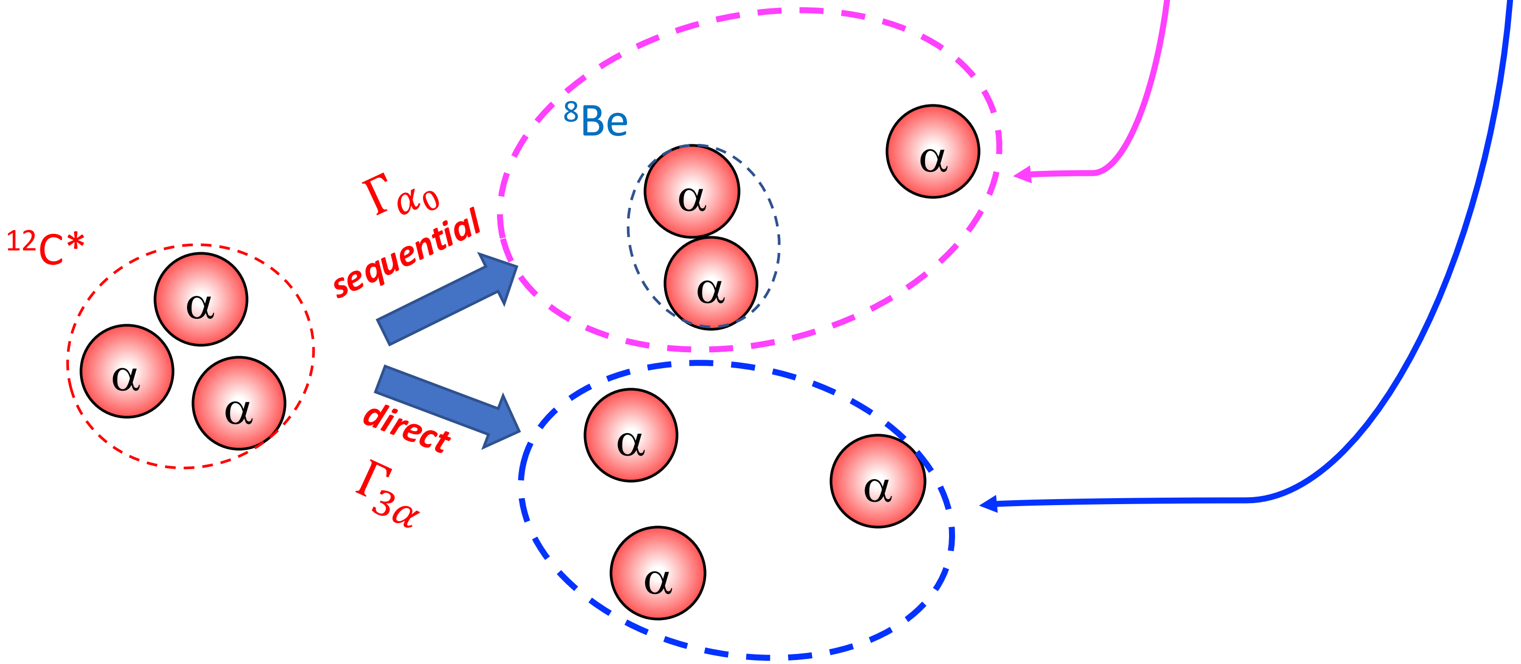
Using Dalitz Plot in α-clustering analysis. An MC study

From the talk of Dell'Aquila at the FOOT meeting in Naples

In the last part of his talk, D. Dell'Aquila underlined one of the questions that we often mentioned in our presentations at the FOOT meetings: which is the fraction of sequential vs direct decay in the full α-fragmentation of ^{12}C?



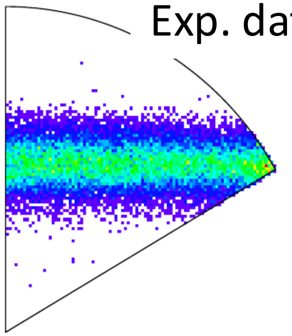
Results from Dell’Aquila et al.

D. Dell’Aquila et al., Phys. Rev. Lett. 119 (2017) 132501

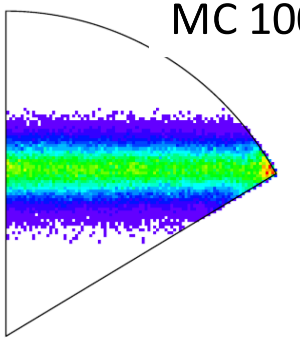
- In this paper, the authors analyzed the “Hoyle state” (7.65 MeV excitation energy) for $^{12}C \rightarrow 3\alpha$ produced in the reaction $^{14}N(d,\alpha)^{12}C$, using 10.5 MeV deuterons

(MeV, not MeV/u)

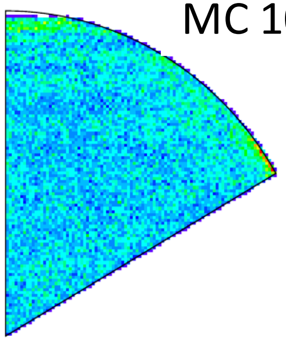
- They used a (symmetryzed) Dalitz plot analysis to show that <u>their experimental data actually excluded the direct decay channel</u>



Exp. data



MC 100% Sequential decay



MC 100% direct decay

How do we read these plots?
Which are the variables?
How do we interpret them?

Today:

- Recap of Dalitz plot theory
- Application of Dalitz plot to the study of clustering (so to understand the figure by Dell'Aquila et al.)
- Use of Dalitz Plot to analyze the predictions obtained by the pure FLUKA MC generator at FOOT energies

The Dalitz Plot

- The Dalitz Plot is an analysis tool invented to investigate a 3-body decay process: $M \rightarrow m_1+m_2+m_3$
- It is an attempt to represent the essential information on the kinematics and dynamics of the decay process in a 2-dimensional plot: is it possible to reduce the intrinsic large number of variables?

3 particles = three 4-vectors = 12 variables.

- You can reduce this number by considering 4-momentum conservation, knowledge of masses, etc.

3-body phase space

In textbooks you can find that:

$$d\phi_3 = \prod d^4 p_i\, \delta(p_i^2 - m^2)\, \delta^4\left(\sum p_i - P\right) =$$

three 3-vectors:
9 variables

$$= \delta(M - \sum E_i)\delta^3(\vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2 + \vec{\boldsymbol{p}}_3)\frac{d^3\boldsymbol{p}_1}{2E_1}\frac{d^3\boldsymbol{p}_2}{2E_2}\frac{d^3\boldsymbol{p}_3}{2E_3}$$

1 constraint

3 constraints

9 variables
- 4 constraints
➔ 5 independent variables

However, it is possible to integrate over some less interesting degrees of freedom.
For example, integrating over directions so to remain with the 3 energies.

Since $M=E_1+E_2+E_3$ ➔ only 2 independent energies remain

This is the idea at the basis of the Dalitz Plot:

combine a 3 energy measurements in a 2-dim plot

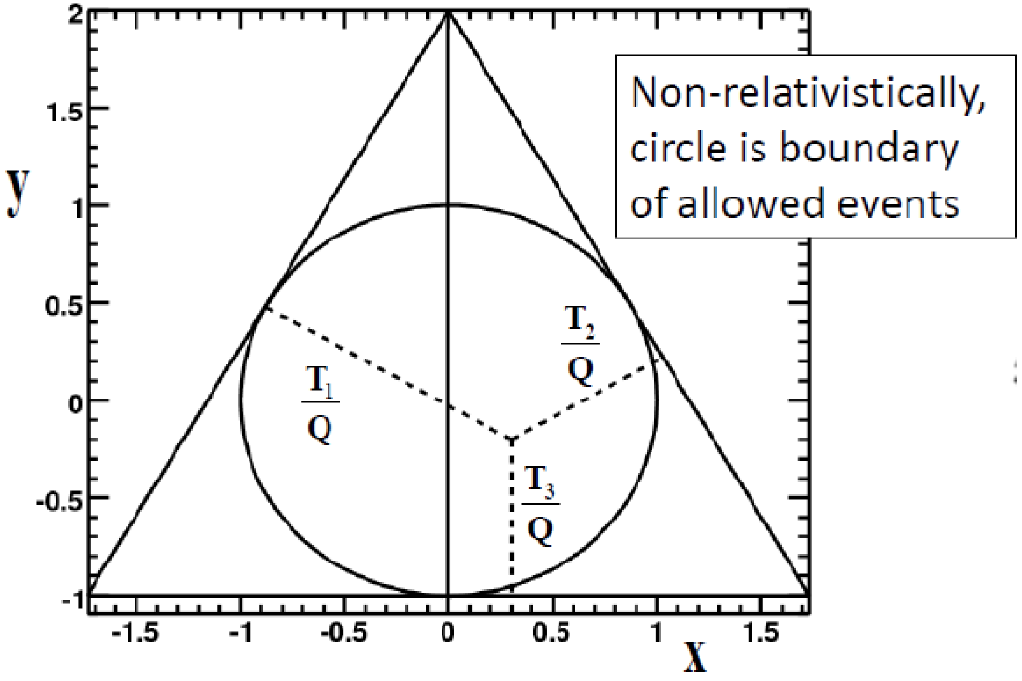
The Dalitz Plot

Dalitz, R. H. (1953). CXII. On the analysis of τ-meson data and the nature of the τ-meson. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, *44*(357), 1068–1080

In the CM frame of the decaying particle!!!

