Spindles, Toric Orbifolds and Holography

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based on [2210.16128] and [2402.08724] with D.Martelli, F.Faedo

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Outline

1. Motivation: Evolution of Physics and Black Holes
2. Framework (Supersymmetry, Superstring, Supergravity, AdS/CFT)
3. The Spindle
4. Actual Research (Branes Wrapped on Spindles, Orbifolds and Holography)
5. Recap and Comments

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- Electromagnetism, Weak and Strong forces $\rightarrow$ QFT $\rightarrow$ Standard Model;
  - Many precision tests at LHC $\sim 10^4$ GeV!
- Gravity $\rightarrow$ classical field theory $\rightarrow$ General Relativity;

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  - Effective theory valid until $M_p = \sqrt{\frac{\hbar c}{G_N}} \sim 10^{19}$ GeV.

$$S_{\text{GR}} \simeq M_p^2 \int \mathrm{d}^4 x \sqrt{g} R \to R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 0\,.$$

Black Holes as tests for Quantum Gravity

- What is gravity above M_p ?
- What is the meaning of a quantum-fluctuating geometry at very small scales $l_p = \sqrt{\frac{G_N \hbar}{c^3}} \sim 10^{-35}$m?

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$$T_H = \frac{\hbar}{k_B}\frac{\kappa}{2\pi}\,, \qquad -2\kappa^2 \equiv \nabla^\mu \xi^\nu \nabla_\mu \xi_\nu \big|_{\text{hor}}\,,$$

$$\delta M_{BH} = \frac{\kappa}{8\pi}\delta A_{\text{hor}} + \phi Q + .. \underset{\text{Hawking radiation}}{\rightarrow} S_{BH} = \frac{A_{\text{hor}}}{4G_N}\,.$$

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- Since "$S_{BH} \gg 1 +$ no-hair theorem", **where are all the microstates?**

String theory in a nutshell

- A change of perspective is necessary:
  - $\underbrace{\text{Special Relativity}}_{x^\mu=(\vec{x},t)} + \underbrace{\text{Quantum Mechanics}}_{\hat{x}(t),\hat{p}(t)} = \underbrace{\text{QFT}}_{\hat{\phi}(\vec{x},t)}$;

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- This is not the only way to combine SR and QM!

$$\hat{\phi}(x,t) \to \text{QFT}$$

$$\text{QM: } (\hat{x}(t), t)$$

$$(\hat{x},\hat{t}) = (\hat{x}(\tau,\sigma),\hat{t}(\tau,\sigma)) \to \text{String Theory}$$

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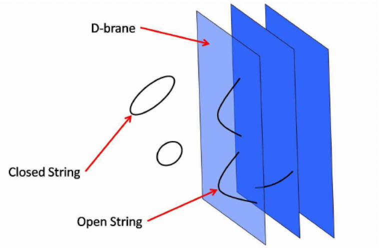
$$\text{QM: } (\hat{x}(t),t) \begin{cases} \hat{\phi}(x,t) \to \text{QFT} \\ (\hat{x},\hat{t}) = (\hat{x}(\tau,\sigma),\hat{t}(\tau,\sigma)) \to \text{String Theory} \end{cases}$$



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$$\underbrace{\Box_{\vec{x},y}\phi(x^\mu, y) = 0}_{\text{scalar in } d+1=d(x^\mu)+1(y)} \quad \underbrace{, \phi(x^\mu, y) = \sum_n \varphi_n(x^\mu) e^{iny/R}}_{y \text{ is compact on } S^1_R} \implies \underbrace{\sum_n \left[\Box_{x^\mu} - \frac{n^2}{R^2}\right]\varphi_n(x^\mu) = 0}_{\infty \text{ massive scalars in } d} .$$

