **short scale** details

this is **universality**



this is why physicists love to toy with toy models does it work also in biology?

correlation ✓

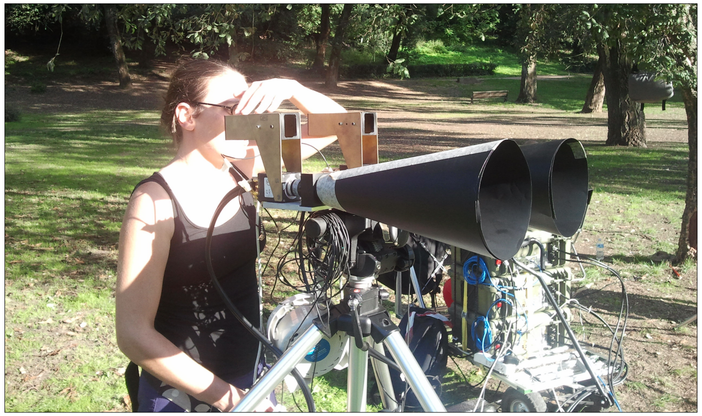
scaling ✓

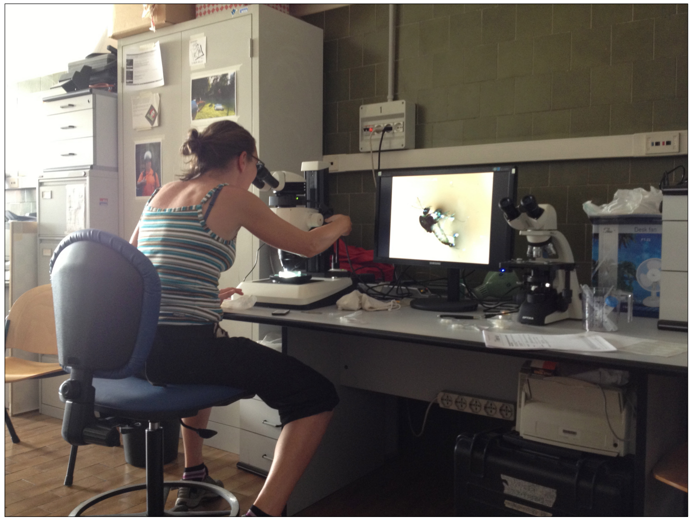
$\boxed{\text{renormalization group}}$

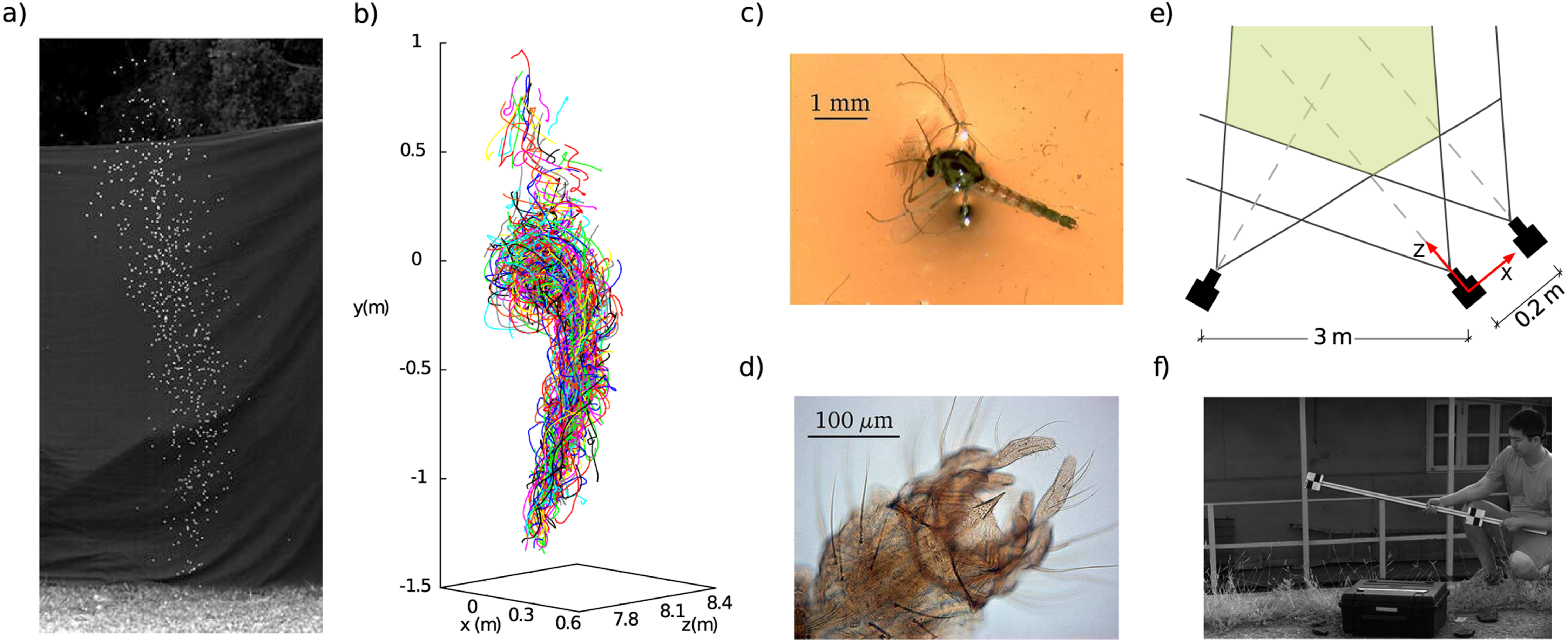
universality

experiments

experiments

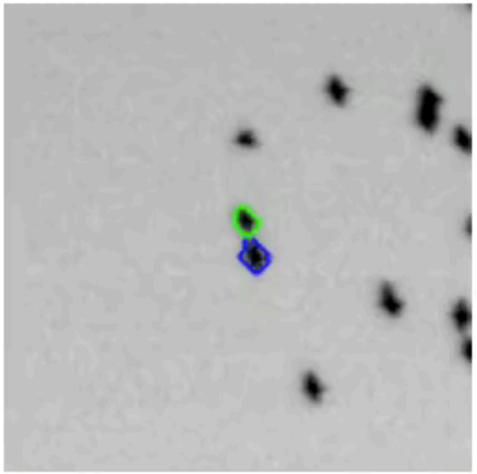


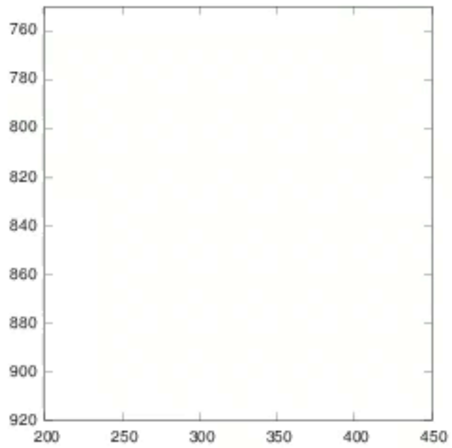

a)
b)
1
0.5
0
y(m)
-0.5
-1
-1.5
0
0.3
0.6
x (m)
7.8
8.1
8.4
z(m)
c)
1 mm
d)
100 μm
e)
z
x
3 m
0.2 m
f)