- *Given the 3 kinetic energies T_1, T_2 and T_3, then $Q = T_1 + T_2 + T_3 = M - m_1 - m_2 - m_3$ is the energy release in the decay*
- *Define the x, y variables:*

$$x = \frac{\sqrt{3}(T_1 - T_2)}{Q}; \qquad y = \frac{2T_3 - T_1 - T_2}{Q}$$



- *For each point, the distances from the triangle sides are T_1/Q, T_2/Q, T_3/Q*
- *In the non-relativistic limits, all points lie within the boundary of a circle of unit radius (see later)*

3-Body decay and Dalitz Plot

Let's consider a 3-body decay process where a particle (resonance) of mass M decays in 3 particles:
$M \rightarrow m_1+m_2+m_3$ with $m_1=m_2=m_3=m$

Matrix element

3-body differential phase space

Decay rate: $$d\Gamma = \frac{(2\pi)^4}{2M} |\mathrm{M}|^2\, d\phi_3(P; p_1, p_2, p_3)$$

Since: $$d\phi_3 = constant\; dE_1 dE_2$$

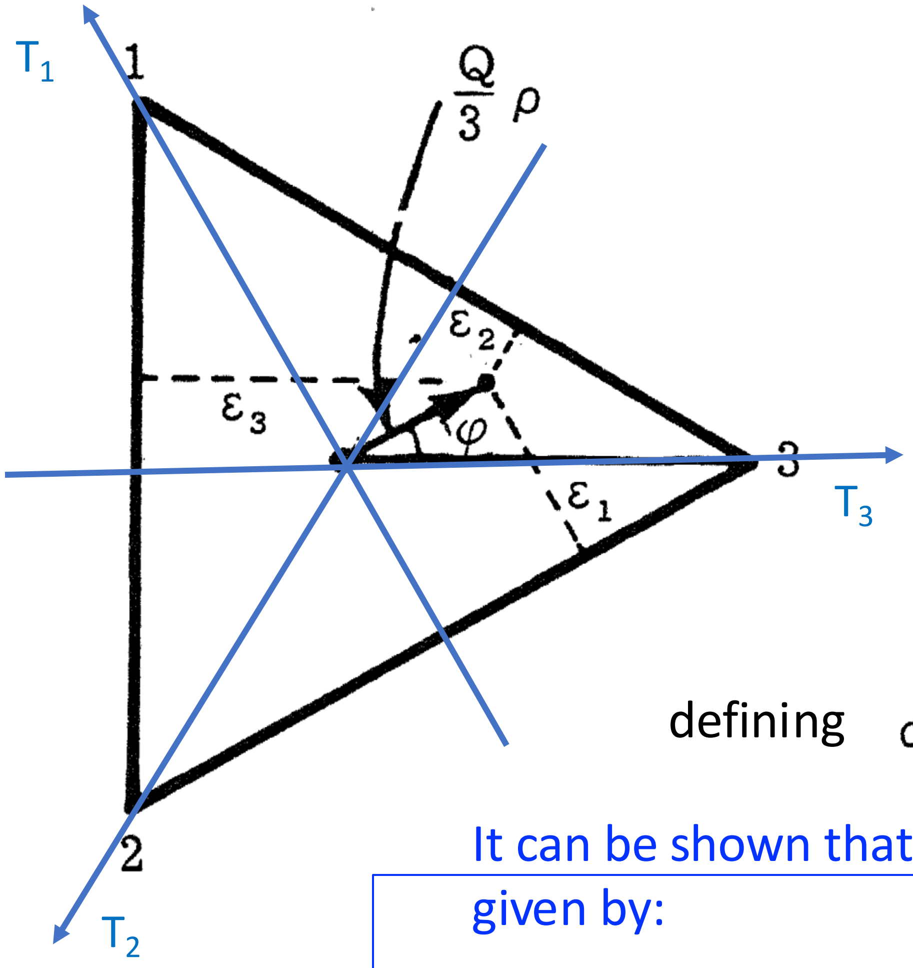
Unit area on a Dalitz plot is proportional to the corresponding Lorentz -invariant phase space

Therefore, the density of plotted points is proportional to the square of the matrix element.

For a pure phase space distribution (no intermediate states, no resonances, etc., i.e., constant squared matrix element) the density of points in the Dalitz plot has to be uniform.

Any deviation from uniformity must be attributed to the dynamics contained in the matrix element

Understanding the Dalitz triangle



$$\varepsilon_1 = \frac{Q}{3}(1 + \rho \cos \varphi_1) \qquad \varphi_1 = \varphi - \frac{2\pi}{3}$$

$$\varepsilon_2 = \frac{Q}{3}(1 + \rho \cos \varphi_2) \qquad \varphi_2 = \varphi + \frac{2\pi}{3}$$

$$\varepsilon_3 = \frac{Q}{3}(1 + \rho \cos \varphi)$$

defining $\sigma \equiv Q/M$

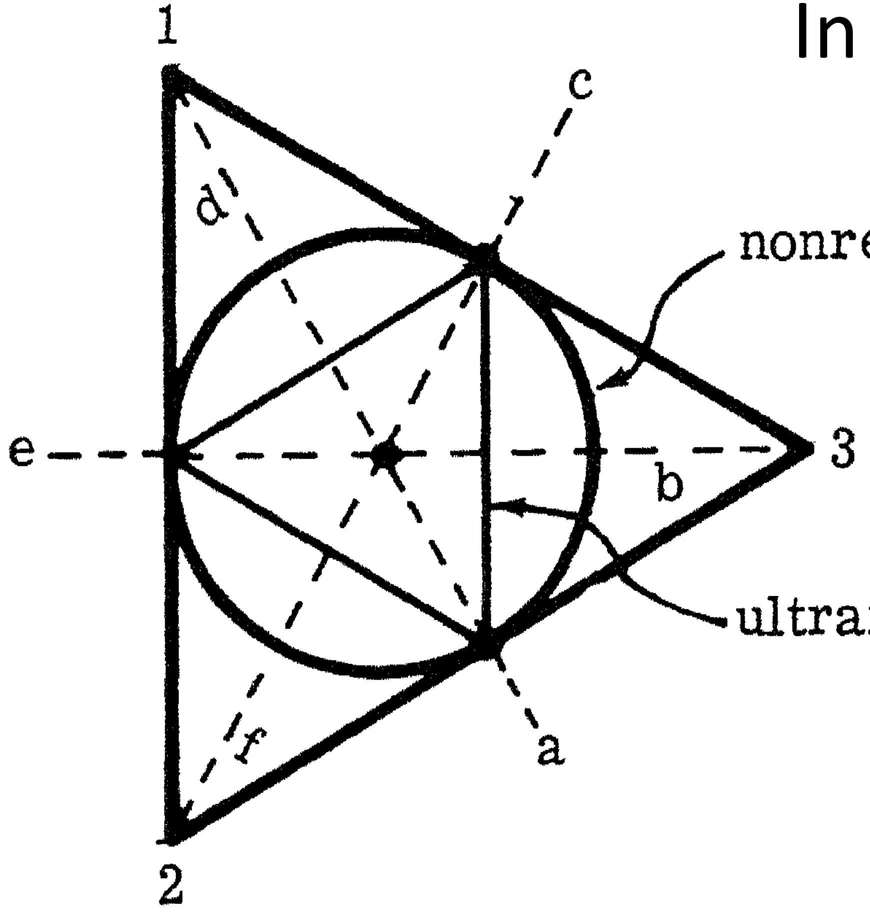
It can be shown that the Boundary Curve is given by:

$$\boxed{\rho^2 = \frac{1}{1 + \frac{2\sigma}{(2-\sigma)^2}(1 + \rho \cos 3\varphi)}}$$

Source:
R. Hagedorn, “Relativistic kinematics”
W.A.Benjamin Inc., 1964 edition

The boundary curve

In the CM frame



nonrelativistic $\varepsilon_i \ll m$ — Inscribed circle

$$\sigma = \frac{Q}{M} = 0$$

ultrarelativistic $\varepsilon_i \gg m$ — Inscribed triangle

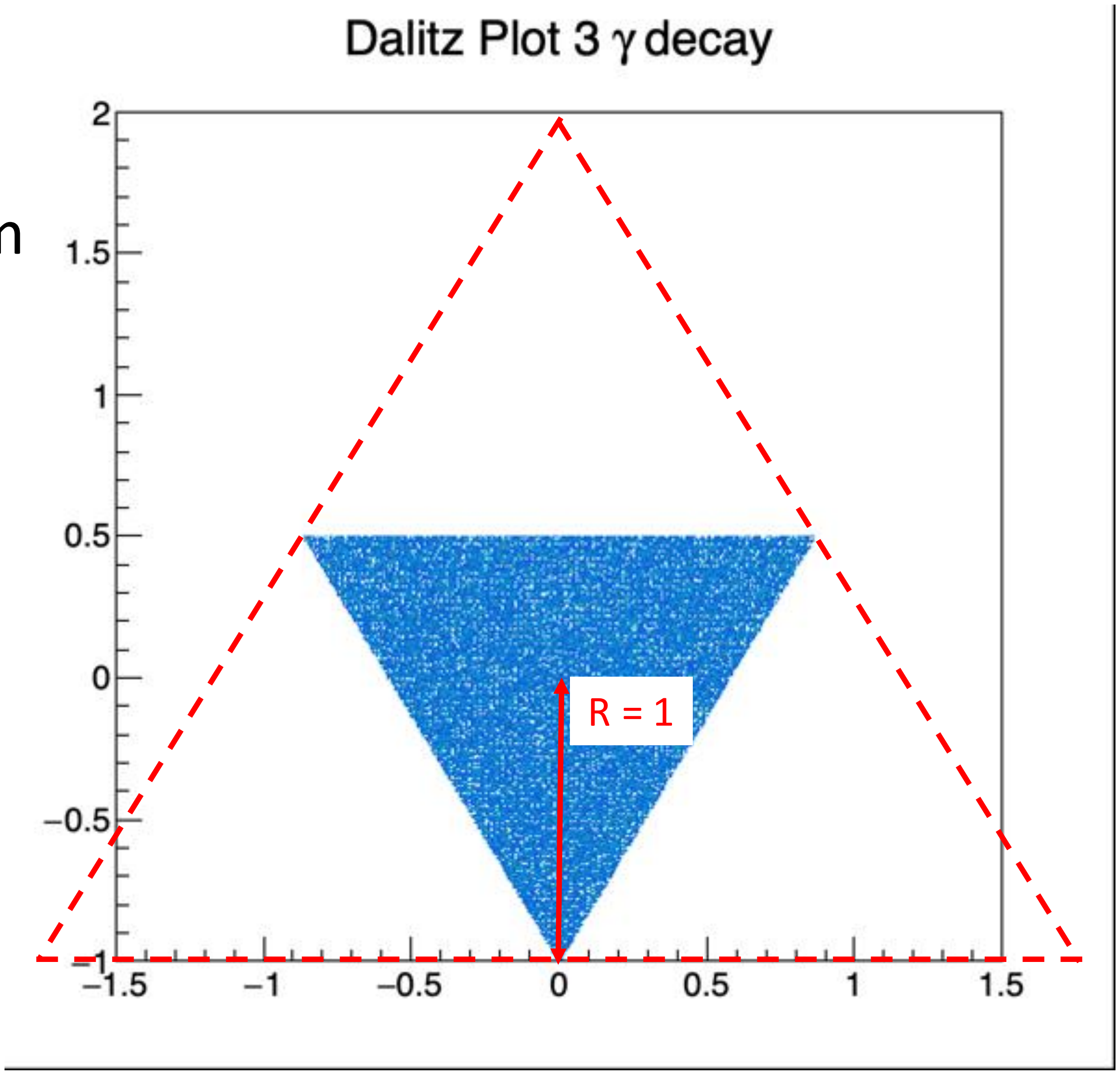
$$\sigma = \frac{Q}{M} = 1$$

i.e. when masses of decay products can be neglected

In the nuclear fragmentation case, since masses are heavy, we are in the <u>non-relativistic case</u>