AdS/CFT

- Anti-de-Sitter$_d$:
  - maximally symmetric space time: $R_{\mu\nu\rho\sigma} = L^{-2}(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma})$;
  - (maximally supersymmetric s.t.);
  - Einstein s.t.: $R_{\mu\nu} \propto -L^{-2}(d-1)g_{\mu\nu} \implies R < 0$;
  - from the embedding in $\mathbb{R}^{2,d-1}$: $-x_0^2 + \sum_{i=1}^{d-1} x_i^2 - x_d^2 = -L^2$;
    $(\mathrm{d}s^2_{\mathbb{R}^2} = \mathrm{d}x^2 + \mathrm{d}y^2,\ x^2 + y^2 = L^2 \implies \mathrm{d}s^2_{S^1} = L^2\mathrm{d}\theta^2)$;
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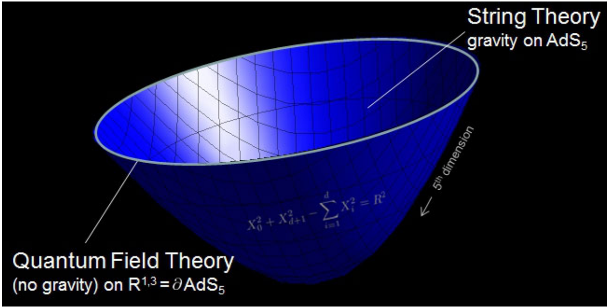
  - invariant under $SO(2, d-1)$;
- Conformal Field Theory$_d$
  - QFT with Poincarrè invariance (Lorentz+translations);
  - CFT=QFT+dilatations+special conformal transformations $\implies SO(2, d)$;

$$(x')^\mu = \Lambda^\mu{}_\nu x^\nu + x^\mu + a^\mu + \lambda x^\mu + \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2},$$

AdS/CFT

AdS/CFT correspondence proposal by J. Maldacena. Holography: "there is a correspondence between gravity in AdS_{d+1} and a CFT_d on the boundary of AdS_{d+1}";



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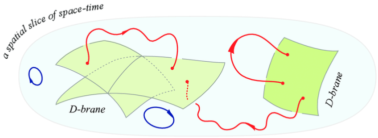


Maldacena motivation:

- The near horizon limit of a stack of **N** D3-branes, which is the maximally supersymmetric background $\mathrm{AdS}_5 \times S^5$;
- $\mathcal{N} = 4, d = 4$ SYM with $G = SU(\mathbf{N})$;

Have the same symmetries!

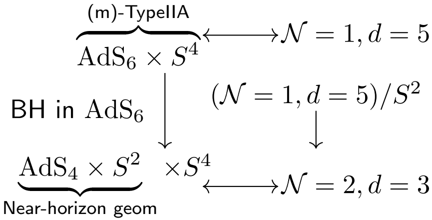
$$\text{gravity: } 32\,\epsilon \times SO(2,4) \times SO(6)\,,$$
$$\text{CFT: } (16+16)\,\text{supercharges} \times \text{conf. symmetry} \times \text{R-sym.}\,.$$

Given a particular gravity solution, how to identify the dual (quantum/conformal) theory?

AdS/CFT

Branes are the key: whenever a solution can be interpreted as the NHL of some configuration of branes, we have (at least..) a hint of the parent theory. How? We start by the well known dualities

M-theory, M2	$\mathrm{AdS}_4 \times S^7$	$\mathcal{N}=6, d=3$ ABJM
Type IIB, D3	$\mathrm{AdS}_5 \times S^5$	$\mathcal{N}=4, d=4$ SYM
Type IIA, D4	$\mathrm{AdS}_6 \times S^4$	$\mathcal{N}=1, d=5$ SCFT
M-theory, M5	$\mathrm{AdS}_7 \times S^4$	$\mathcal{N}=(2,0), d=6$ SCFT



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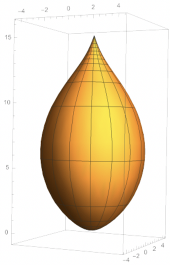
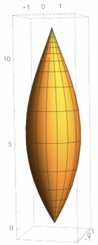
What's next sphere: Spindles

- A spindle Σ is the orbifold $\mathbb{WCP}^1_{[n_-,n_+]} \equiv S^3/U(1)_{n_\pm}$ where $\lambda(z_1, z_2) = (\lambda^{n_-} z_1, \lambda^{n_+} z_2)$, $\lambda \in U(1)$;





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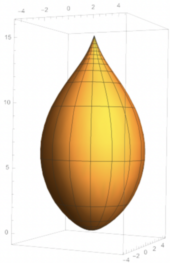
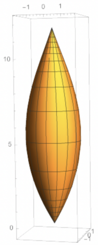
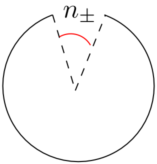
- Locally modelled by $\mathbb{C}/\mathbb{Z}_{n_\pm}$ singularities: $\mathrm{d}S^2 \underset{\theta\sim\theta_\pm}{\simeq} \mathrm{d}\theta_\pm^2 + \frac{\theta_\pm^2}{n_\pm^2}\mathrm{d}\phi^2 \;\; \Delta\phi = 2\pi$;







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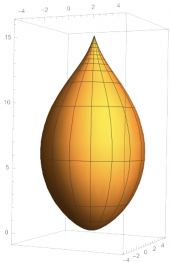
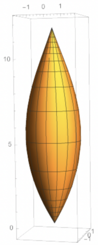
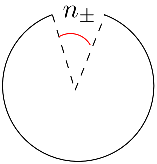
- Physical perspective: BH can accelerate!