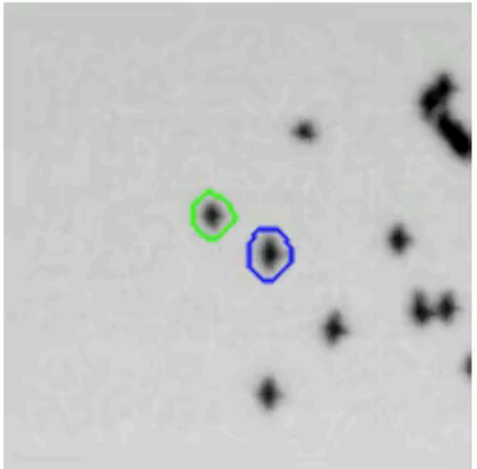
tracking

frame 1

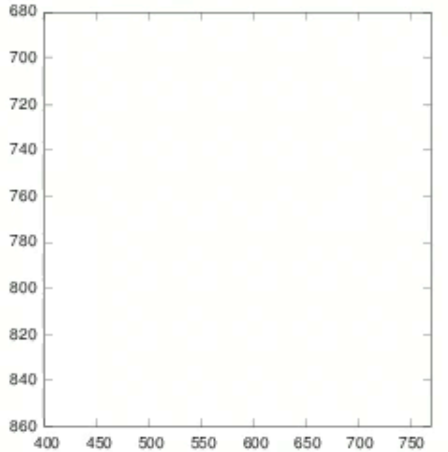


right camera

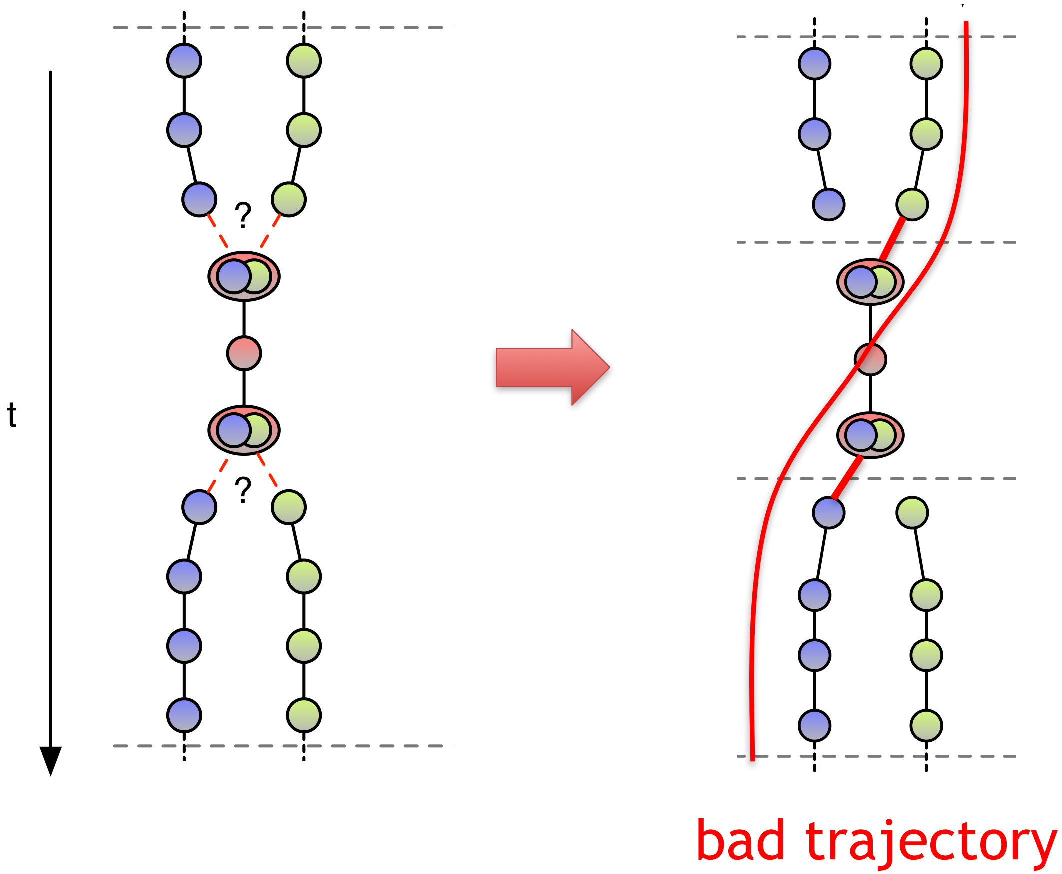




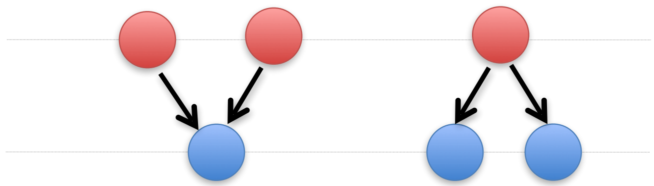
left camera

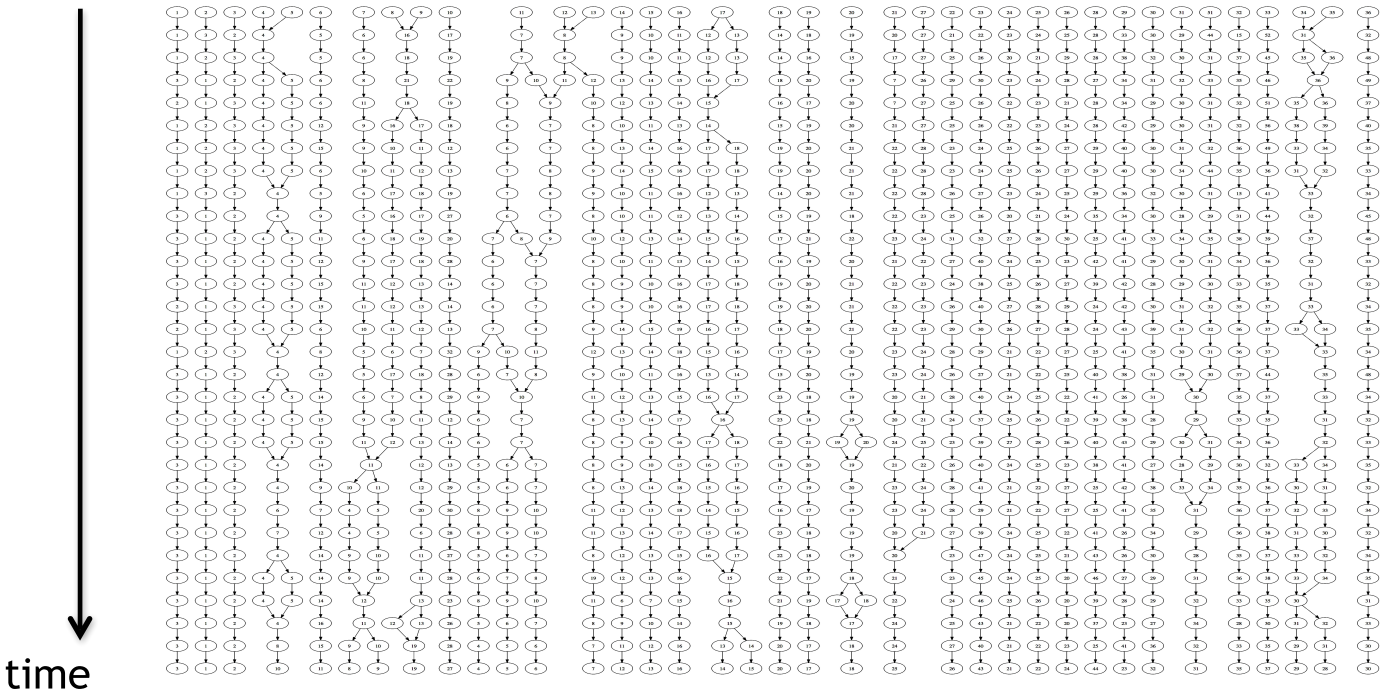


virus

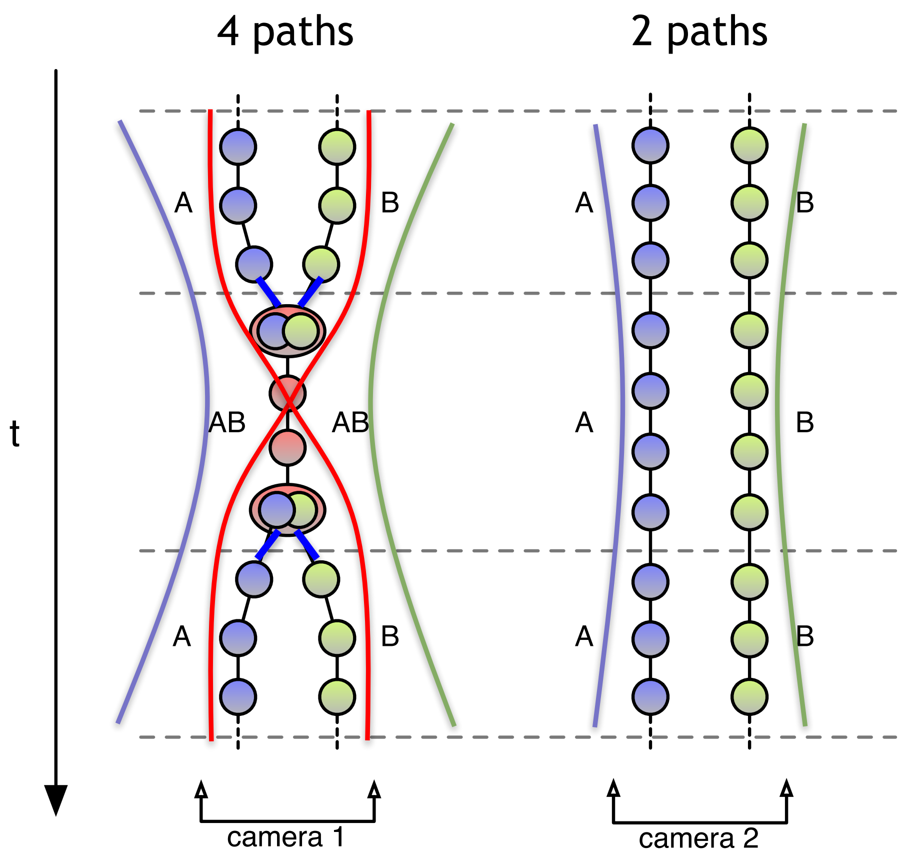


bifurcate - build all paths - define a weigth - optimize





paths optimization

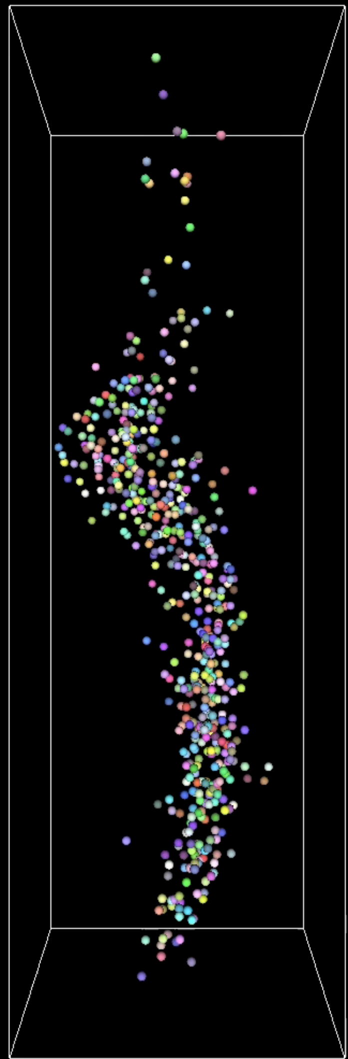


weigth matrix

	A	B
AA	10	4
AB	7	7
BA	7	7
BB	4	10

2 real trajectories