Relativistic example

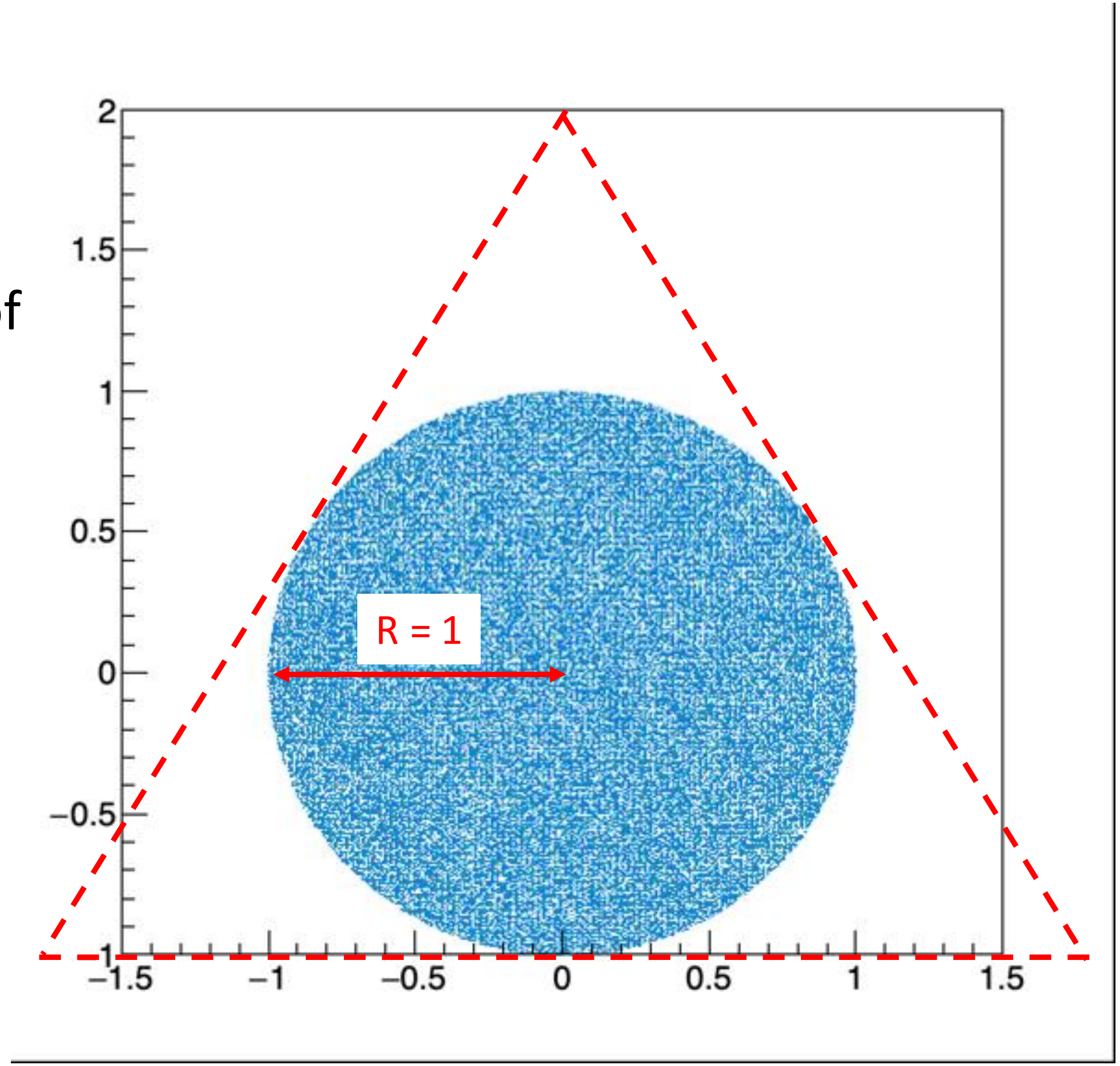
ortho-positronium
(s = 1) decay:
$e^+e^- \rightarrow 3\,\gamma$



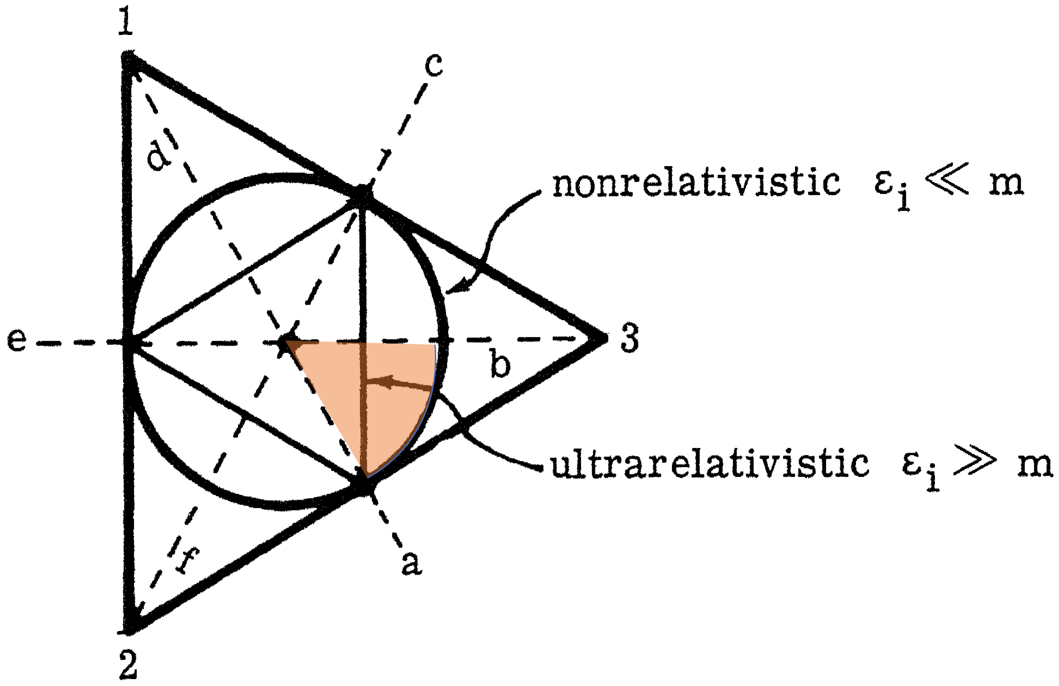
Non-relativistic example

Using a simple toy model for the decay of a nucleus into 3 fragments of with large masses

No intermediate state= constant matrix element = ***uniform phase space***



Symmetries in Dalitz Plot



- The triangular symmetry enables to consider only a sector, for instance the region between lines *a* and *b* (or a different pair)
- Points in other sectors can be reflected (folded) in the chosen sector, therefore increasing the statistics in density by a factor of 6

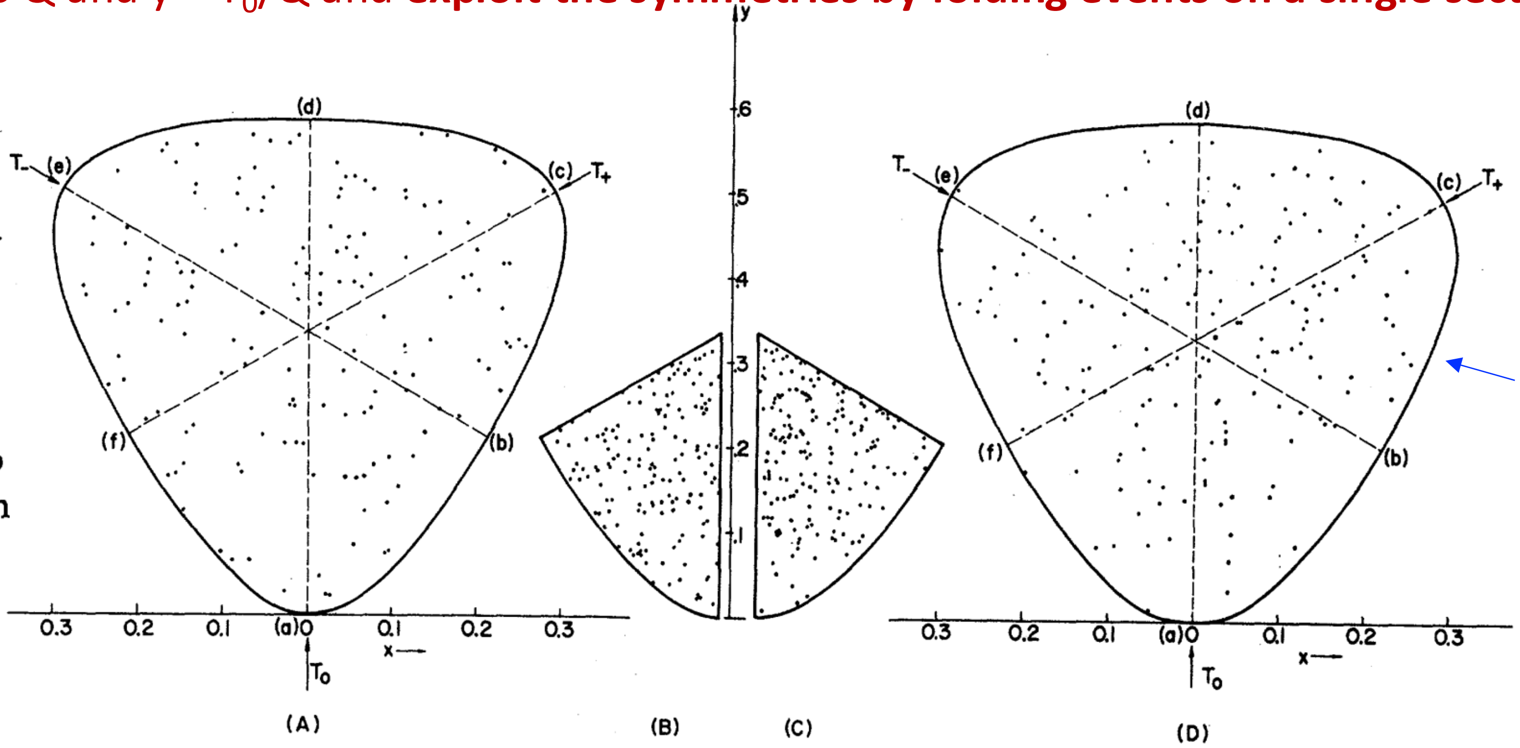
Historical Example

B. C. Maglić, L. W. Alvarez, A. H. Rosenfeld, and M. L. Stevenson, "Evidence for a T=0 Three-Pion Resonance", Phys. Rev. Lett. 7, 178 (1961); M. L. Stevenson, L. W. Alvarez, B. C. Maglić, and A. H. Rosenfeld, "Spin and Parity of the ω Meson", Phys. Rev. 125, 687 (1962)

They searched for $\omega \rightarrow \pi^+ + \pi^- + \pi^0$ in $p+\bar{p} \rightarrow \pi^+ + \pi^- + \pi^+ + \pi^- + \pi^0$

Here they use $x = (T_- - T_+)/\sqrt{3}\,Q$ and $y = T_0/Q$ and **exploit the symmetries by folding events on a single sector**

FIG. 3. (A): Dalitz plot of 171 triplets from the control region ($820 \leq M_3 < 900$); (B): folded control region plot; (D): Dalitz plot for 191 triplets in the peak region, $43 \pm 7\%$ of which are due to ω mesons; (C) folded peak region plot. T_+, T_-, and T_0 are kinetic energies of the π^+, π^-, and π^0, respectively.