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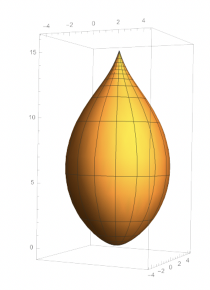
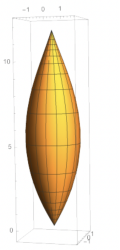
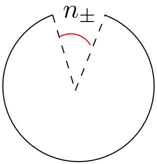
- Physical perspective: BH can accelerate!
- Spindle solutions (meaning $\mathrm{AdS}_D \times \Sigma$ spacetimes) have been built for $D = 2, 3, 4, 5$ In the last few years.







Spindles: arXiv:2011.10579

- Minimal gauged supergravity in $D = 5$ has eom and susy variation

$$R_{\mu\nu} = -4g_{\mu\nu} - \frac{2}{3}F_{\mu\rho}F^{\rho}_{\ \nu} - \frac{1}{9}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\,,$$

$$d \star F = -\frac{2}{3}F \wedge F\,, \quad F_{\mu\nu} = (\mathrm{d}A)_{\mu\nu}\,,$$

$$\left[\nabla_{\mu} - \frac{\mathrm{i}}{12}(\Gamma^{\nu}{}_{\rho\mu} - 4\delta^{\nu}_{\mu}\Gamma^{\rho})F_{\nu\rho} - \frac{1}{2}\Gamma_{\mu} - \mathrm{i}A_{\mu}\right]\epsilon = 0$$

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- Near-horizon-like geometry

$$\mathrm{d}s^2_5 = \frac{4y}{9}\mathrm{d}s^2_{\mathrm{AdS}_3} + \mathrm{d}s^2_{\Sigma} \,, \quad A = \frac{1}{4}\left(1 - \frac{a}{y}\right)\mathrm{d}z,$$
$$ds^2_{\Sigma} = \frac{y}{q(y)}\mathrm{d}y^2 + \frac{q(y)}{36y^2}\mathrm{d}z^2 \implies \Delta z = ?\,, \quad q(y) = 4y^3 - 9y^2 + 6ay - a^2.$$

Spindles: arXiv:2011.10579

if

$$a = \frac{n_-^2 - n_+^2}{n_-^2 + n_+^2}\,, \qquad \Delta z = \frac{\sqrt{2}\sqrt{n_-^2 + n_+^2}}{n_- n_+}\pi\,,$$

then

$$\frac{1}{2\pi}\int_\Sigma F = \frac{n_- - n_+}{2n_- n_+}\,, \qquad \chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma R_\Sigma \, \mathrm{vol}_\Sigma = \frac{n_- + n_+}{n_- n_+}\,.$$

N D3 at the sing. of the CY_3 cone over SE_5: $\mathrm{AdS}_5 \times SE^5 \longleftrightarrow \mathcal{N} = 1, d = 4$

BH in AdS_5 $\downarrow$ $\qquad (\mathcal{N} = 1, d = 4)/\Sigma$ $\downarrow$

$\mathrm{AdS}_3 \times \Sigma \times SE^5 \longleftrightarrow \mathcal{N} = (2,0), d = 2$

$$S_{BH}\Big|_{\text{grav}} = \log(\Omega)\Big|_{\text{CFT}}$$

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- In $D = 6$ gauged Supegravity we found (arXiv: 2210.16128) $\mathrm{AdS}_2 \times \Sigma_{\mathrm{g}} \ltimes \mathbb{\Sigma}_2$ and more surprisingly $\mathrm{AdS}_2 \times \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$;

$$\mathrm{d}s^2 = (y^2 h_1 h_2)^{1/4}\Bigg[\frac{x^2}{4}\,\mathrm{d}s^2_{\mathrm{AdS}_2} + \overbrace{\frac{x^2}{q(x)}\,\mathrm{d}x^2 + \frac{q(x)}{4x^2}\,\mathrm{d}\psi^2}^{\mathbb{\Sigma}_{[m_-,m_+]}} + \underbrace{\frac{y^2}{f(y)}\,\mathrm{d}y^2 + \frac{f(y)}{h_1 h_2}\Big(\mathrm{d}z}_{\mathbb{\Sigma}_{[n_-,n_+]}} - \boxed{\frac{1}{2m}\Big(1-\frac{\mathtt{a}}{x}\Big)\,\mathrm{d}\psi}\Big)^2\Bigg],$$