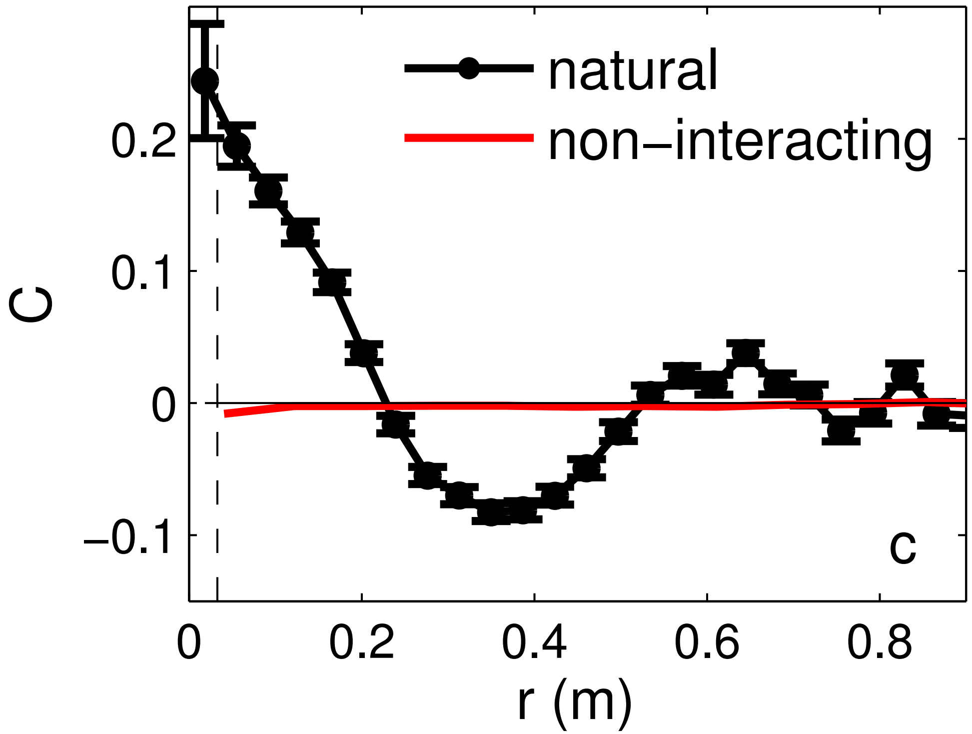
A. Attanasi, A. Cavagna, L. Del Castello, I. Giardina, A. Jelić, S. Melillo, L. Parisi, F. Pellacini, E. Shen, E. Silvestri, et al., GReTA-a novel global and recursive tracking algorithm in three dimensions, IEEE Trans. Pattern Anal. Mach. Intell. 37 (12) (2015) 2451–2463.

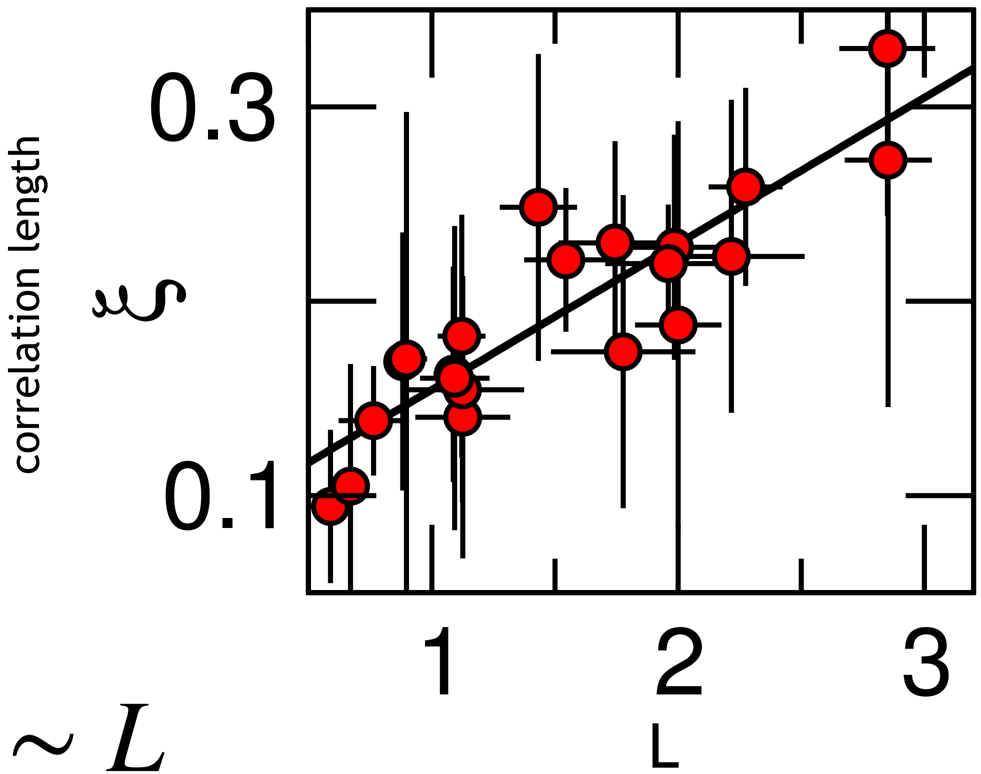


spatial correlations

equal-time velocity correlation function

$$C(r) = \langle \delta\vec{v}(x_0, t_0) \cdot \delta\vec{v}(x_0 + r, t_0) \rangle$$