Intermediate case between non-relativistic and ultra-relativistic conditions

For a Isospin=0 particle, they expected a maximum at the center and a depopulation at the border in points b,d,f

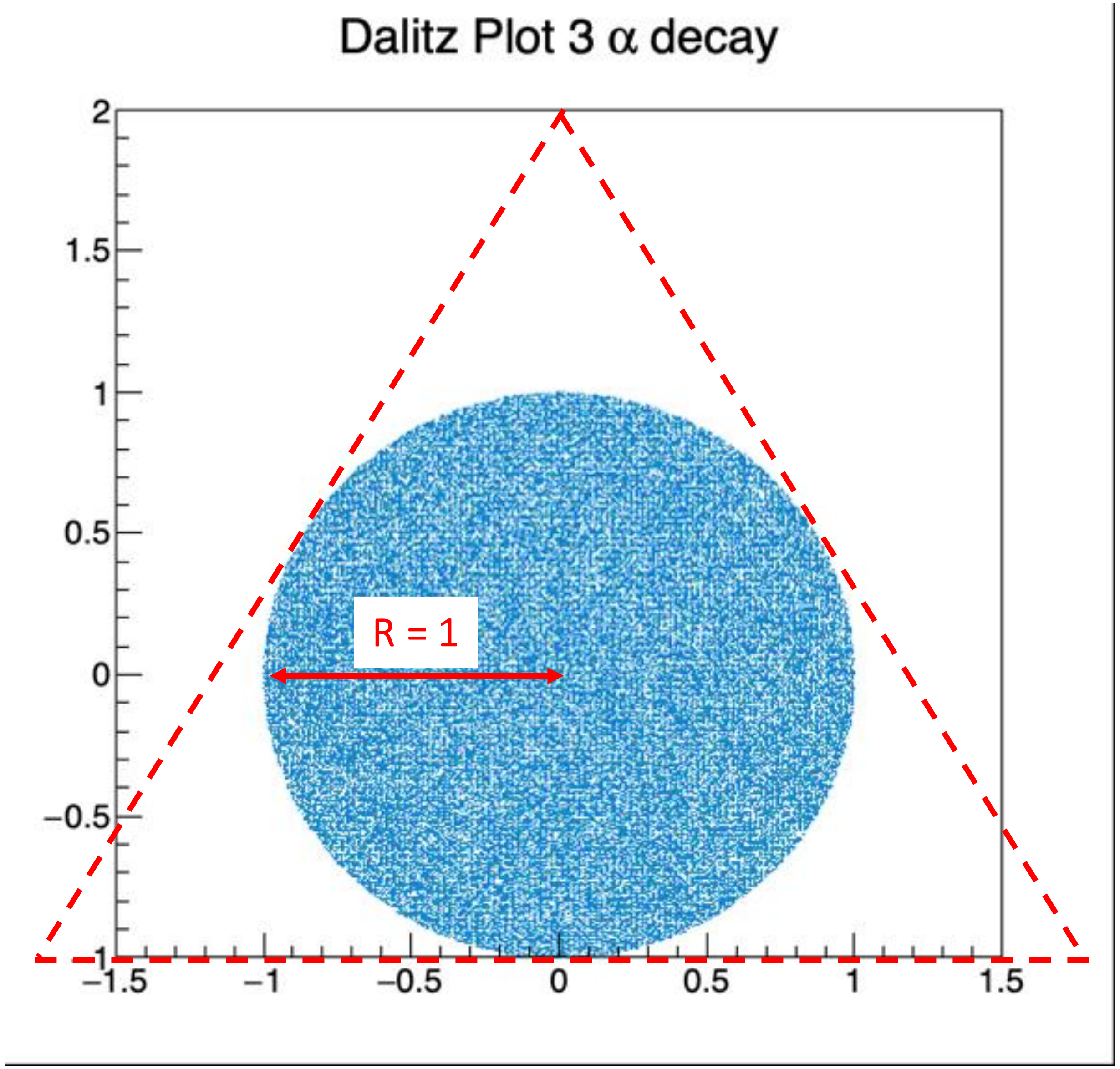
Application of Dalitz plot in clustering study

- In our case this technique can be exploited for the analysis of 3α's final state in ^{12}C or ^{16}O fragmentation
- This requires to perform a Lorentz Transformation of the energies of the 3 alphas from the lab system back to rest frame, i.e. to the frame where their resulting total 3-momentum is null.

Toy generator: direct decay

^{12}C ➔ 3α in the rest frame of ^{12}C using a simple toy model.

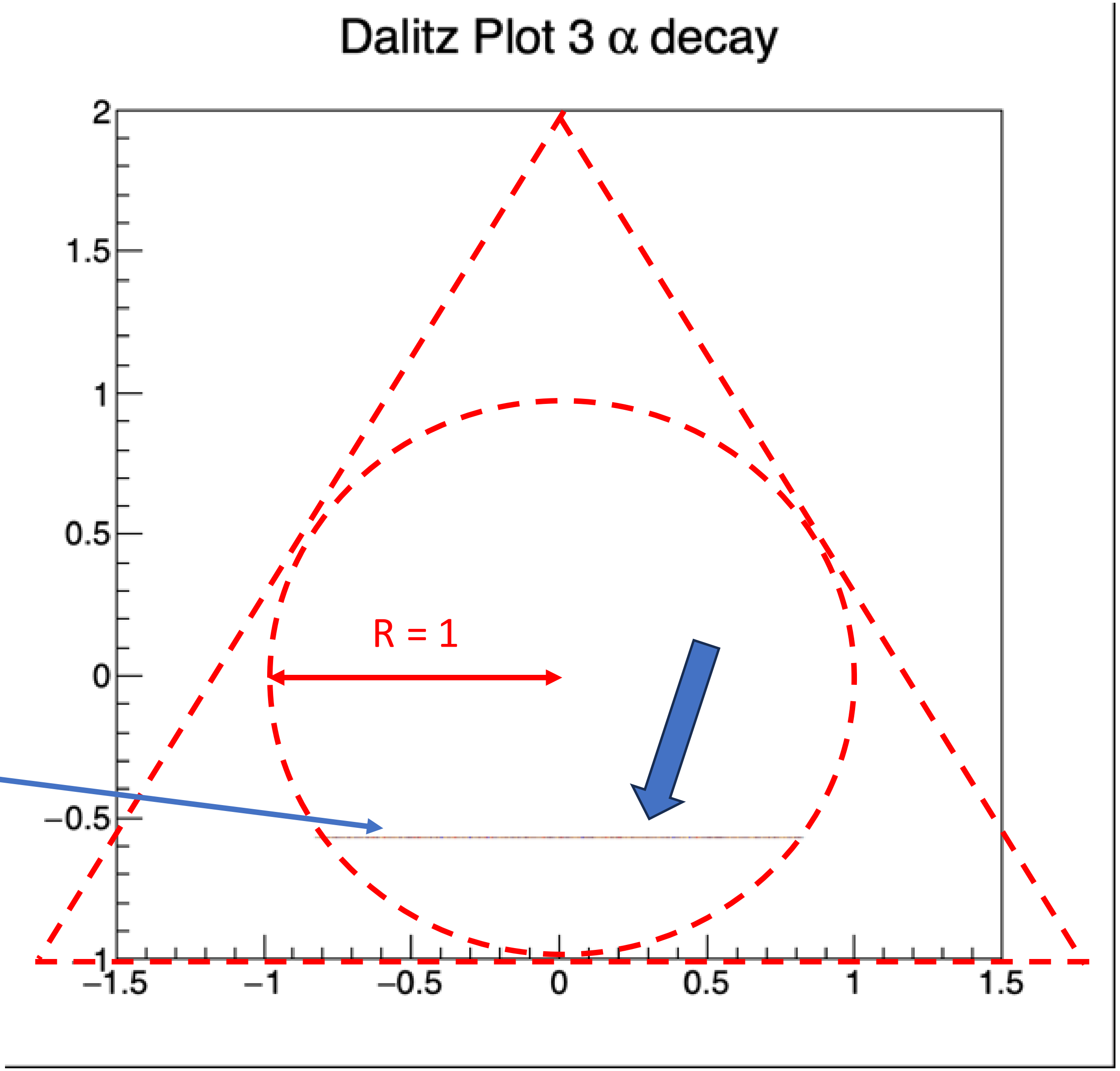
Here just direct decay, i.e. no intermediate state: uniform phase space



Toy generator: sequential decay

$^{12}C \rightarrow 3\alpha$ in the rest frame of ^{12}C using a simple toy model.