$$A_i = -\frac{y^3}{h_i}\Big(\mathrm{d}z - \frac{1}{2m}\Big(1-\frac{\mathtt{a}}{x}\Big)\,\mathrm{d}\psi\Big)\,, \qquad X_i = (y^2 h_1 h_2)^{3/8} h_i^{-1}\,,$$

$$B = \frac{\mathtt{a}y}{2m}\,\mathrm{vol}(\mathrm{AdS}_2)\,,$$

$$h_i(y) = y^3 + q_i\,,\;\; f(y) = m^2 h_1 h_2 - y^4\,,\;\; q(x) = x^4 - 4x^2 + 4\mathtt{a}x - \mathtt{a}^2\,.$$

- In $D=6$ gauged Supegravity we found (arXiv: 2210.16128) $\mathrm{AdS}_2 \times \Sigma_{\mathrm{g}} \ltimes \mathbb{\Sigma}_2$ and more surprisingly $\mathrm{AdS}_2 \times \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$;
- These can be uplifted to $D=10$ solutions of massive type IIA sugra: no singularities, thus allowed AdS/CFT computations!

$$\begin{aligned} \mathrm{d}s^2 = & (y^2 h_1 h_2)^{1/4} \Bigg[\frac{x^2}{4} \mathrm{d}s^2_{\mathrm{AdS}_2} + \overbrace{\frac{x^2}{q(x)} \mathrm{d}x^2 + \frac{q(x)}{4x^2} \mathrm{d}\psi^2}^{\mathbb{\Sigma}_{[m_-,m_+]}} \\ & + \underbrace{\frac{y^2}{f(y)} \mathrm{d}y^2 + \frac{f(y)}{h_1 h_2} \Big(\mathrm{d}z}_{\mathbb{\Sigma}_{[n_-,n_+]}} - \boxed{\frac{1}{2m}\Big(1-\frac{\mathtt{a}}{x}\Big)\mathrm{d}\psi} \Big)^2 \Bigg] , \end{aligned}$$

$$A_i = -\frac{y^3}{h_i}\Big(\mathrm{d}z - \frac{1}{2m}\Big(1-\frac{\mathtt{a}}{x}\Big)\mathrm{d}\psi\Big) , \qquad X_i = (y^2 h_1 h_2)^{3/8} h_i^{-1} ,$$

$$B = \frac{\mathtt{a}y}{2m} \mathrm{vol}(\mathrm{AdS}_2) ,$$

$$h_i(y) = y^3 + q_i , \;\; f(y) = m^2 h_1 h_2 - y^4 , \;\; q(x) = x^4 - 4x^2 + 4\mathtt{a}x - \mathtt{a}^2 .$$

- After uplifting to massive (m)type IIA sugra $\rightarrow$ near-horizon limit of a system of N D4-branes and N_f D8-branes wrapped on $\mathbb{M} = \Sigma_1 \ltimes \Sigma_2$;

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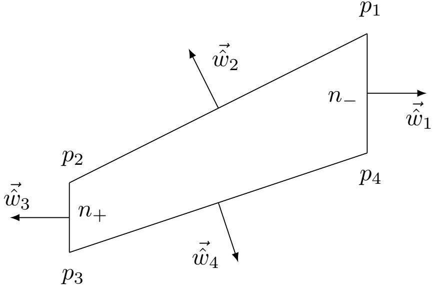
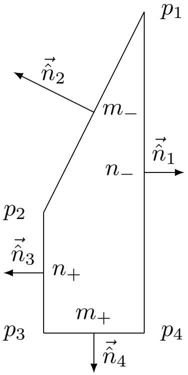
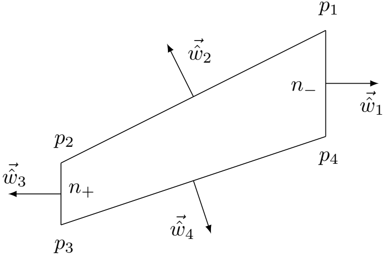
- The dual CFT will (should..) be again obtained as $(\mathcal{N} = 1, d = 5)/\mathbb{M}$;
- **Difficult in practice!** (Sometimes nearly impossible...)
- We take advantage from the toric properties of our solution:



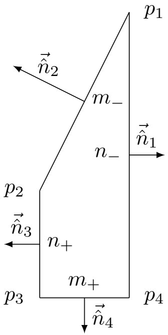
(a) $\gcd(t, m_\pm) = 1$.



