Which ball? Which box? Where did the ball stop?

— A playful introduction to Bayesian reasoning —

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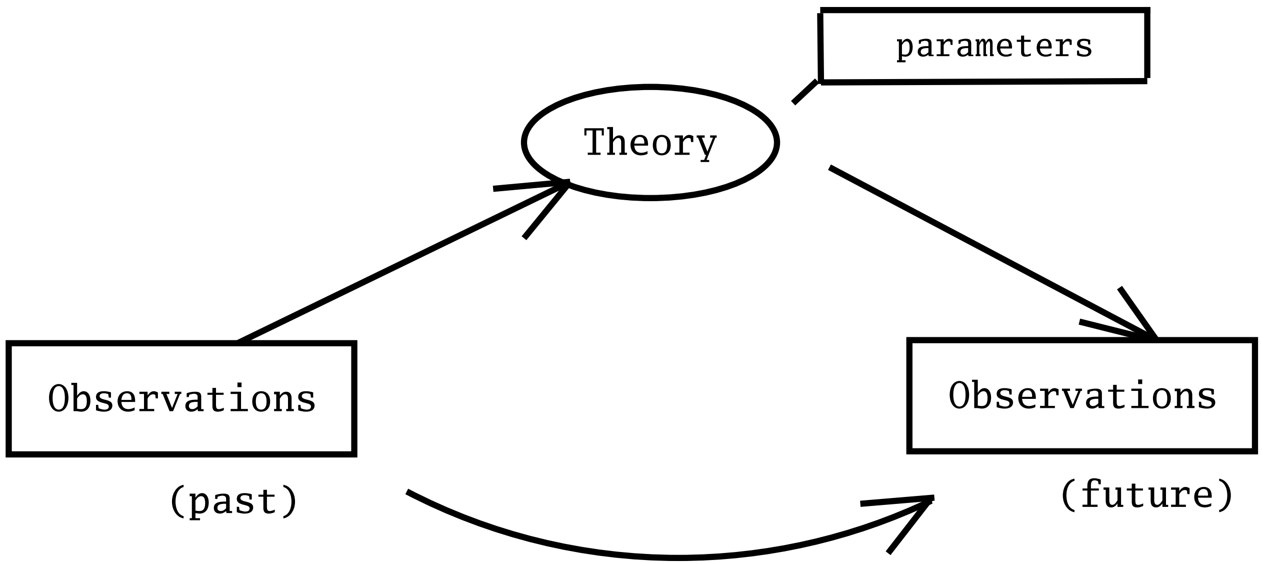
"Probability is good sense reduced to a calculus" (Laplace)

Doing Physics [Science in general]



Task of physicists:

- Describe/understand the physical world
  $\Rightarrow$ inference of laws and their parameters
- Predict observations
  $\Rightarrow$ forecasting