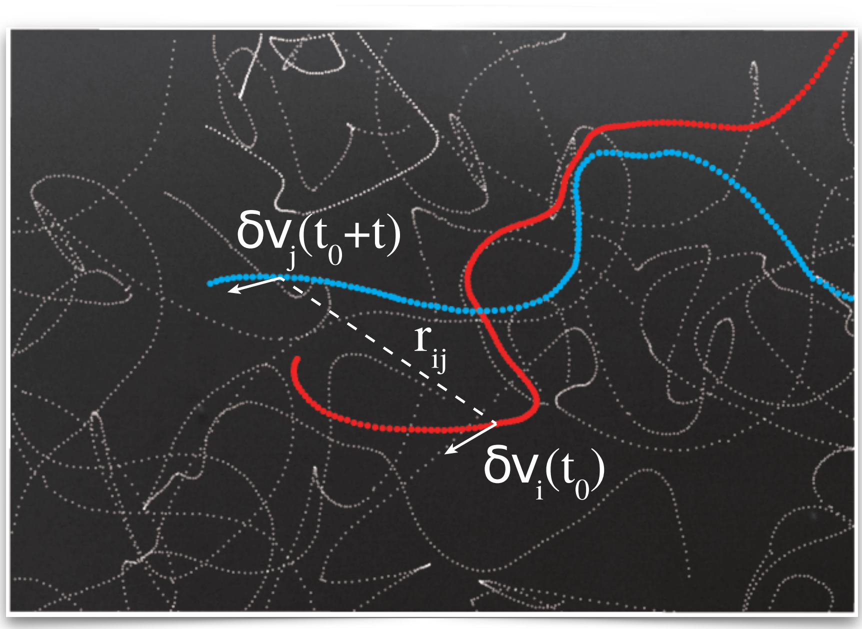


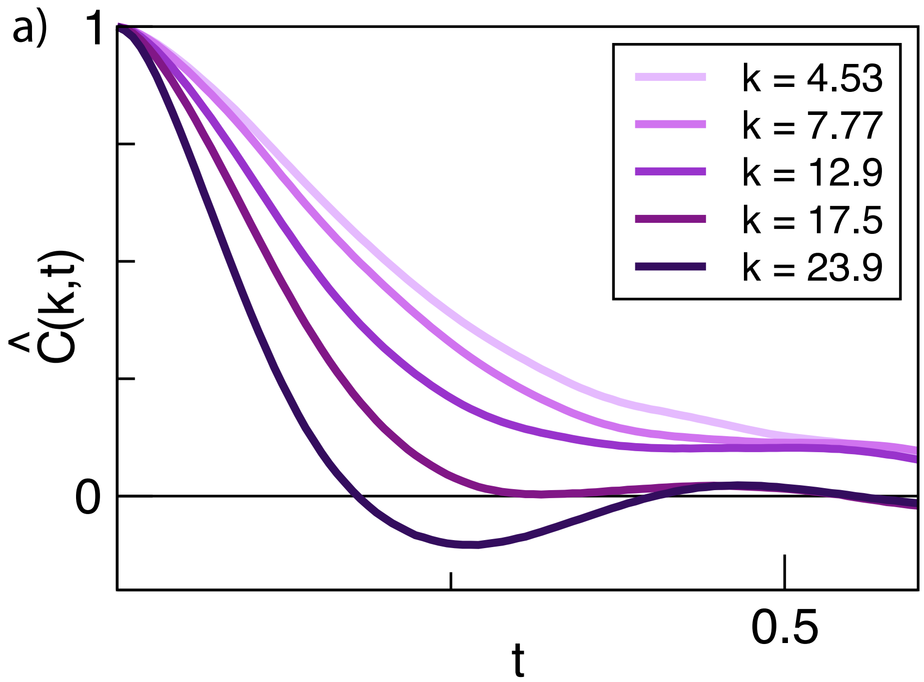
$$\xi \sim L$$

temporal correlations

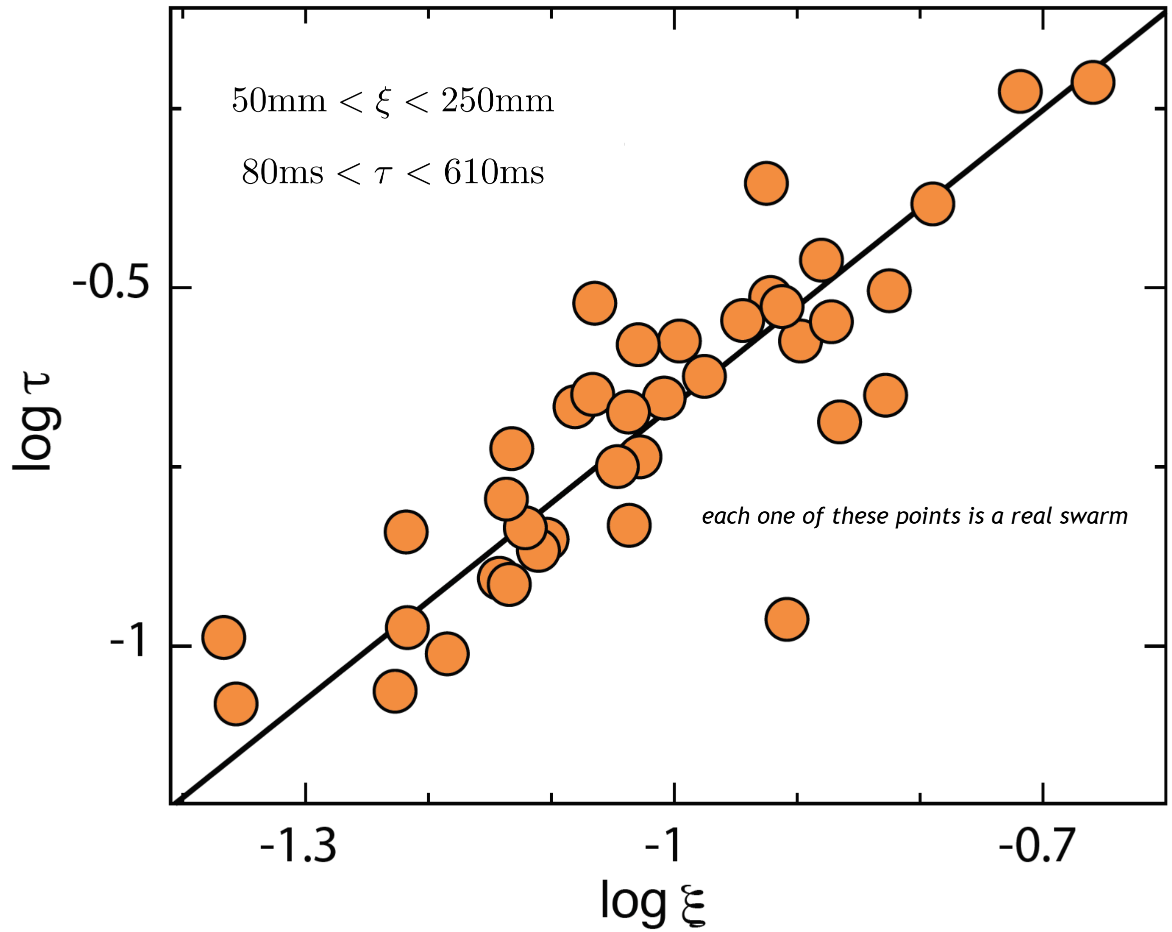
space-time correlation function

$$C(r,t) = \langle \delta\vec{v}(x_0, t_0) \cdot \delta\vec{v}(x_0 + r, t_0 + t) \rangle$$





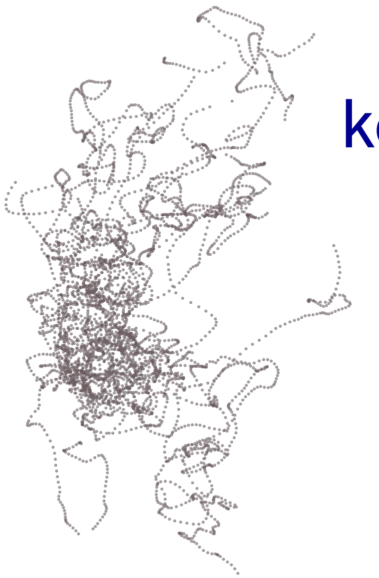
critical slowing down



$$\tau \sim \xi^z$$

dynamical critical exponent z

$$\boxed{z = 1.37 \pm 0.11}$$



key experimental facts about natural swarms:

- scale-free correlations: $\xi \sim L$
- critical slowing down: $\tau \sim \xi^z$
- dynamical critical exponent: $z = 1.37 \pm 0.11$

theory

ingredient #1

imitation

(aka ferromagnetism)

simple ferromagnets

$$\frac{d\boldsymbol{\sigma}_i}{dt} = J \sum_j n_{ij}\, \boldsymbol{\sigma}_j + \boldsymbol{\zeta}_i \quad \xrightarrow{\text{coarse-graining}} \quad \text{Model A} \quad \frac{\partial \boldsymbol{\psi}(x,t)}{\partial t} = -\,\Gamma \frac{\delta \mathcal{H}}{\delta \boldsymbol{\psi}} + \boldsymbol{\theta}(x,t)$$

noise

$$|\boldsymbol{\sigma}_i| = 1$$

Landau-Ginzburg Hamiltonian: $\mathcal{H} = \int d^d x \, \{(\nabla \boldsymbol{\psi})^2 + r\boldsymbol{\psi}^2 + u\boldsymbol{\psi}^4\}$

RG flow (on the critical manifold, because $\xi \sim L$)

$$z \approx 2$$

Wilson, Fisher (1972)

Halperin, Hohenberg, Ma (1972)

RG - ferromagnetism

$$z \approx 2$$

Halperin, Hohenberg, Ma (1972)

something is missing

$$z_{\mathrm{exp}} = 1.37 \pm 0.11$$

experiments

what a low critical exponent is telling us?

$$\tau \sim \xi^z \longleftrightarrow \text{space} \sim \text{time}^{1/z}$$

the smaller is z, the more effective is the transport of fluctuations across the system

$$z = 1.37 \qquad \text{vs} \qquad z \approx 2$$

an exponent $z \ll 2$ implies that fluctuations propagate much more effectively than mere diffusion

ingredient #2

activity

active ferromagnets: the Vicsek model

$$\begin{cases} \dfrac{d\boldsymbol{\sigma}_i}{dt} = J \displaystyle\sum_j n_{ij}(t)\, \boldsymbol{\sigma}_j + \boldsymbol{\zeta}_i \\ \dfrac{d\boldsymbol{r}_i}{dt} = v_0\, \boldsymbol{\sigma}_i \end{cases}$$