(b) $t = m_+ m_- \bar{t}$.



(a) $\gcd(t, m_\pm) = 1$.



(b) $t = m_+ m_- \bar{t}$.

$$S = \frac{\sqrt{2}\sqrt{m_+^2 + m_-^2} - (m_+ + m_-)}{2m_+m_-} \frac{n_+ + n_-}{n_+n_-} \frac{\sqrt{3}\pi N^{5/2}}{\sqrt{8 - N_f}} \frac{[3\mu(x^2+1) - x(x^2+5)]^{3/2}}{x(x^2+3)(\mu - x)^{1/2}} ,$$

$$t_i \equiv \frac{1}{2\pi}\int_{\Sigma_2} \mathrm{d}A_R = \frac{+n_+ + n_-}{n_+n_-} , \quad s_i^\pm \equiv \frac{1}{2\pi}\int_{\Sigma_1^\pm} \mathrm{d}A_R = \frac{+m_+ - m_-}{m_+m_-} \pm \frac{t/n_\pm}{m_+m_-} ,$$

- Idea: Use a "geometrical" approach to find the entropy functions from the toric data! For "'toric-symplectic'" orbifolds we conjectured an off-shell free energy.

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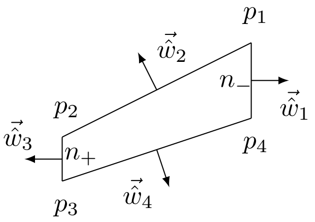

$$\mathfrak{p}_i^1 = \mathfrak{p}_i^3 = t_i\,, \quad \mathfrak{p}_i^2 = \mathfrak{p}_i^4 = \frac{s_i^+ + s_i^-}{2}\,,$$

$$\varphi_1+\varphi_2 - \frac{\mu}{2}\varepsilon_1 + \Big[\frac{\chi_1}{2} - \frac{\tilde{t}}{2m_+m_-}\frac{\mu}{2}\Big]\varepsilon_2 = 2\,,$$

$$\sigma^a = (+,+,+,-)\,, \quad \mu = \frac{n_+ - n_-}{n_+ n_-}\,.$$

$$\overbrace{F(\varphi_i,\varepsilon_I;\mathfrak{p}_i^a) \propto \sum_a \frac{\eta_d^a}{d_{a,a+1}}\frac{\mathcal{F}_d(\Phi_i^a)}{\varepsilon_1^a\varepsilon_2^a}}^{\text{Orbifold grav. blocks}}\,, \quad \mathcal{F}_d(\Phi_i^a) \propto N^{(d-1)/2}(\Phi_1^a\Phi_2^a)^{(d-3)/2}\,,$$

$$\Phi_i^a = \varphi_i - \mathfrak{p}_i^a\varepsilon_1^a - \mathfrak{p}_i^{a+1}\varepsilon_2^a\,, \quad \varepsilon_1^a = -\frac{\det(\vec{\hat{w}}_{a+1},\vec{\varepsilon})}{d_{a,a+1}}\,, \quad \varepsilon_2^a = \frac{\det(\vec{\hat{w}}_a,\vec{\varepsilon})}{d_{a,a+1}}\,,$$

$$\varphi_1 + \varphi_2 - \det(\vec{W},\vec{\varepsilon}) = 2\,, \quad \mathfrak{p}_1^a + \mathfrak{p}_2^a = \sigma^a + \det(\vec{W},\vec{\hat{w}}_a)\,,$$

$$\mathfrak{q}_i^a = \sum_a D_{ab}\mathfrak{p}_i^b\,, \quad \sum_a \mathfrak{q}_i^a\vec{\hat{w}}_a = 0 \implies \mathfrak{q}_R^a = \sum_b D_{ab}\sigma^b\,,$$