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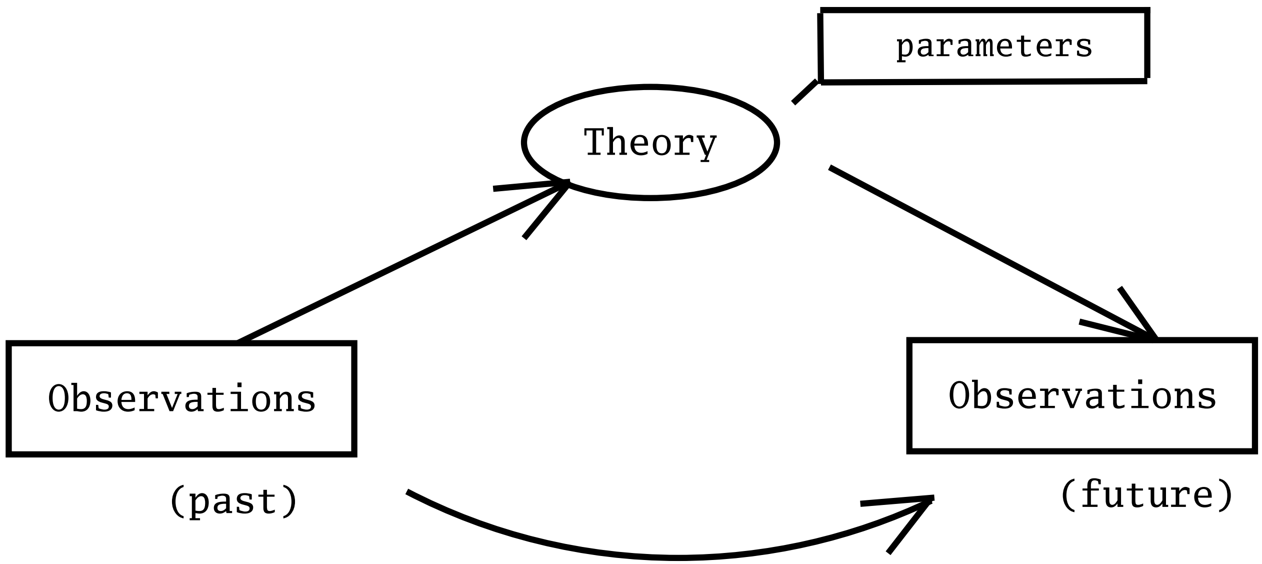
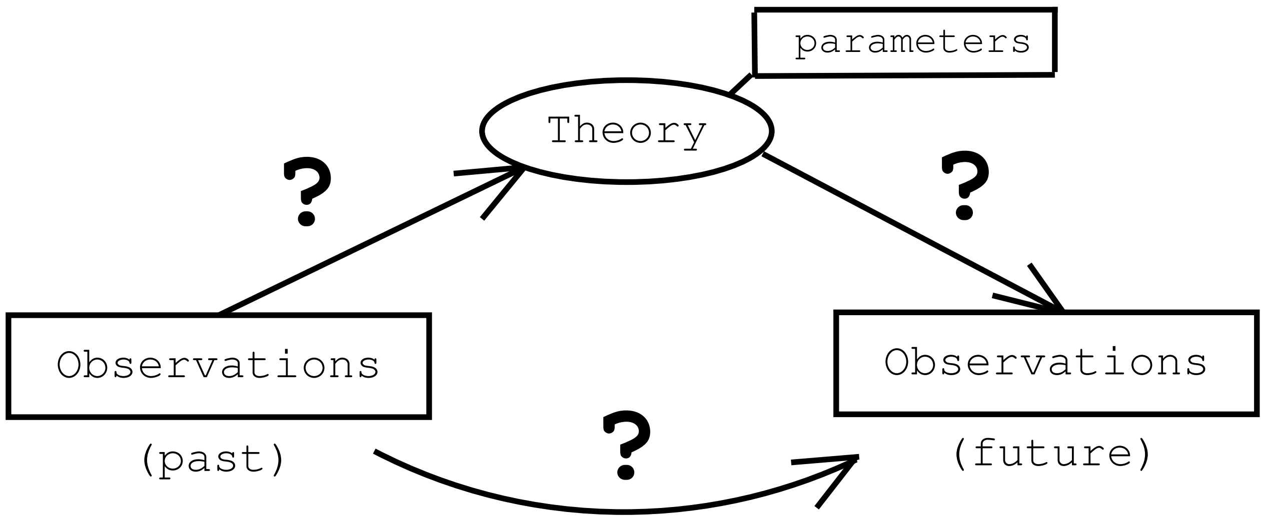


Process

- neither automatic
- nor purely contemplative
  - $\rightarrow$ ‘scientific method’
  - $\rightarrow$ planned experiments (‘actions’) $\Rightarrow$ decision.

Doing Physics [Science in general]