$v_0\, \boldsymbol{\sigma}_i = \boldsymbol{v}_i$ is the velocity

Ferromagnets meet Navier-Stokes: the Toner-Tu field theory

Vicsek
$$\begin{cases} \dfrac{d\boldsymbol{\sigma}_i}{dt} = J \displaystyle\sum_j n_{ij}(t)\, \boldsymbol{\sigma}_j + \boldsymbol{\zeta}_i \\ \dfrac{d\boldsymbol{r}_i}{dt} = v_0\, \boldsymbol{\sigma}_i \end{cases}$$

$\xrightarrow{\text{coarse-graining}}$

Toner-Tu
$$\begin{cases} D_t\boldsymbol{v}(x,t) = -\Gamma \dfrac{\delta \mathscr{H}}{\delta \boldsymbol{v}} - \nabla P + \boldsymbol{\theta}(x,t) \\ \dfrac{\partial \rho(x,t)}{\partial t} + \nabla(\rho\, \boldsymbol{v}(x,t)) = 0 \end{cases}$$

$$\mathscr{H} = \int d^d x\ \{(\nabla \boldsymbol{v})^2 + r\boldsymbol{v}^2 + u\boldsymbol{v}^4\}$$

material derivative

$$D_t\, \boldsymbol{v}(x,t) = \partial_t \boldsymbol{v} + \lambda(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}$$

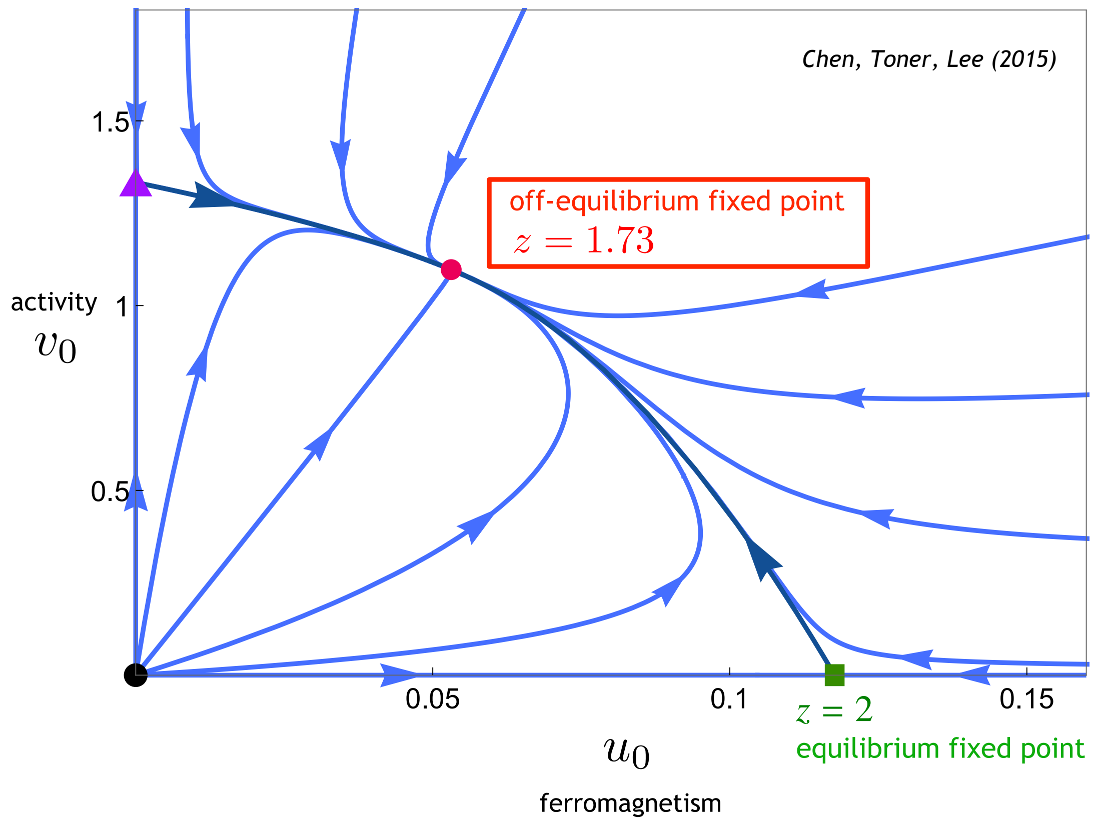
incompressible case: $\rho(x,t) = \rho_0$

$$\begin{cases} D_t\boldsymbol{v}(x,t) = -\Gamma \dfrac{\delta \mathscr{H}}{\delta \boldsymbol{v}} - \nabla P + \boldsymbol{\theta}(x,t) \\ \nabla \cdot \boldsymbol{v} = 0 \end{cases}$$

Chen, Toner, Lee (2015)

see also Forster, Nelson, Stephens (1977)

activity brings the RG flow to a new fixed point



RG - ferromagnetism

$z \approx 2$

Halperin, Hohenberg, Ma (1972)

RG - ferromagnetism and activity

$z = 1.73$

Chen, Toner, Lee (2015)

something is still missing

$z_{\rm exp} = 1.37 \pm 0.11$

experiments

let's go back to the experimental evidence

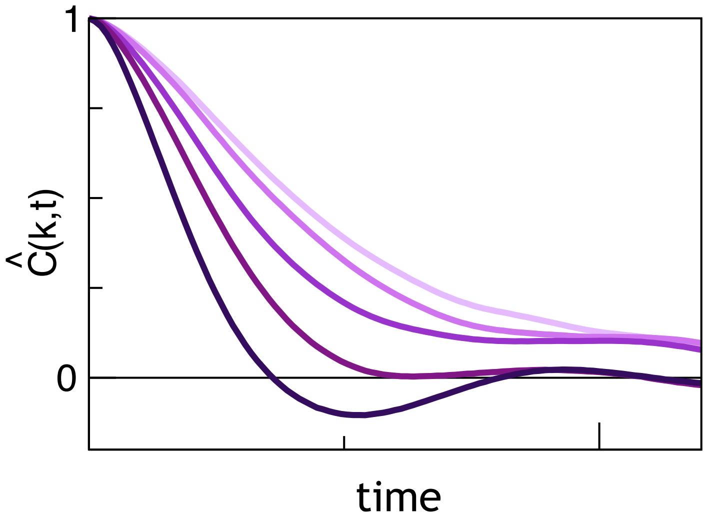
and remember:

a smaller exponent z suggests that

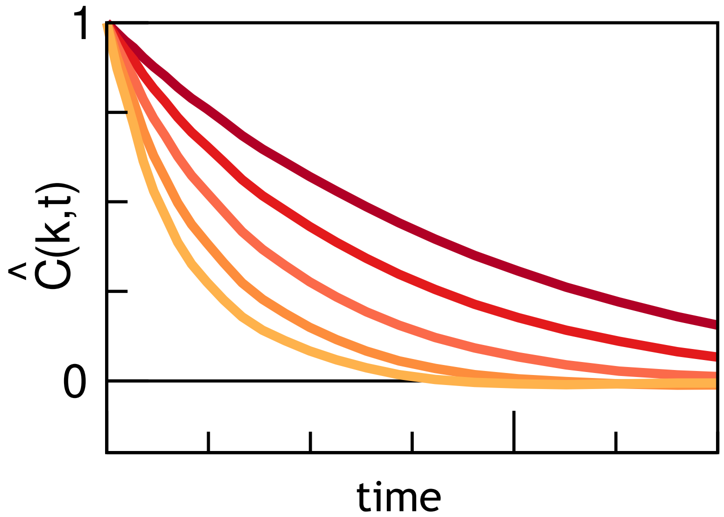
a more efficient propagation mechanism is at play

temporal relaxation in natural swarms looks underdamped

natural swarms - experiments

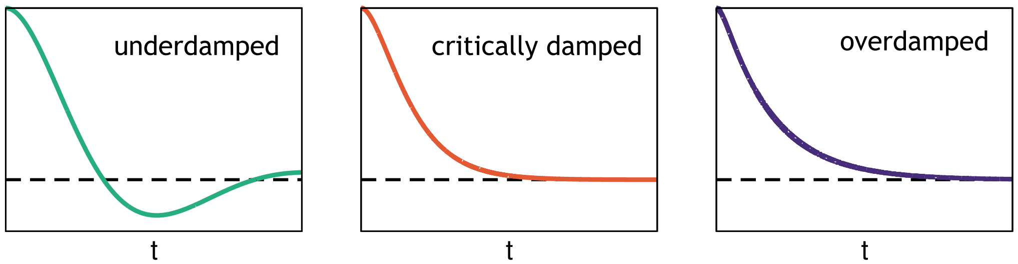


theory up to now - simulations



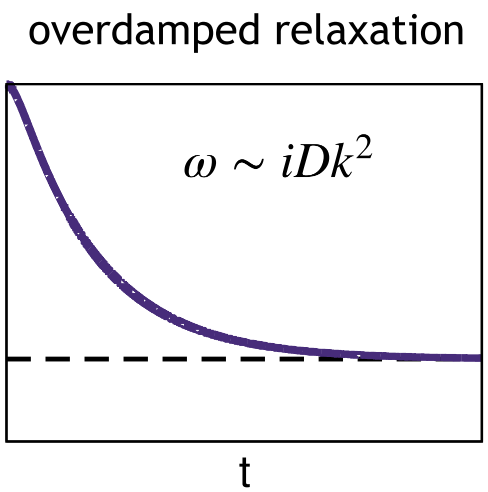
toy example:

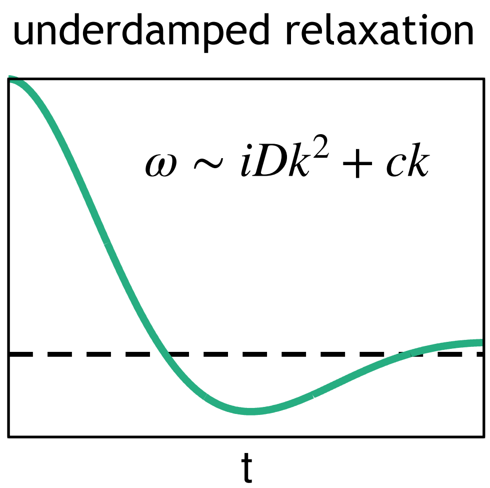
$$m\ddot{q} + \eta\dot{q} + kq = \zeta$$



why this should help?

underdamping requires a real part of the frequency





because alignment requires a Laplacian, a real part of ω indicates there are *two* derivatives in time:

$$\frac{\partial}{\partial t} \sim \nabla^2 \quad \longrightarrow \quad i\omega \sim k^2 \quad \longrightarrow \quad z_{\text{naive}} \sim 2$$

$$\frac{\partial^2}{\partial t^2} \sim \nabla^2 \quad \longrightarrow \quad \omega^2 \sim k^2 \quad \longrightarrow \quad z_{\text{naive}} \sim 1$$

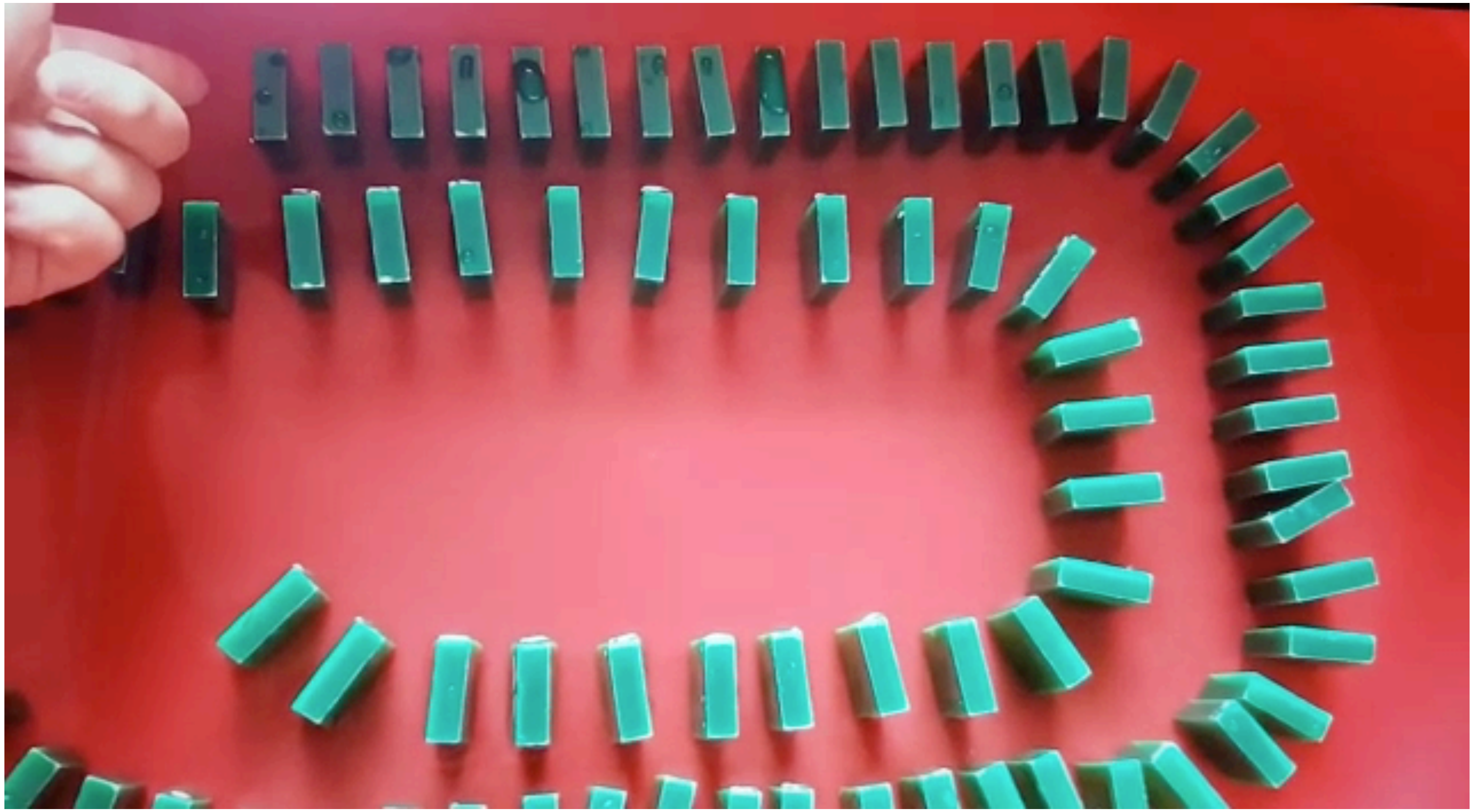
looks promising!

in the competition between friction and inertia

inertia *enhances* propagation

viscosity hinders it

competition between viscosity and inertia



if viscosity dominates over inertia information may not propagate at all

ingredient #3

inertia

conjugate variables and their Poisson relations

$$\{p_\alpha, p_\beta\} = 0$$

$$\{p_\alpha, q_\beta\} = \delta_{\alpha\beta}$$

p is the generator of the translations of q

$$\begin{cases} \dot{q} = \boxed{p} \\ \dot{p} = \boxed{-\dfrac{\partial H}{\partial q}} \boxed{-\eta p + \theta} \end{cases}$$

reversible   irreversible - relax

$$\xrightarrow{\text{overdamped limit}} \quad \eta \dot{q} = \boxed{-\frac{\partial H}{\partial q} + \theta}$$

we need to restore the generator of the *rotations* of the polarization field ψ

this is the *internal* angular momentum, aka **spin** s:

$$\{s_\alpha, s_\beta\} = \epsilon_{\alpha\beta\gamma} s_\gamma$$

$$\{s_\alpha, \psi_\beta\} = \epsilon_{\alpha\beta\gamma} \psi_\gamma$$

$$\begin{cases} \dot{\psi} = \boxed{\psi \times s} \\ \dot{s} = \boxed{-\psi \times \dfrac{\partial H}{\partial \psi}} \boxed{-\eta s + \theta} \end{cases}$$

?

$$\xleftarrow{\text{back to underdamped}} \quad \dot{\psi} = \boxed{-\frac{\partial H}{\partial \psi} + \theta}$$

Theory of dynamic critical phenomena

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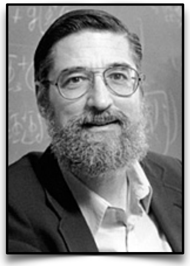
Bell Laboratories, Murray Hill, New Jersey 07974
and Physik Department, Technische Universität München, 8046, Garching, W. Germany

B. I. Halperin*

Department of Physics, Harvard University, Cambridge, Mass. 02138



Reviews of Modern Physics, July 1977

CONTENTS

Halperin-Hohenberg's Model G

Model G

$$\begin{cases} \dfrac{\partial \boldsymbol{\psi}(x,t)}{\partial t} = \boxed{+ g\,\boldsymbol{\psi} \times \dfrac{\delta \mathscr{H}}{\delta \boldsymbol{s}}} \boxed{- \Gamma \dfrac{\delta \mathscr{H}}{\delta \boldsymbol{\psi}} + \boldsymbol{\theta}_{\psi}(x,t)} \\ \dfrac{\partial \boldsymbol{s}(x,t)}{\partial t} = \boxed{- g\,\boldsymbol{\psi} \times \dfrac{\delta \mathscr{H}}{\delta \boldsymbol{\psi}}} \boxed{- \Lambda \dfrac{\delta \mathscr{H}}{\delta \boldsymbol{s}} + \boldsymbol{\theta}_{s}(x,t)} \end{cases}$$

reversible terms relaxational irreversible terms

overdamped limit $\longrightarrow$

Model A

$$\frac{\partial \boldsymbol{\psi}(x,t)}{\partial t} = \boxed{- \Gamma \frac{\delta \mathscr{H}}{\delta \boldsymbol{\psi}} + \boldsymbol{\theta}(x,t)}$$

relaxational irreversible terms

the rotational symmetry of the dynamics implies that the spin $\boldsymbol{s}(x,t)$ *is a conserved quantity*