From 2402.08724:

$$\mathrm{d}s^2 = (H_1 H_2)^{1/(D-2)} \left(\frac{x^2 y^2}{a^2} \, \mathrm{d}s^2_{\mathrm{AdS}_{D-4}} + \mathrm{d}s^2_{\mathbb{M}_4} \right),$$

where

$$\begin{aligned}\mathrm{d}s^2_{\mathbb{M}_4} = \frac{(H_1 H_2)^{-1}}{\Xi^2 (x^2 - y^2)} \Big[& (V_y^2 \Delta_x - V_x^2 \Delta_y) \, \mathrm{d}\psi^2 + (\widetilde{V}_y^2 \Delta_x - \widetilde{V}_x^2 \Delta_y) \, \mathrm{d}\phi^2 \\ & - \frac{4 c_1 c_2 \widetilde{c}_1 \widetilde{c}_2}{a^2} \left(\frac{N_y \Delta_x}{y^{D-5}} - \frac{N_x \Delta_y}{x^{D-5}} \right) \mathrm{d}\psi \, \mathrm{d}\phi \Big] + \frac{y^2 - x^2}{\Delta_y} \, \mathrm{d}y^2 + \frac{x^2 - y^2}{\Delta_x} \, \mathrm{d}x^2 \, .\end{aligned}$$

The scalar fields are

$$X_i = (H_1 H_2)^{(D-3)/2(D-2)} H_i^{-1} \, ,$$

the gauge potentials are given by

$$\begin{aligned} A_1 &= \frac{2N_y s_1}{\Xi(x^2-y^2)H_1 y^{D-5}}\left(c_1\widetilde{c}_2 V_x^{(1)}\,\mathrm{d}\psi - c_2\widetilde{c}_1 \widetilde{V}_x^{(1)}\,\mathrm{d}\phi\right) \\ &- \frac{2N_x s_1}{\Xi(x^2-y^2)H_1 x^{D-5}}\left(c_1\widetilde{c}_2 V_y^{(1)}\,\mathrm{d}\psi - c_2\widetilde{c}_1 \widetilde{V}_y^{(1)}\,\mathrm{d}\phi\right), \\ A_2 &= \frac{2N_y s_2}{\Xi(x^2-y^2)H_2 y^{D-5}}\left(c_2\widetilde{c}_1 V_x^{(2)}\,\mathrm{d}\psi - c_1\widetilde{c}_2 \widetilde{V}_x^{(2)}\,\mathrm{d}\phi\right) \\ &- \frac{2N_x s_2}{\Xi(x^2-y^2)H_2 x^{D-5}}\left(c_2\widetilde{c}_1 V_y^{(2)}\,\mathrm{d}\psi - c_1\widetilde{c}_2 \widetilde{V}_y^{(2)}\,\mathrm{d}\phi\right), \end{aligned}$$

and the $(D-4)$-form is

$$B = -\frac{4s_1 s_2 (N_y x^3 - N_x y^3)}{a^2(x^2-y^2)} \, \mathrm{vol}(\mathrm{AdS}_2) \qquad (D=6)\,,$$

$$B = -\frac{2s_1 s_2 (N_y x^4 - N_x y^4)}{a^3(x^2-y^2)} \, \mathrm{vol}(\mathrm{AdS}_3) - \frac{2N_y g s_1 s_2 x}{\Xi a^2 y^2} \, \mathrm{d}\psi \wedge \mathrm{d}\phi \wedge \mathrm{d}x - \frac{2N_x g s_1 s_2 y}{\Xi a^2 x^2} \, \mathrm{d}\psi \wedge \mathrm{d}\phi \wedge \mathrm{d}y \qquad (D=7)\,.$$

The functions are

$$\Delta_y = -y^2 + a^2 - \frac{2N_y}{y^{D-5}} + g^2\left(y^2 - \frac{2N_y s_1^2}{y^{D-5}}\right)\left(y^2 - \frac{2N_y s_2^2}{y^{D-5}}\right) - a^2 g^2 y^2 - \frac{2N_y a^2 g^2 s_1^2 s_2^2}{y^{D-5}}\,,$$

$$\Delta_x = -x^2 + a^2 - \frac{2N_x}{x^{D-5}} + g^2\left(x^2 - \frac{2N_x s_1^2}{x^{D-5}}\right)\left(x^2 - \frac{2N_x s_2^2}{x^{D-5}}\right) - a^2 g^2 x^2 - \frac{2N_x a^2 g^2 s_1^2 s_2^2}{x^{D-5}}\,,$$

$$V_y^2 = V_y^{(1)} V_y^{(2)}\,, \qquad \widetilde{V}_y^2 = \widetilde{V}_y^{(1)} \widetilde{V}_y^{(2)}\,, \qquad V_R^2 = V_R^{(1)} V_R^{(2)}\,, \qquad \widetilde{V}_R^2 = \widetilde{V}_R^{(1)} \widetilde{V}_R^{(2)}\,,$$

$$V_y^{(i)} = 1 - g^2\left(y^2 - \frac{2N_y s_i^2}{y^{D-5}}\right), \qquad \widetilde{V}_y^{(i)} = 1 - \frac{1}{a^2}\left(y^2 - \frac{2N_y s_i^2}{y^{D-5}}\right),$$