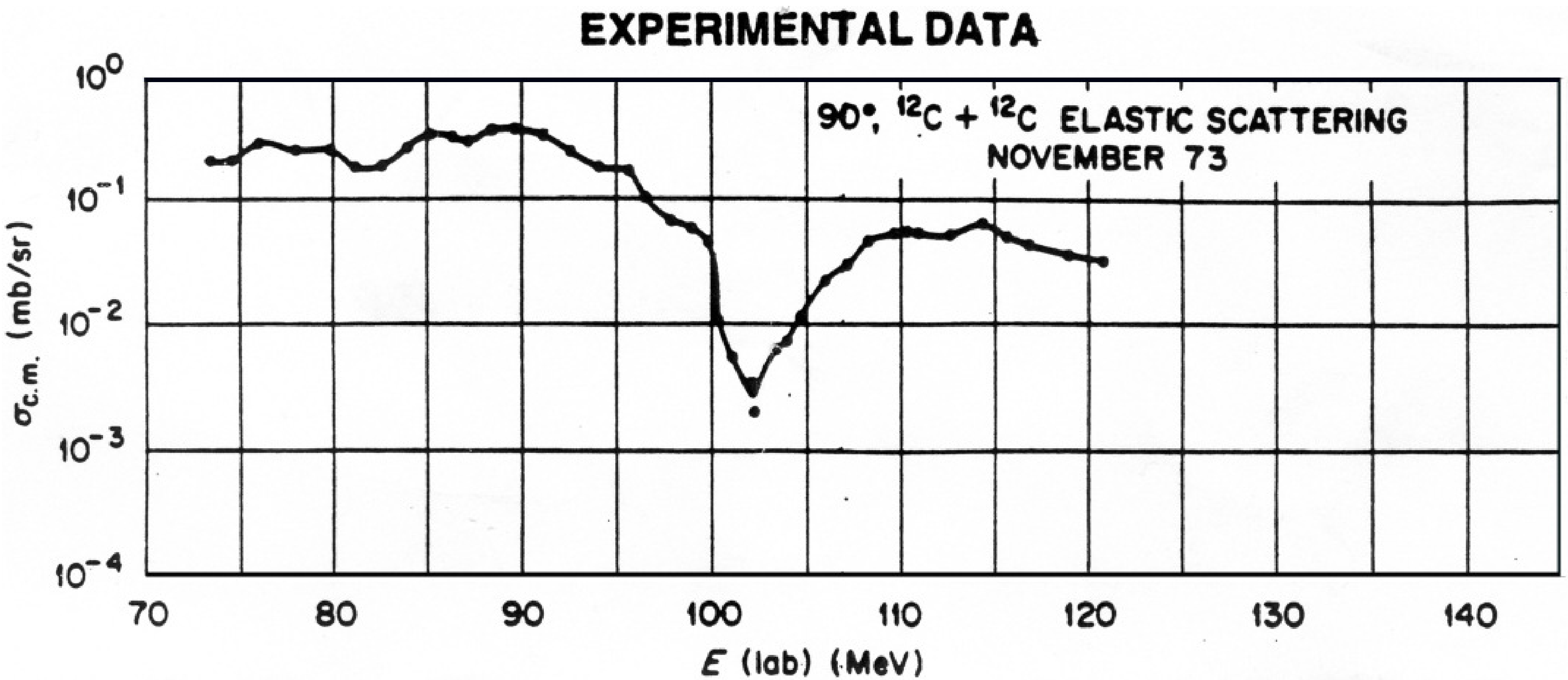


⇒ Uncertainty:

1. Given the past observations, in general we are not sure about the theory parameters (and/or the theory itself)

2. Even if we were sure about theory and parameters, there could be internal (e.g. Q.M.) or external effects (initial/boundary conditions, ‘errors’, etc) that make the forecasting uncertain.