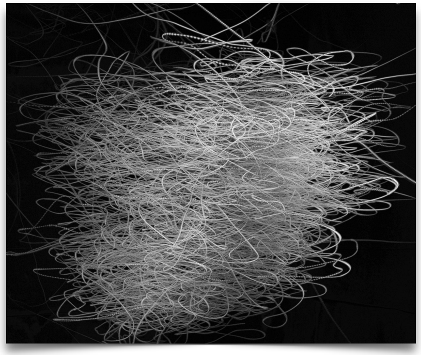
spin conservation law $\Rightarrow$ $\omega = iDk^2 \pm ck$ and $z = 1.5$

our three fundamental ingredients:

imitation

activity

inertia



fundamental ingredients:

city

river

the new theory is: *self-propelled Model G*

Equilibrium Model G:

$$\begin{cases} \dfrac{\partial \boldsymbol{v}(x,t)}{\partial t} = +\, g\, \boldsymbol{v} \times \dfrac{\delta \mathscr{H}}{\delta \boldsymbol{s}} - \Gamma \dfrac{\delta \mathscr{H}}{\delta \boldsymbol{v}} + \boldsymbol{\theta}_v(x,t) \\ \dfrac{\partial \boldsymbol{s}(x,t)}{\partial t} = -\, g\, \boldsymbol{v} \times \dfrac{\delta \mathscr{H}}{\delta \boldsymbol{v}} - \Lambda \dfrac{\delta \mathscr{H}}{\delta \boldsymbol{s}} + \boldsymbol{\theta}_s(x,t) \end{cases}$$

go active: $\begin{cases} \partial_t \boldsymbol{v} \;\rightarrow\; D_t \boldsymbol{v} = \partial_t \boldsymbol{v} + \gamma_{\boldsymbol{v}} (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} \\ \partial_t \boldsymbol{s} \;\rightarrow\; D_t \boldsymbol{s} = \partial_t \boldsymbol{s} + \gamma_{\boldsymbol{s}} (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{s} \end{cases}$

$$\boxed{\begin{cases} D_t \boldsymbol{v}(x,t) = +\, g\, \boldsymbol{v} \times \dfrac{\delta \mathscr{H}}{\delta \boldsymbol{s}} - \Gamma \dfrac{\delta \mathscr{H}}{\delta \boldsymbol{v}} - \nabla P + \boldsymbol{\theta}_v(x,t) \\ D_t \boldsymbol{s}(x,t) = -\, g\, \boldsymbol{v} \times \dfrac{\delta \mathscr{H}}{\delta \boldsymbol{v}} - \Lambda \dfrac{\delta \mathscr{H}}{\delta \boldsymbol{s}} + \boldsymbol{\theta}_s(x,t) \end{cases}}$$

to be studied in the swarm phase

plus incompressibility: $\nabla \cdot \boldsymbol{v} = 0$

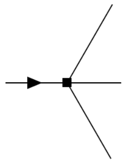
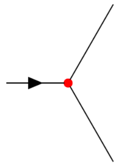
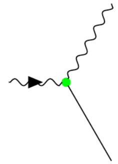
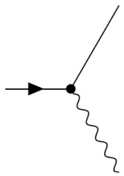
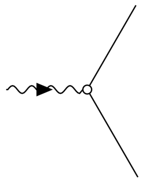
$$\mathscr{H} = \int d^d x\, [(\nabla \boldsymbol{v})^2 + r \boldsymbol{v}^2 + u \boldsymbol{v}^4] + \frac{1}{2} \boldsymbol{s}^2$$

4 dynamical fields and 5 non-linear couplings

$\mathbf{v}(x,t)$ $\mathbf{s}(x,t)$ 

$\hat{\mathbf{v}}(x,t)$  $\hat{\mathbf{s}}(x,t)$ 

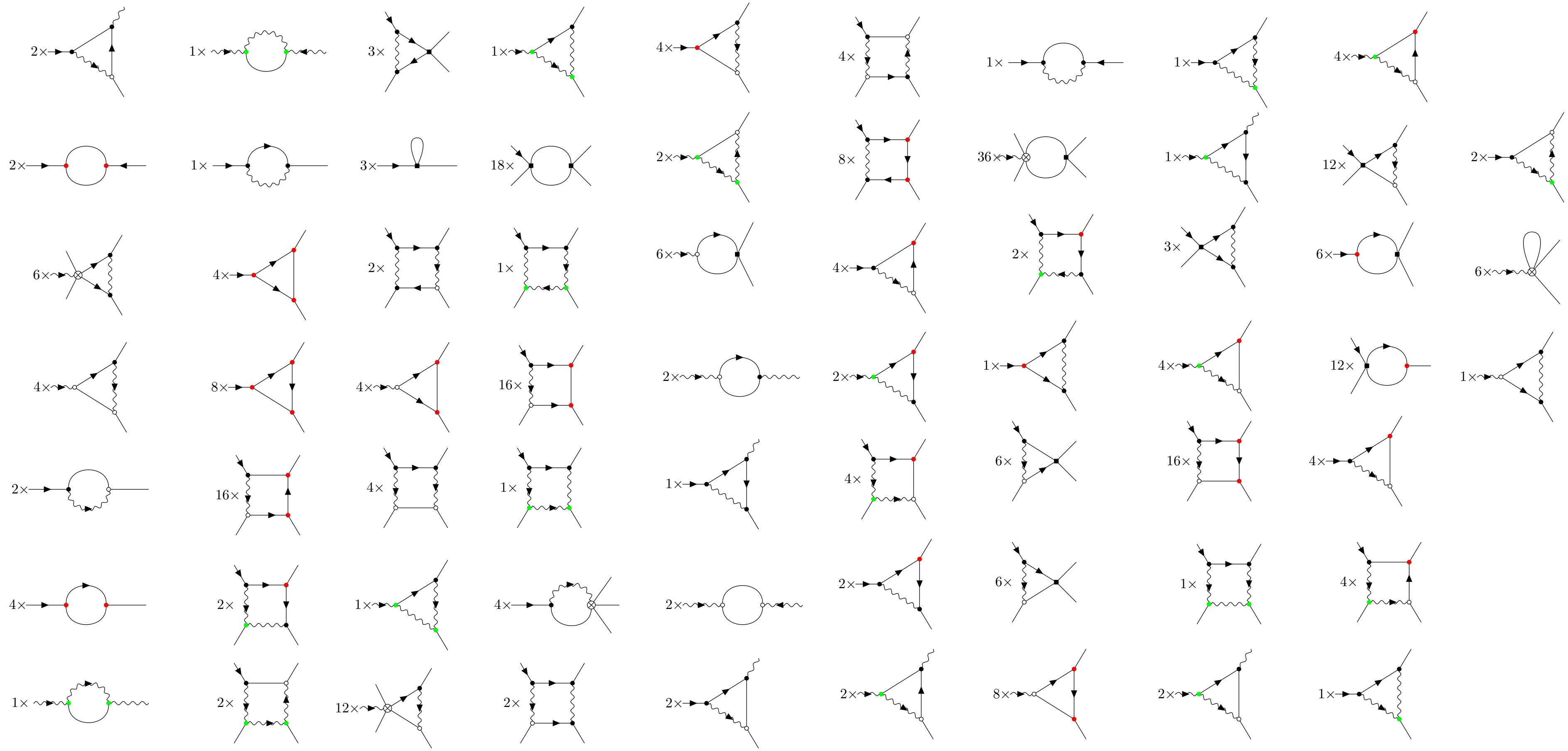
Martin-Siggia-Rose/Janssen-De Dominicis-Peliti

- ferromagnetic interaction: 
- active transport of the velocity and of the spin:  
- inertial spin-velocity couplings:  