Pure sequential decay through $^{8}Be + \alpha$



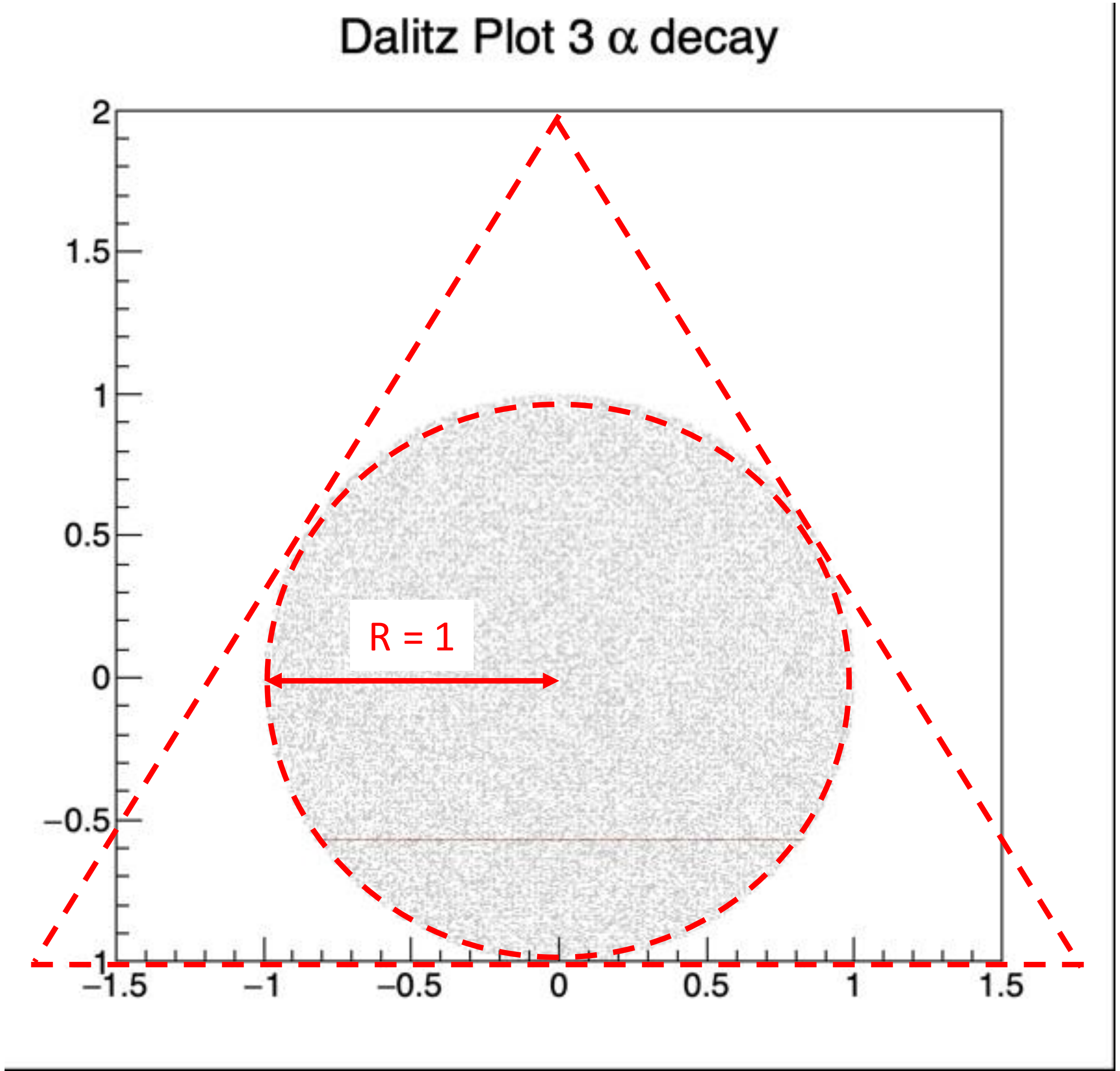
That line is very sharp just because with this generator we are omitting nuclear recoil and other phenomena. A complete MC simulation will produce a visible smearing

Toy generator: direct + sequential decay

$^{12}C \rightarrow 3\alpha$ in the rest frame of ^{12}C using a simple toy model.

- 50% Sequential decay through $^{8}Be + \alpha$

- 50% Direct Decay



From the toy model to advanced MC generator

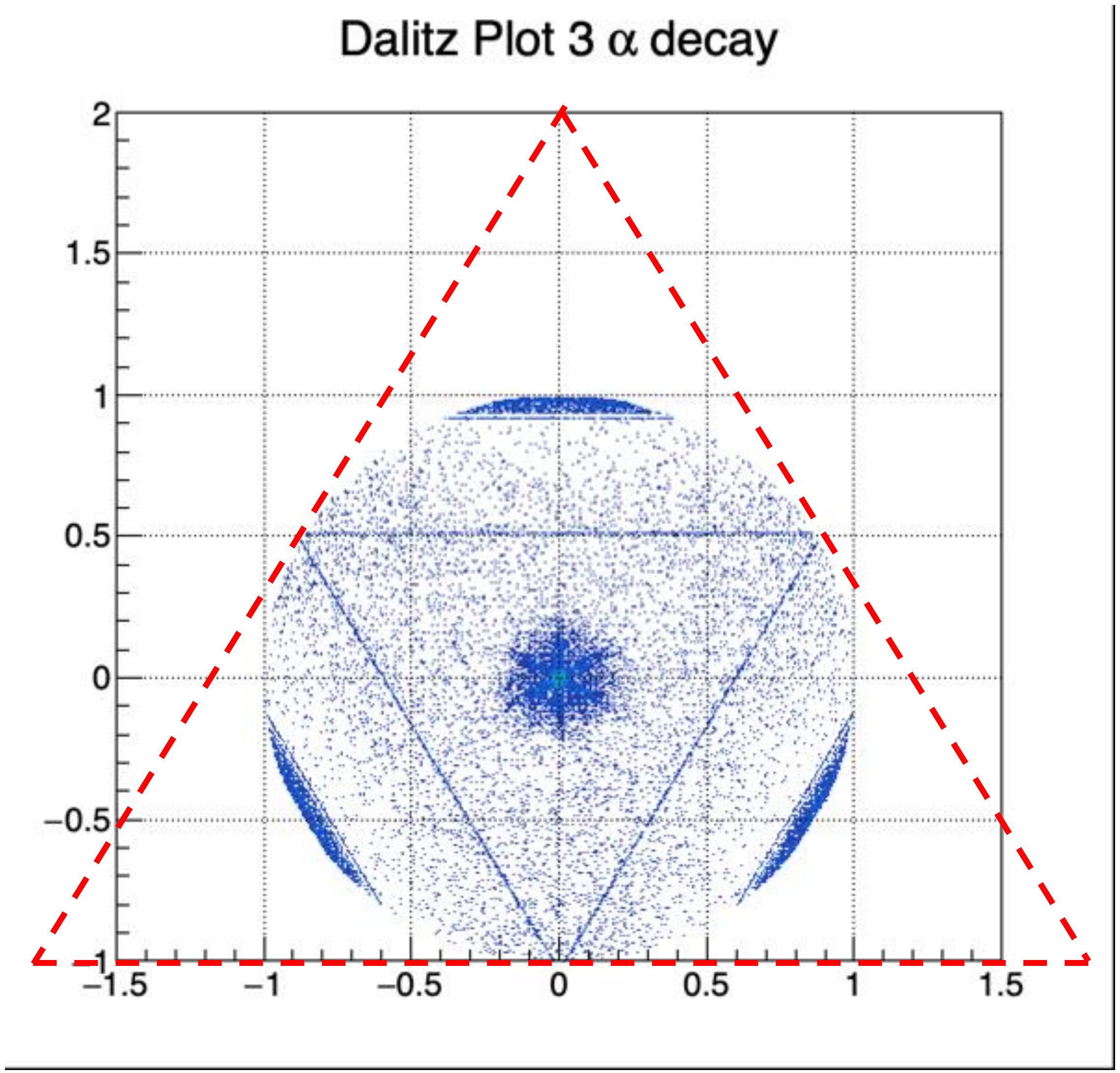
Using the MC stand-alone generator of FLUKA, 10^6 interactions of $^{12}C + ^{12}C$ at 200 MeV/u have been produced, and events where the projectile undergoes $^{12}C \rightarrow 3\alpha$ fragmentation were selected

29480 selected events

This selection includes all possible intermediate states, i.e. no “a priori” selection on possible formation of intermediate state (such as ^{8}Be) is performed

Note: with respect to the full MC, this stand-alone generator stops before some steps following the primary interactions, such as nuclear recoil, which determine some smearing of energies and other kinematic variables. For example, energy levels will remain with very thin widths

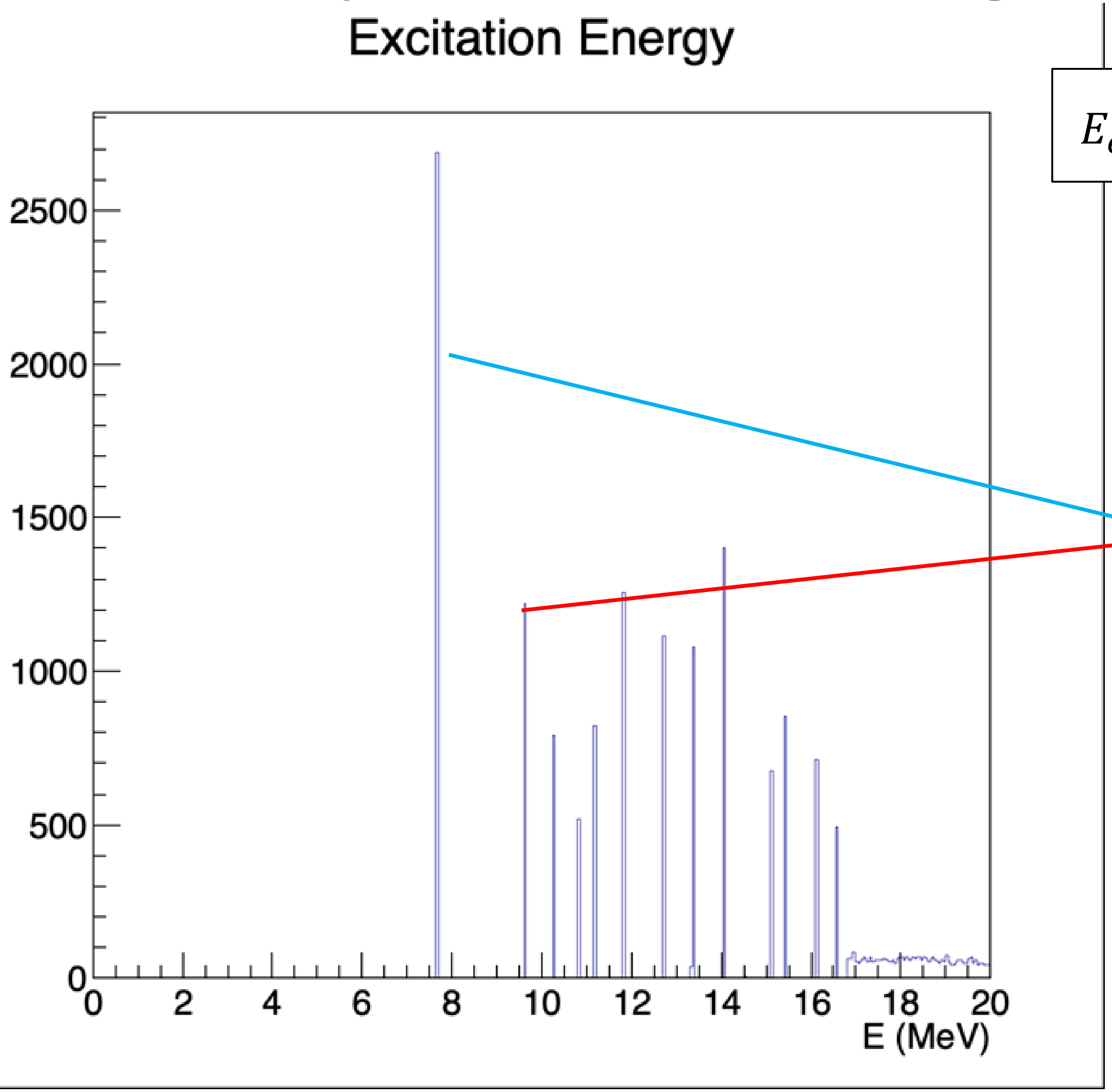
Resulting Dalitz Plot (all events)