Inferential-predictive process



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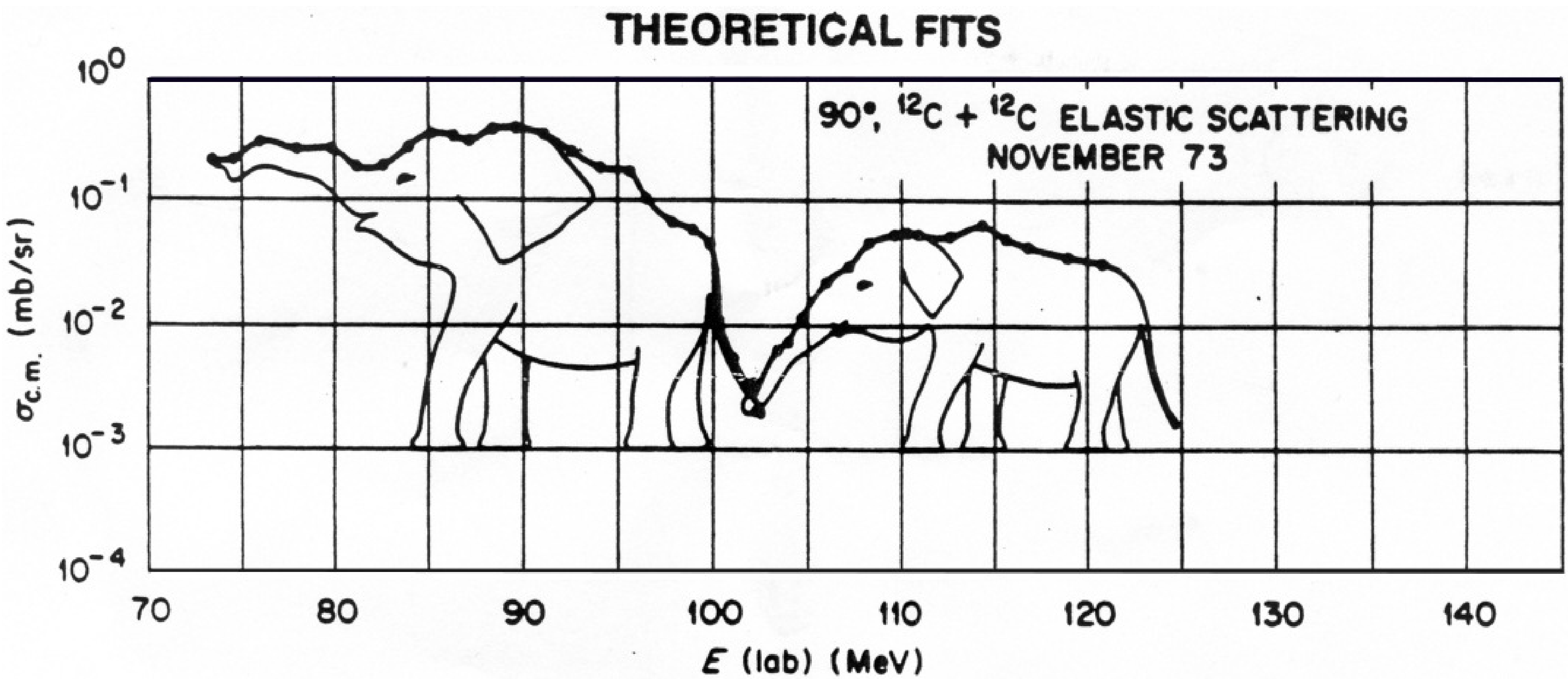