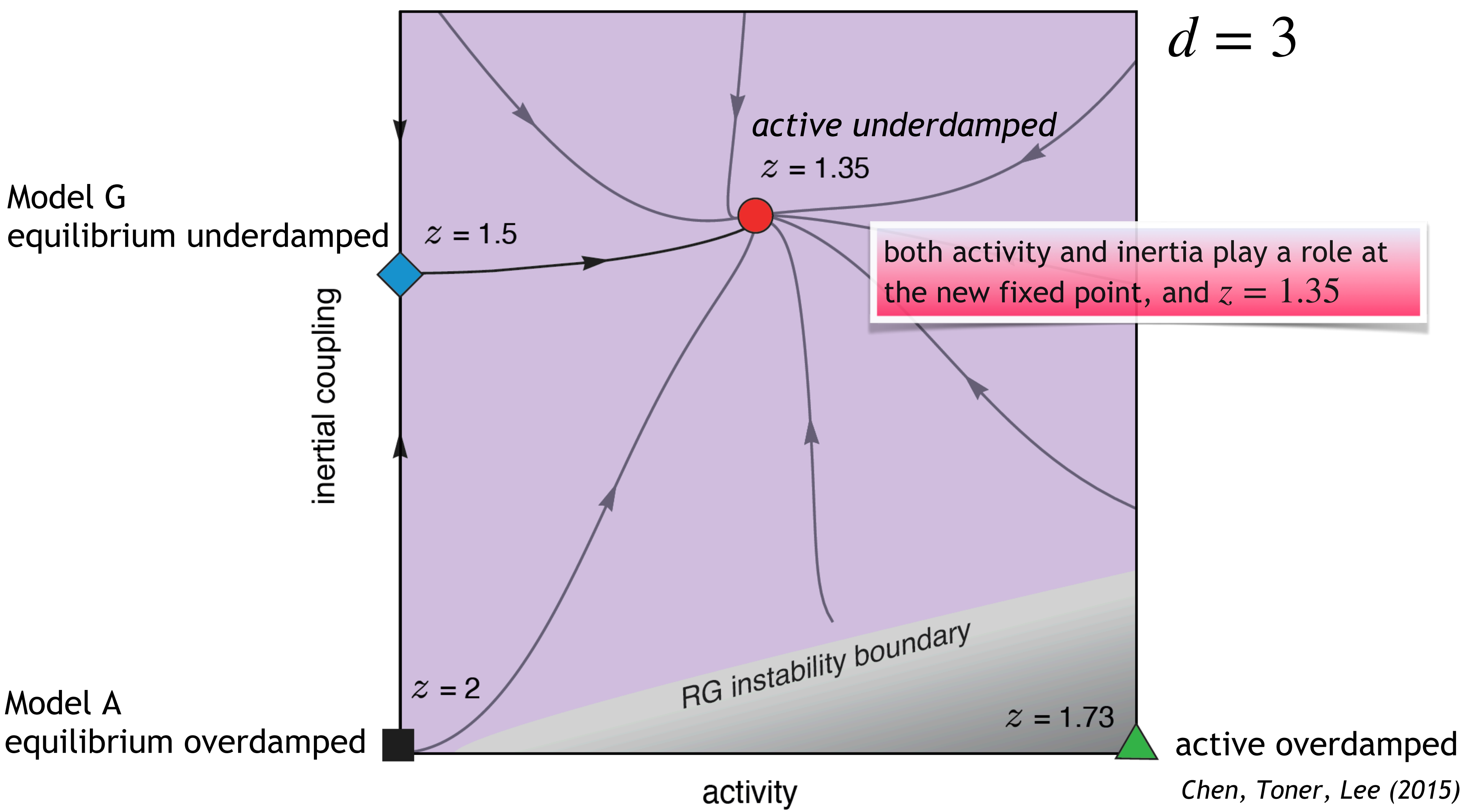
all coupling constants have RG scaling dimension equal to $\varepsilon = 4 - d$, hence:

$$\boxed{\text{expansion for } d = 4 - \varepsilon}$$

a handful of 1-loop diagrams



a novel fixed point emerges



Chen, Toner, Lee (2015)

RG - ferromagnetism

$z \approx 2$

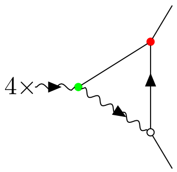
Halperin, Hohenberg, Ma (1972)

RG - ferromagnetism and activity

$z = 1.73$

Chen, Toner, Lee (2015)

a fair agreement



RG - ferromagnetism, activity and inertia

$z = 1.35$

this work

$z_{\text{exp}} = 1.37 \pm 0.11$

experiments

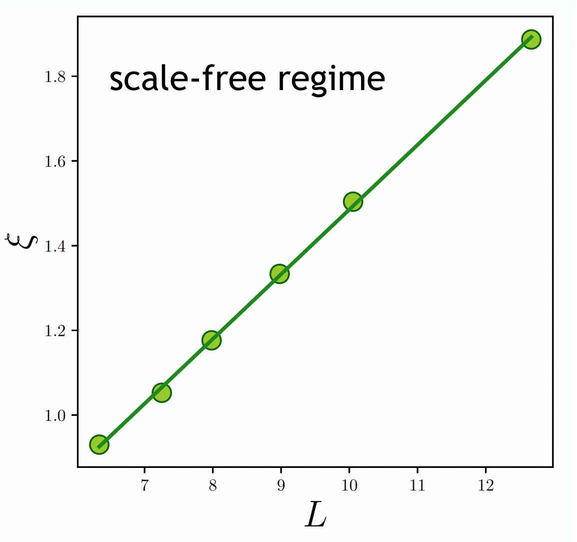
numerical simulations

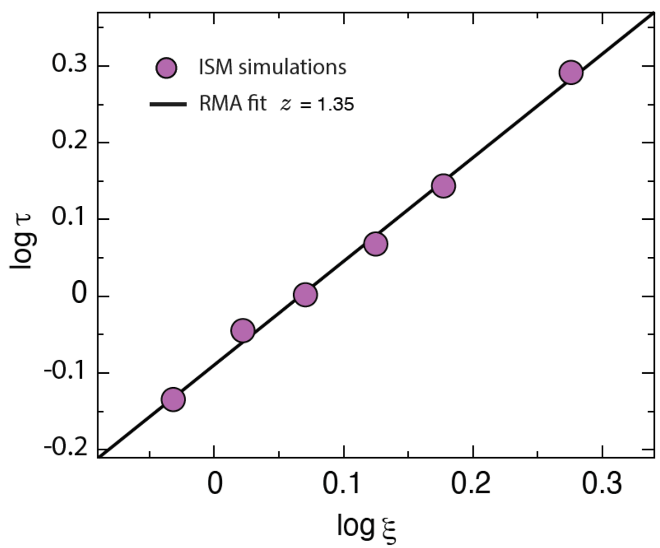
numerical simulations - Inertial Spin Model

$$\begin{cases} \dfrac{d\boldsymbol{v}_i}{dt} = \dfrac{1}{\chi}\boldsymbol{s}_i \times \boldsymbol{v}_i \\ \dfrac{d\boldsymbol{s}_i}{dt} = \boldsymbol{v}_i \times \dfrac{J}{n_i}\sum_j n_{ij}(t)\boldsymbol{v}_j - \dfrac{\eta}{\chi}\boldsymbol{s_i} + \boldsymbol{v}_i \times \boldsymbol{\zeta}_i \\ \dfrac{d\boldsymbol{r}_i}{dt} = \boldsymbol{v}_i \end{cases}$$

$$\langle \boldsymbol{\zeta}_i(t) \cdot \boldsymbol{\zeta}_j(t') \rangle = 2dT\,\eta\,\delta_{ij}\delta(t-t')$$

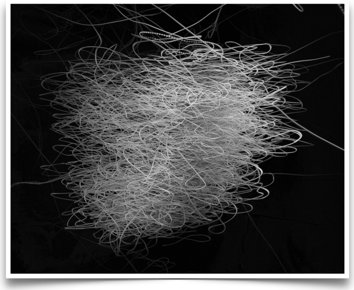
Cavagna et al 2015





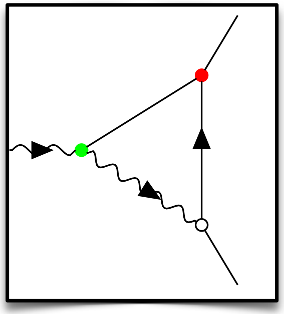
$$z_{\rm sim} = 1.35 \pm 0.04$$

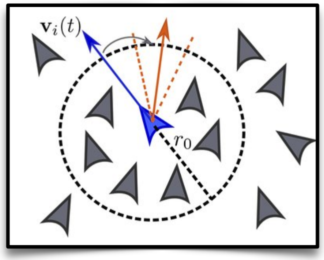
final comparison



$$z_{\rm exp} = 1.37 \pm 0.11$$

$$z_{\rm RG} = 1.35$$





$$z_{\rm sim} = 1.35 \pm 0.04$$

no free parameters

the 3.99 group

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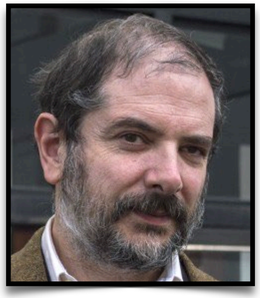

Tomas Grigera
La Plata

“Natural Swarms in 3.99 Dimensions”
Nature Physics, 2023

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