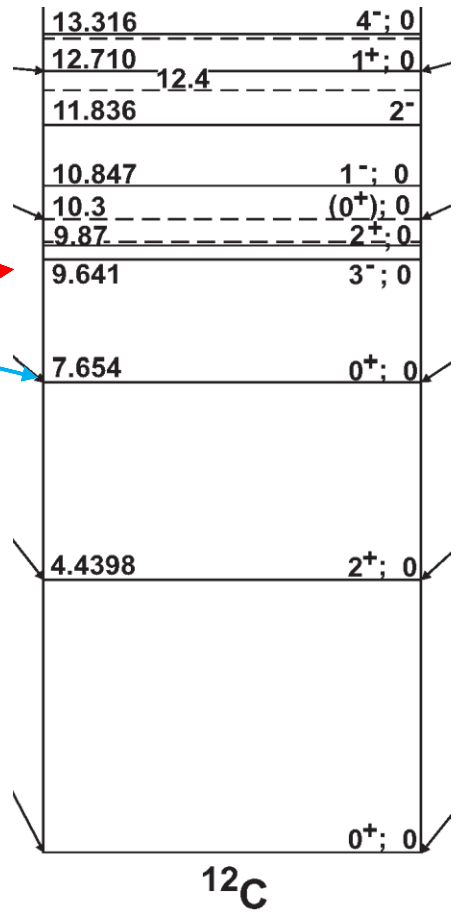
There are structures:
In this generator, the process $^{12}C \rightarrow 3\alpha$ proceeds mostly through sequential intermediate states.
The structures are undoubtedly connected to intermediate states in the decay:
a decay $^{12}C \rightarrow 3\alpha$ at each different excitation energy E_{ex} can be thought as a different resonance

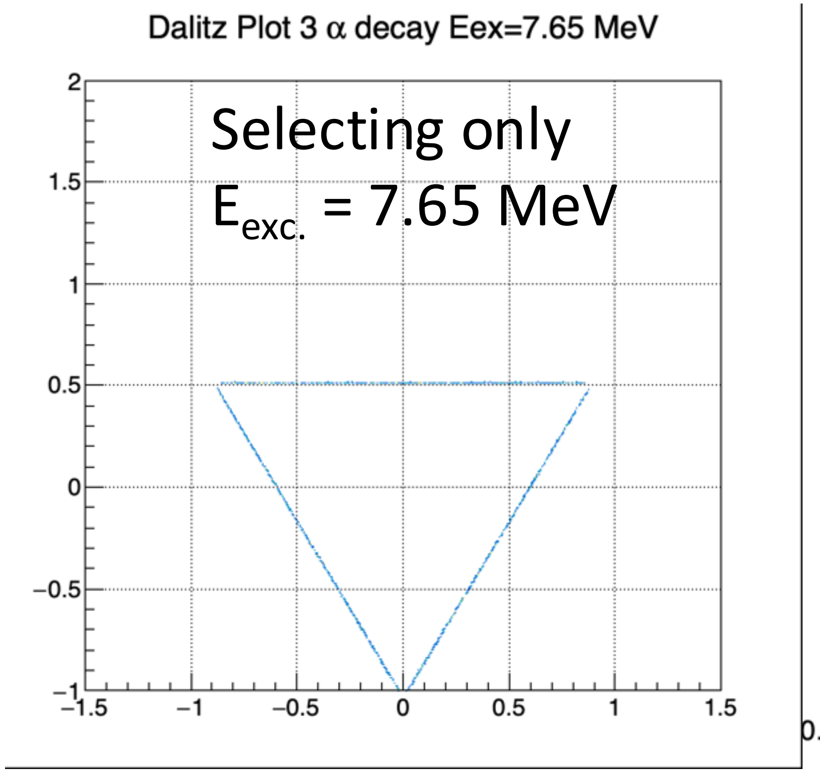
Notice: in this exercises the order of occurrence of the 3 alphas has been randomized, since in the FLUKA generator the most energetic particle will always appear as the first

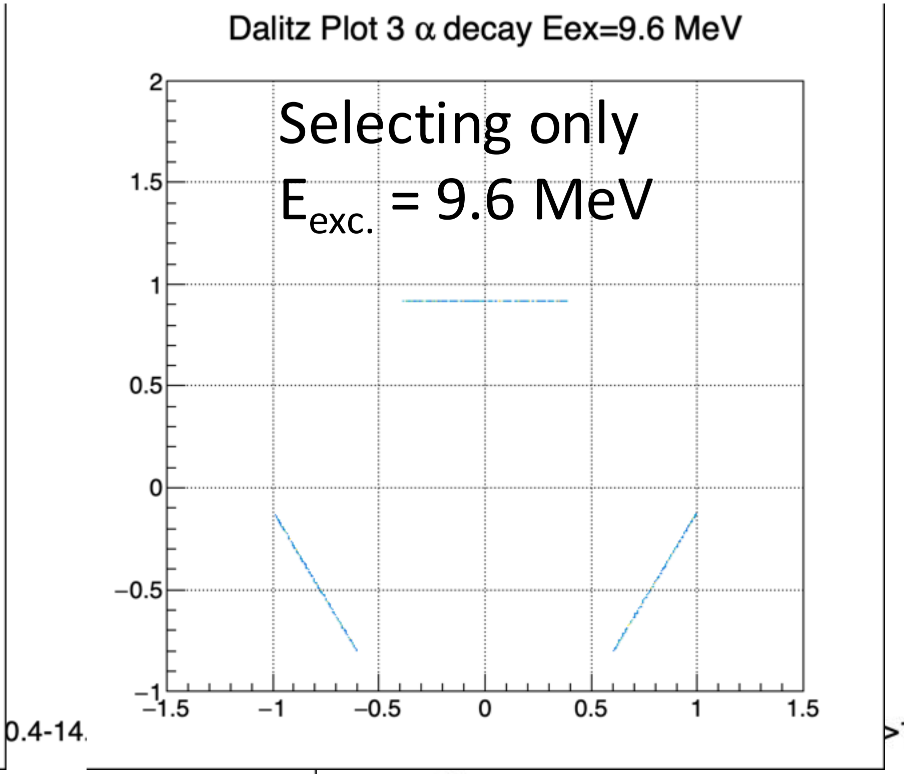
Interpretation using Excitation Energy

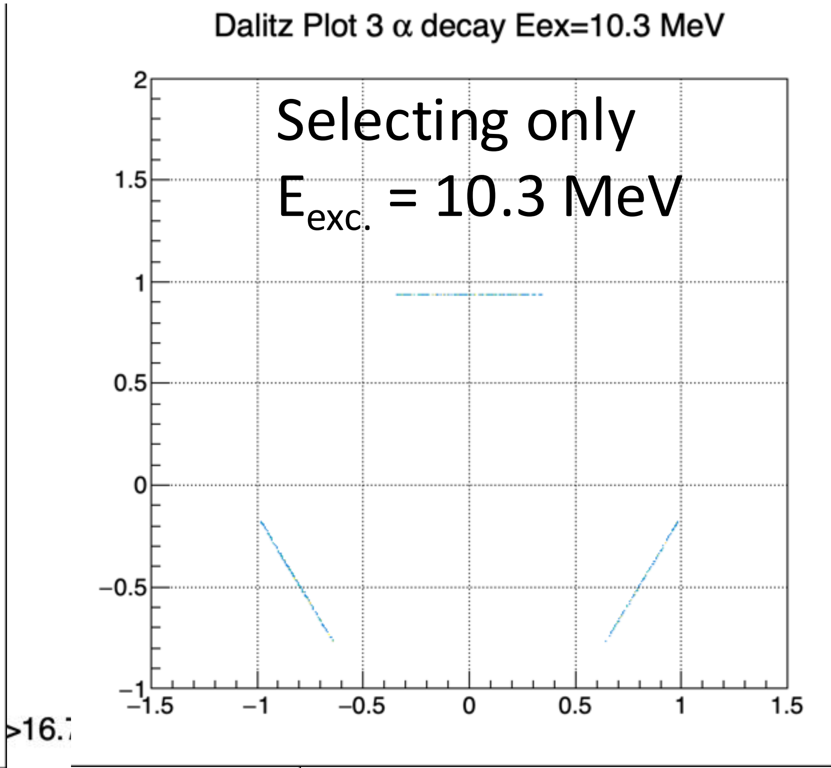


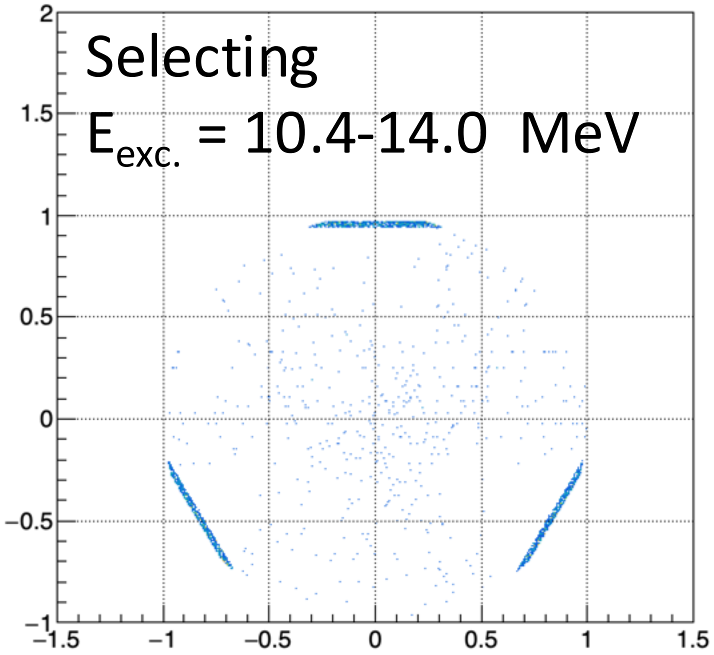
$$E_{ex} = \sqrt{(E_i^{kin} + E_j^{kin} + E_k^{kin} + 3m_\alpha)^2 - (\overrightarrow{p_i} + \overrightarrow{p_j} + \overrightarrow{p_k})^2} - m_{^{12}C}$$