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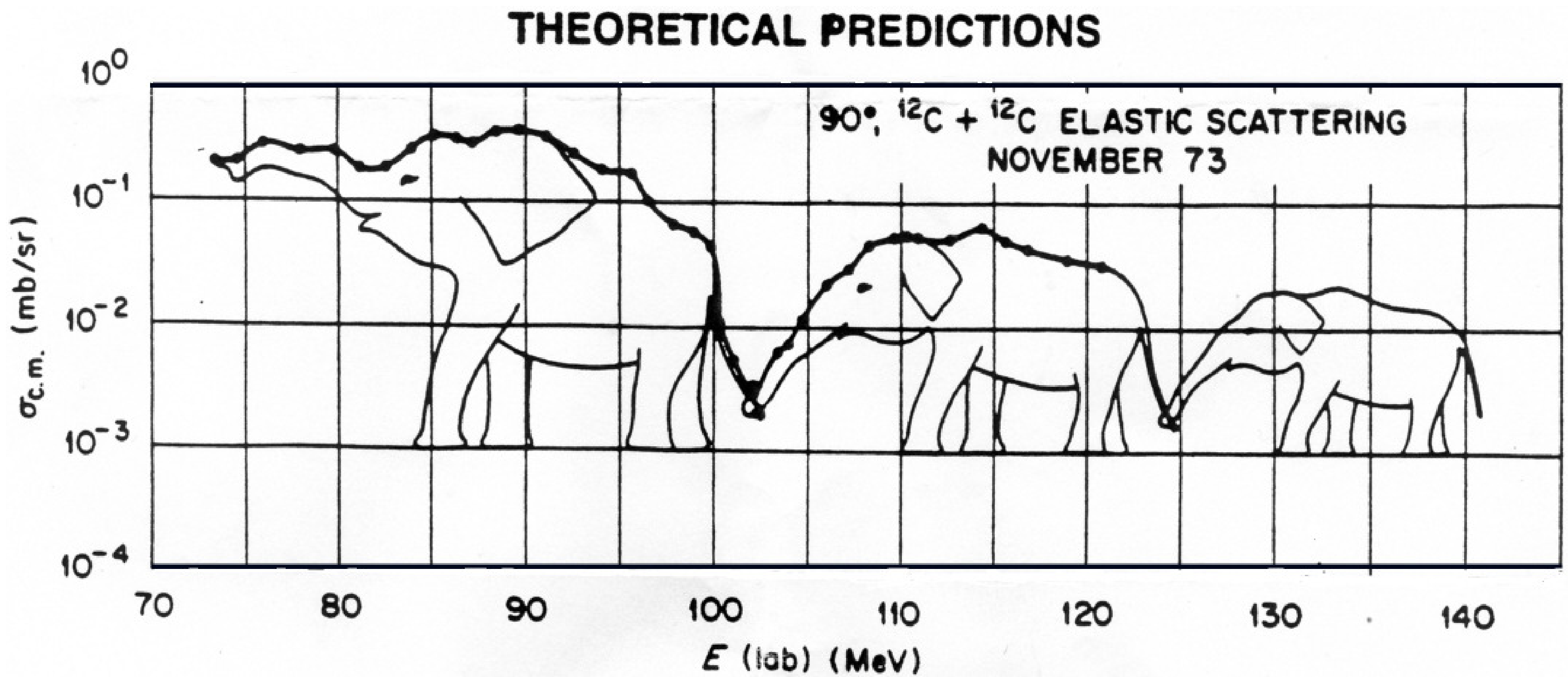
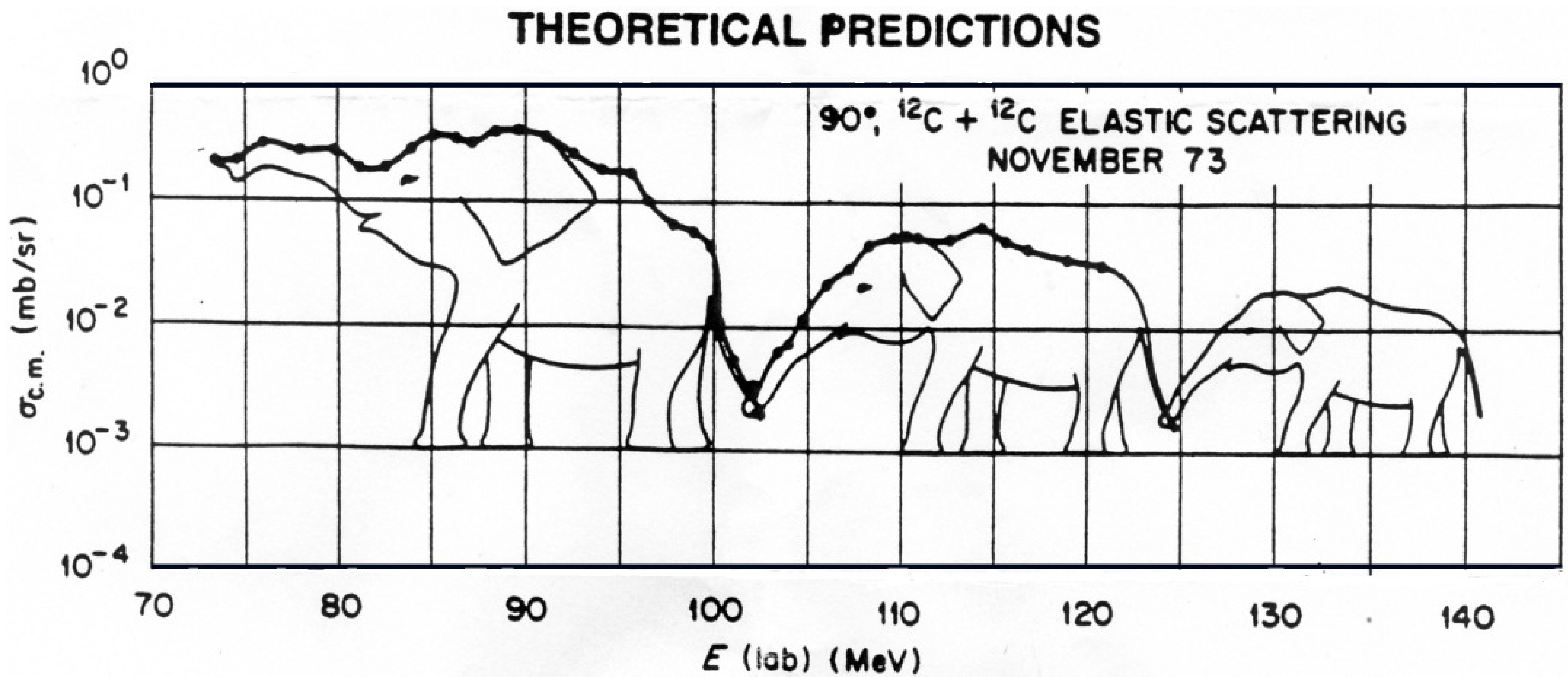