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$$H_i = 1 + \frac{2 s_i^2}{x^2 - y^2}\left(\frac{N_y}{y^{D-5}} - \frac{N_x}{x^{D-5}}\right),$$

$$\Xi = 1 - a^2 g^2\,, \qquad s_i = \sinh \delta_i\,, \quad c_i = \cosh \delta_i\,, \quad \widetilde{c}_i = \sqrt{1 + a^2 g^2 s_i^2}\,.$$

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5 Recap and Comments

Comments:

- Our conjecture is valid also for M5-branes wrapped over $\mathbb{M}_4$ with minor tweaks and for an arbitrary number of fixed points;
- It works also for the $\mathrm{AdS}_{2,3} \times \mathbb{M}_4$ solutions of arXiv:2402.08724;
- sum over fixed points $\implies$ equivariant integration at work $\implies$ equivariant volume ([Colombo-Faedo-Martelli-Zaffaroni])/ localization of the action (Genolini-Gauntlett-Sparks);
- Proved: M2, D3 and M5 on Σ, M5 on $\mathbb{M}_4$;

Some open problems:

- D4 branes?
- Are there other interesting equivariant quantities?
- Black objects with $\mathbb{M}_4$ horizon in $D = 6, 7$ gauged supergravity?
- Existence of solutions?

$$\mathcal{S}_{\pm}(\varphi_i, \varepsilon; p_i, n_{\pm}) \equiv \frac{1}{\varepsilon}\left[F_d(\varphi_i + p_i\varepsilon) \pm F_d(\varphi_i - p_i\varepsilon)\right] ,$$

$$\sum_i \varphi_i - \frac{n_- - \sigma n_+}{n_- n_+}\varepsilon = 2 , \quad \sum_i p_i = \frac{n_- + \sigma n_+}{n_- n_+} ,$$

Thanks for your attention!

Back-up slides

$$\mathbb{S}^3(a,b;m_\pm;t) \equiv \mathbb{S}^3(a,b;m_\pm)/\mathbb{Z}_t = (\mathbb{C}_1/\mathbb{Z}_{m_-} \times \mathbb{C}_2/\mathbb{Z}_{m_+})/\mathbb{Z}_t\,, \quad \frac{|z_1|^2}{a^2} + \frac{|z_2|^2}{b^2} = 1$$

with the identification

$$(z_1, z_2) \sim e^{\frac{2\pi \mathrm{i}}{t}} \circ (e^{\frac{2\pi \mathrm{i}}{m_-}} z_1, e^{\frac{2\pi \mathrm{i}}{m_+}} z_2) \sim (e^{\frac{2\pi \mathrm{i}}{t\, m_-}} z_1, e^{\frac{2\pi \mathrm{i}}{t\, m_+}} z_2) \sim (e^{\frac{2\pi \mathrm{i}\, m_+}{t}} z_1, e^{\frac{2\pi \mathrm{i}\, m_-}{t}} z_2)\,.$$

Consider $t = k_+ k_- \bar{t}$ and $m_\pm = k_\pm \overline{m}_\pm$:

$$(e^{\frac{2\pi \mathrm{i}\, m_+}{t}} z_1, e^{\frac{2\pi \mathrm{i}\, m_-}{t}} z_2) = (e^{\frac{2\pi \mathrm{i}\, \overline{m}_+}{\bar{t} k_-}} z_1, e^{\frac{2\pi \mathrm{i}\, \overline{m}_-}{\bar{t} k_+}} z_2) \implies (e^{\frac{2\pi \mathrm{i}\, \overline{m}_+ k_+}{k_-}} z_1, e^{2\pi \mathrm{i}\, \overline{m}_-} z_2)\,.$$

Alternatively

$$\vec{\hat{w}}_1 = (n_-, 0)\,, \quad \vec{\hat{w}}_2 = (t a_+, m_-)\,, \quad \vec{\hat{w}}_3 = (-n_+, 0)\,, \quad \vec{\hat{w}}_4 = (t a_-, -m_+)\,,$$

with $a_+ m_+ + a_- m_- = 1$ (or $r_+ m_+ + r_- m_- = t$ with $r_\pm = t a_\pm$).