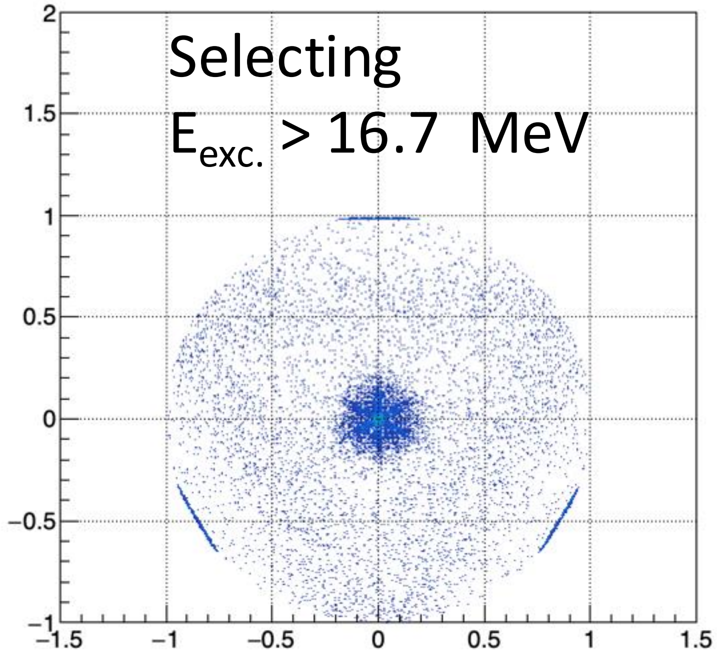












In FLUKA:

- **For low excitation energy, there is pure sequential decay**
- **Direct $^{12}C \rightarrow 3\alpha$ decay (uniform density) seems to appear only at high excitation energy**

Back to Dell’Aquila el at. results

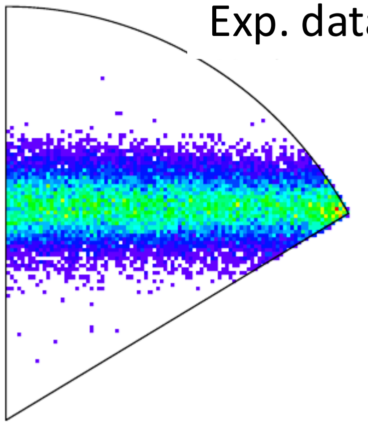
Their variables:

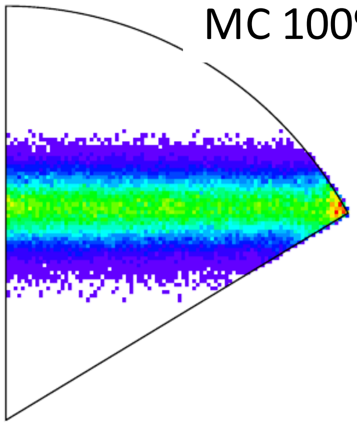
$\epsilon_{i,j,k} = E_{i,j,k}/Q$
$x = \sqrt{3}\,(\epsilon_j - \epsilon_k)$
$y = 2\epsilon_i - \epsilon_j - \epsilon_k$

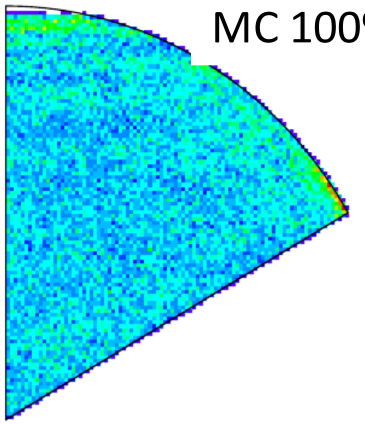
They exploited the Dalitz plot symmetry simply by ordering the energies of the 3 α’s:

$\epsilon_i \geq \epsilon_j \geq \epsilon_k$

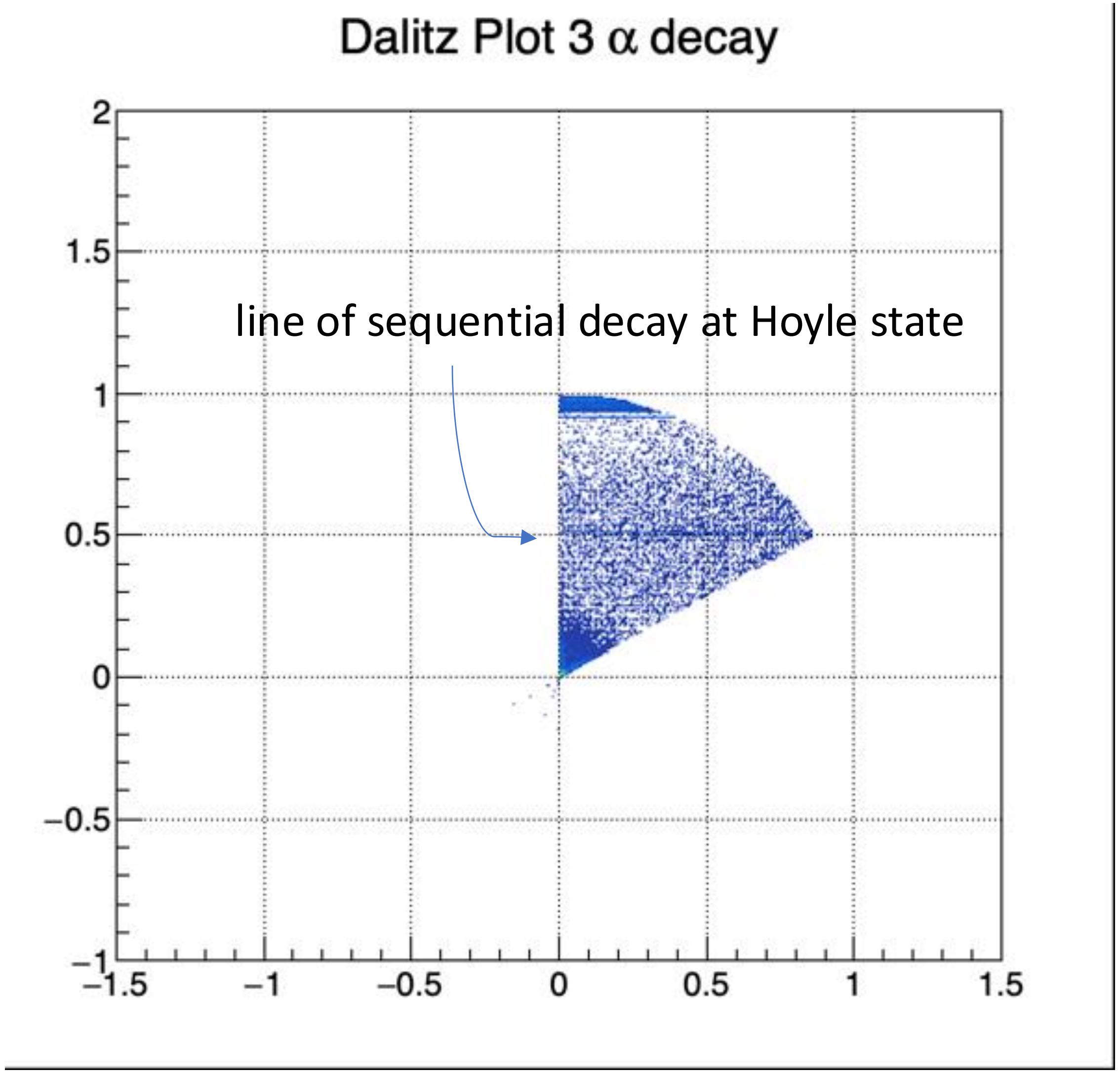
In this way you automatically obtain the folding of the unit radius circle into one of its sextants


Exp. data


MC 100% Sequential decay


MC 100% direct decay

Generator exercise: symmetric Dalitz Plot

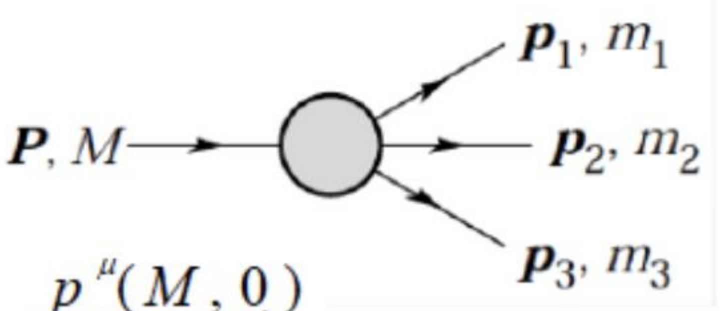


Here the same ordering of Dell'Aquila et al. has been used.

Again, it is possible to perform an analysis in terms of the excitation energy of the 3α system

Alternative approach to Dalitz Plot

3-body decay relativistic kinematics



$$p^\mu(M, 0)$$
$$p_1^\mu(E_1, \vec{p}_1)$$
$$p_2^\mu(E_2, \vec{p}_2)$$
$$p_3^\mu(E_3, \vec{p}_3)$$

Invariant square masses of pairs of final particles:

$$m_{12}^2 = (p_1^\mu + p_2^\mu)^2 = (p^\mu + p_3^\mu)^2 = M^2 + m_3^2 - 2ME_3$$

$$m_{23}^2 = (p_2^\mu + p_3^\mu)^2 = (p^\mu + p_1^\mu)^2 = M^2 + m_1^2 - 2ME_1$$

$$m_{13}^2 = (p_1^\mu + p_3^\mu)^2 = (p^\mu + p_2^\mu)^2 = M^2 + m_2^2 - 2ME_2$$

Notice that: $dE_1 = -dm_{23}{}^2$ $dE_2 = -dm_{13}{}^2$ $dE_3 = -dm_{12}{}^2$

Therefore: $d\varphi_3 = constant\ dE_1 dE_2 \propto dm_{23}^2 dm_{13}^2$

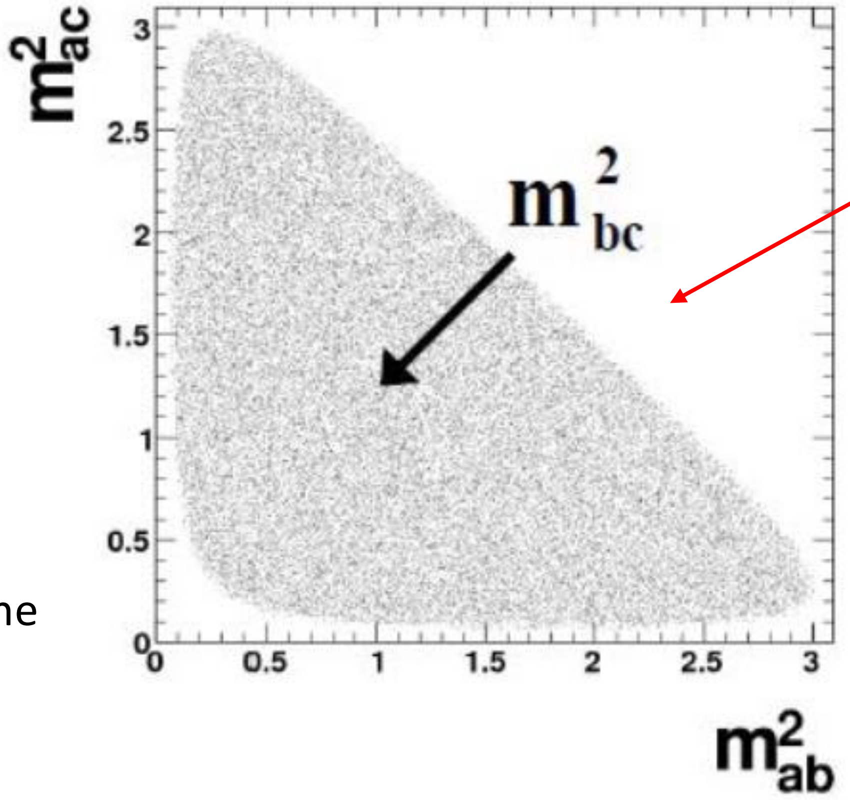
or a similar combination: $dm_{12}^2 dm_{23}^2$; $dm_{12}^2 dm_{13}^2$

relativistic kinematics and Dalitz Plot

M ➔ a+b+c

Use invariant masses of 2 different pairs

$$x = m^2_{ab} = s_{ab} = (p^\mu_a + p^\mu_b)^2$$
$$y = m^2_{ac} = s_{ac} = (p^\mu_a + p^\mu_c)^2$$

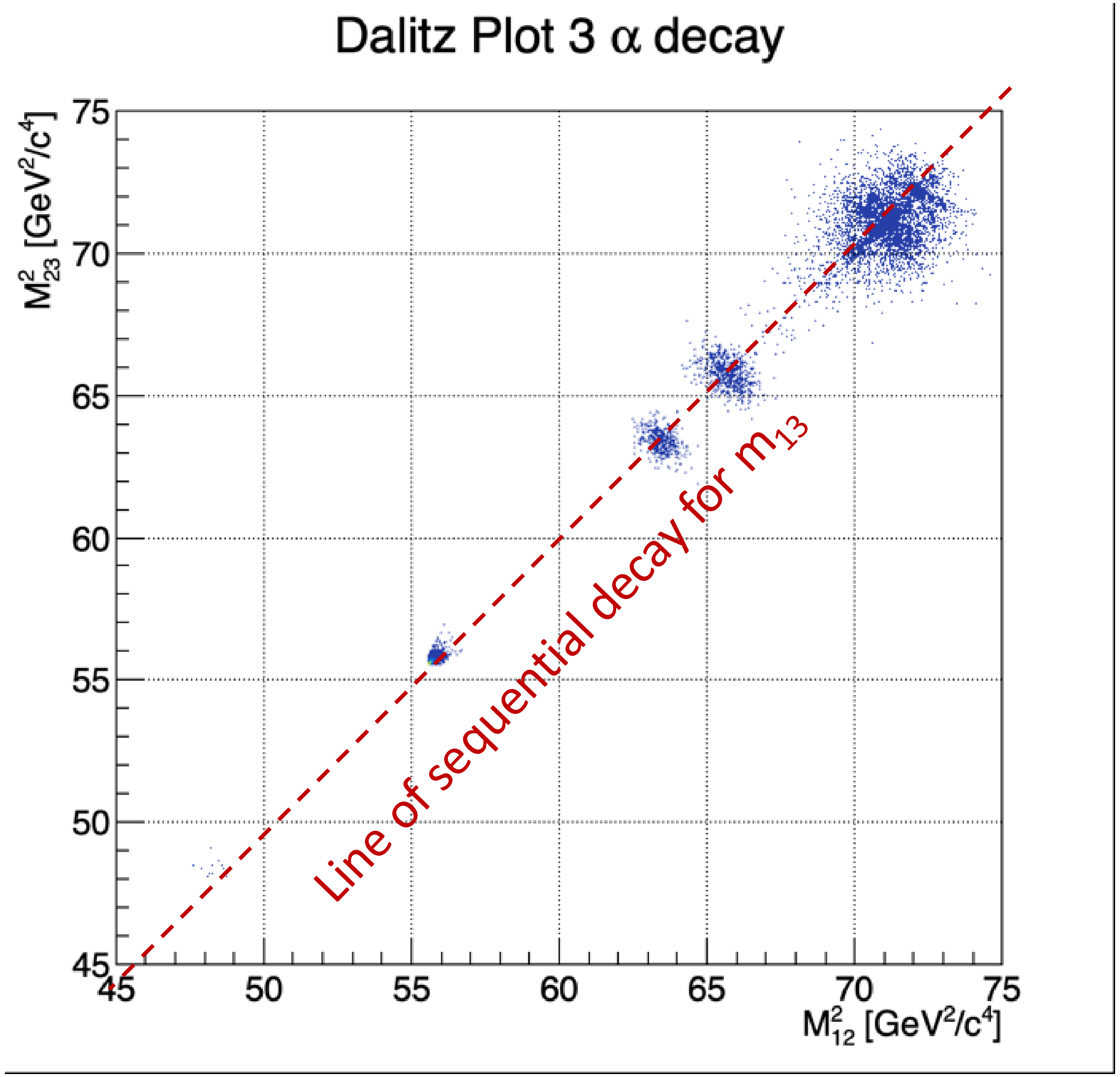


The kinematical constraints produce a ~triangular boundary

Any enhancement in the distribution of points in the Dalitz Plot would corresponds to the existence of a resonance R decaying in 2 of the 3 particles:

In our case: ➔ the presence of a sequential decay through a given level of excitation energy

Exercise with our MC generator:



Building squared invariant masses of pairs of detected α particles

Each excitation energy E_{ex} in the decay $^{12}C \rightarrow 3\alpha$ behaves as a different "resonance" with a given mass value

Conclusions

- The technique of Dalitz Plot analysis is helpful to investigate the possible competition between sequential and direct fragmentation in the $^{12}C \rightarrow 3\alpha$ process
- From our MC generator, it appears natural that for low excitation energy, such a fragmentation should go through a pure sequential channel
- At the very low energies (like those investigated by Dell'Aquila et al.) it should be difficult to excite high energy levels
- At 200 MeV/u and above instead, we could find these higher excitations where direct decay is predicted to occur. The presence or absence of direct decay also for E>200 MeV/u probably would be one of the most interesting results from the α-clustering analysis in FOOT.
- Next step: test the use of Dalitz plot analysis using the full MC simulation

Appendix: 3-d Lorentz Transformation to the rest frame of the 3 α system

In the laboratory system we can build the quantities:

$$P_x^{tot} = \sum P_x^i \quad P_y^{tot} = \sum P_y^i \quad P_z^{tot} = \sum P_z^i$$

$$E^{tot} = \sum (T_i A + m_\alpha)$$

A=4 and T_i are the kinetic energies per nucleon

We can then define the parameters for the 3-D Lorentz Transformation to the rest frame where $\left|\vec{P}^{tot}\right| = 0$

$$\beta_x = \frac{P_x^{tot}}{E^{tot}} \quad \beta_y = \frac{P_y^{tot}}{E^{tot}} \quad \beta_z = \frac{P_z^{tot}}{E^{tot}} \quad \gamma = \frac{1}{\sqrt{1 - \left|\vec{\beta}\right|^2}}$$

Where: $\left|\vec{\beta}\right|^2 = \beta_x^2 + \beta_y^2 + \beta_z^2$

Let's introduce also the 3-vector $\vec{\eta} = \gamma\vec{\beta}$

For each α we can then obtain the kinetic quantities $P^*_{x,y,z}$ and E^* in the frame where $|\vec{P}^{tot}| = 0$ by means of the following Lorentz transformation defined for an arbitrary direction in the 3-d space:

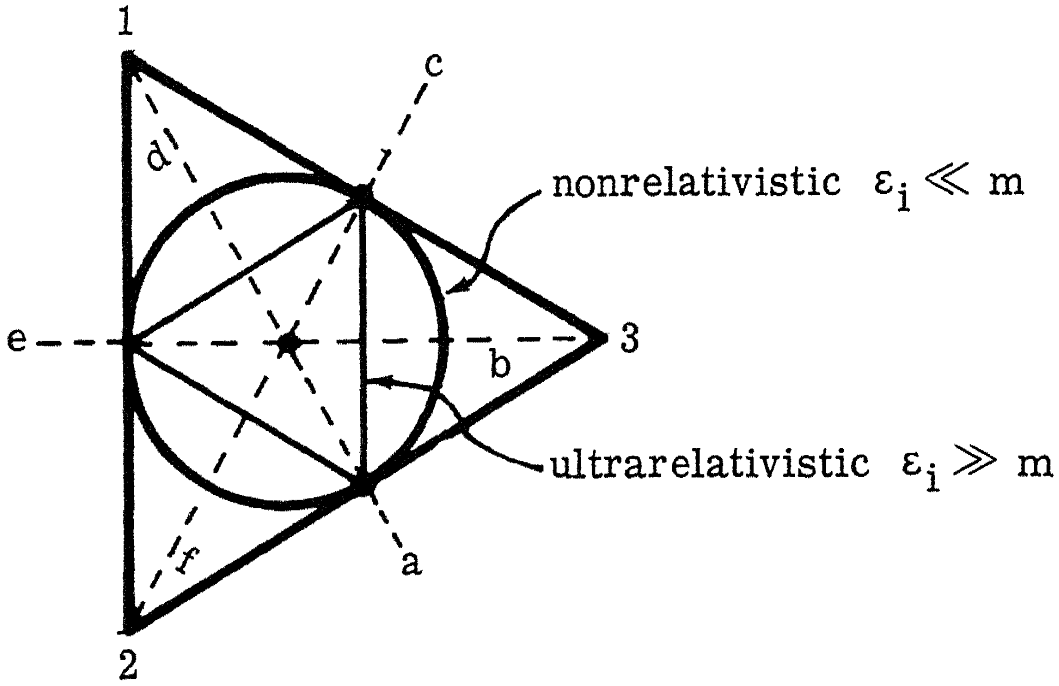
$$E^* = \gamma\, E - \vec{P} \cdot \vec{\eta}$$

$$P_x^* = P_x - \eta_x \left(E - \frac{\vec{P} \cdot \vec{\eta}}{1 + \gamma} \right)$$

$$P_y^* = P_y - \eta_y \left(E - \frac{\vec{P} \cdot \vec{\eta}}{1 + \gamma} \right)$$

$$P_z^* = P_z - \eta_z \left(E - \frac{\vec{P} \cdot \vec{\eta}}{1 + \gamma} \right)$$

Appedix: Symmetries in Dalitz Plot



- The triangular symmetry enables to consider only a sector, for instance the region between lines *a* and *b* (or a different pair)
- Points in other sectors can be reflected (folded) in the chosen sector, and therefore increase the statistics by a factor of 6
- The line center→ a is a symmetry line on which $T_2 = T_3 = ½(Q-T_1)$. For $T_1=Q/3$ you reach the center, for $T_1=0$ you reach the boundary
- The line center→ b is a symmetry line on which $T_1 = T_2$. T_3 varies between Q/3 at the center, and the maximum value at the boundary
- For any of the energies $T_{i,max} = Q(M+m)/(2 M)$
- On the boundary curve, all 3 momenta are collinear