Back-up slides

$$Z_{\mathsf{GC}}^{\mathsf{CFT}}(\varphi_a, \varepsilon_i; p_a) = \sum_{Q_a, J_i} c(Q_a, J_i) \mathrm{e}^{\mathrm{i}(\varphi_a Q_a + \varepsilon_i J_i)} ,$$

$$\mathrm{e}^{S_{\mathsf{BH}}(Q_a, J_i)} = c(Q_a, J_i) = \int \frac{\mathrm{d}\varphi_a}{2\pi} \frac{\mathrm{d}\varepsilon_i}{2\pi} Z_{\mathsf{GC}}^{\mathsf{CFT}}(\varphi_a, \varepsilon_i) \mathrm{e}^{-\mathrm{i}(\varphi_a Q_a + \varepsilon_i J_i)} ,$$

$$S_{\mathsf{BH}}(Q_a, J_i) = \left[\log Z_{\mathsf{GC}}^{\mathsf{CFT}}(\varphi_a, \varepsilon_i) - \mathrm{i}(\varphi_a Q_a + \varepsilon_i J_i) \right] \Big|_{(\varphi_a^*, \varepsilon_i^*)} ,$$

$$\partial_{(\varphi, \varepsilon)} \left[\log Z_{\mathsf{GC}}^{\mathsf{CFT}}(\varphi_a, \varepsilon_i) - \mathrm{i}(\varphi_a Q_a + \varepsilon_i J_i) \right] \Big|_{(\varphi_a^*, \varepsilon_i^*)} \equiv 0 .$$

Moreover

$$Z_{\mathsf{CFT}} = Z_{\mathsf{grav}} = \int \mathcal{D}[\Phi] \mathrm{e}^{-S[\Phi]} \longrightarrow \mathrm{e}^{-S^{\mathsf{OS}}[\Phi]} .$$

Back-up slides

Squashed-branched lens space:

$$\mathbb{S}^3(a,b;m_\pm;t) \equiv \mathbb{S}^3(a,b;m_\pm)/\mathbb{Z}_t = (\mathbb{C}_1/\mathbb{Z}_{m_-} \times \mathbb{C}_2/\mathbb{Z}_{m_+})/\mathbb{Z}_t\,,$$

$$\frac{|z_1|^2}{a^2} + \frac{|z_2|^2}{b^2} = 1\,,$$

with the identification

$$(z_1, z_2) \sim e^{\frac{2\pi \mathrm{i}}{t}} \circ (e^{\frac{2\pi \mathrm{i}}{m_-}} z_1, e^{\frac{2\pi \mathrm{i}}{m_+}} z_2) \sim (e^{\frac{2\pi \mathrm{i}}{t\, m_-}} z_1, e^{\frac{2\pi \mathrm{i}}{t\, m_+}} z_2) \sim (e^{\frac{2\pi \mathrm{i}\, m_+}{t}} z_1, e^{\frac{2\pi \mathrm{i}\, m_-}{t}} z_2)\,.$$

Consider $t = k_+ k_- \bar{t}$ and $m_\pm = k_\pm \overline{m}_\pm$:

$$(e^{\frac{2\pi \mathrm{i}\, m_+}{t}} z_1, e^{\frac{2\pi \mathrm{i}\, m_-}{t}} z_2) = (e^{\frac{2\pi \mathrm{i}\, \overline{m}_+}{\bar{t} k_-}} z_1, e^{\frac{2\pi \mathrm{i}\, \overline{m}_-}{\bar{t} k_+}} z_2) \implies (e^{\frac{2\pi \mathrm{i}\, \overline{m}_+ k_+}{k_-}} z_1, e^{2\pi \mathrm{i}\, \overline{m}_-} z_2)\,.$$

S^3

$$x^2 + y^2 + z^2 + t^2 = 1\,, \quad z_1 = x + \mathrm{i}y\,, \quad z_2 = z + \mathrm{i}t \quad |z_1|^2 + |z_2|^2 = 1\,,$$

Choosing $(z_1, z_2) = (\cos\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}(-\phi/2+\psi)}, \sin\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}(\phi/2+\psi)})$ the metric appears as the Hopf fibration

$$\mathrm{d}S^2 = \frac{1}{4}\mathrm{d}S^2_{S^2}(\theta,\phi) + (\mathrm{d}\psi - \frac{1}{2}\cos\theta\,\mathrm{d}\phi)^2\,, \qquad \theta \in [0,\pi]\,,\ \phi \in [0, 2\pi)\,,\ \psi \in [0, 2\pi)\,.$$

$$\frac{1}{2\pi}\int \frac{2\pi}{\Delta\psi}\mathrm{d}\left[\mathrm{d}\psi - \frac{1}{2}\cos\theta\,\mathrm{d}\phi\right] = 1\,,$$