(S. Raman, *Science with a smile*)

Inferential-predictive process



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Even if the (*ad hoc*) model fits perfectly the data,
we do not believe the predictions
because we don't trust the model!

[Many 'good' models are *ad hoc* models!]

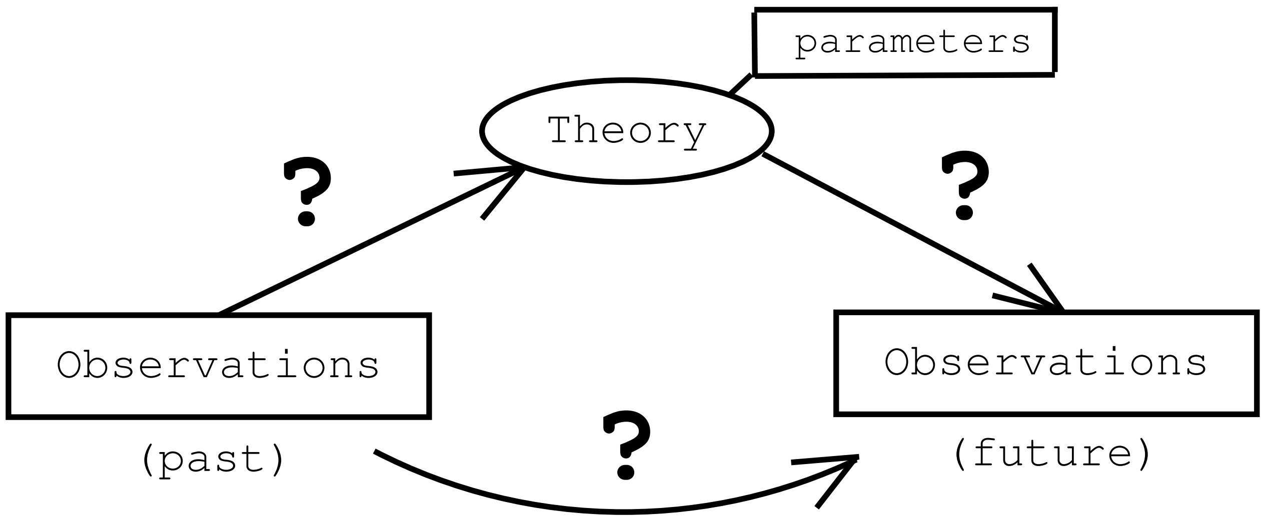
2011 IgNobel prize in Mathematics

- D. Martin of USA (who predicted the world would end in 1954)
- P. Robertson of USA (who predicted the world would end in 1982)
- E. Clare Prophet of the USA (who predicted the world would end in 1990)
- L.J. Rim of KOREA (who predicted the world would end in 1992)
- C. Mwerinde of UGANDA (who predicted the world would end in 1999)
- H. Camping of the USA (who predicted the world would end on September 6, 1994 and later predicted that the world will end on October 21, 2011)

2011 IgNobel prize in Mathematics

"For teaching the world to be careful when making mathematical assumptions and calculations"

Deep source of uncertainty



Uncertainty:

Theory	— ? $\longrightarrow$	Future observations
Past observations	— ? $\longrightarrow$	Theory
Theory	— ? $\longrightarrow$	Future observations

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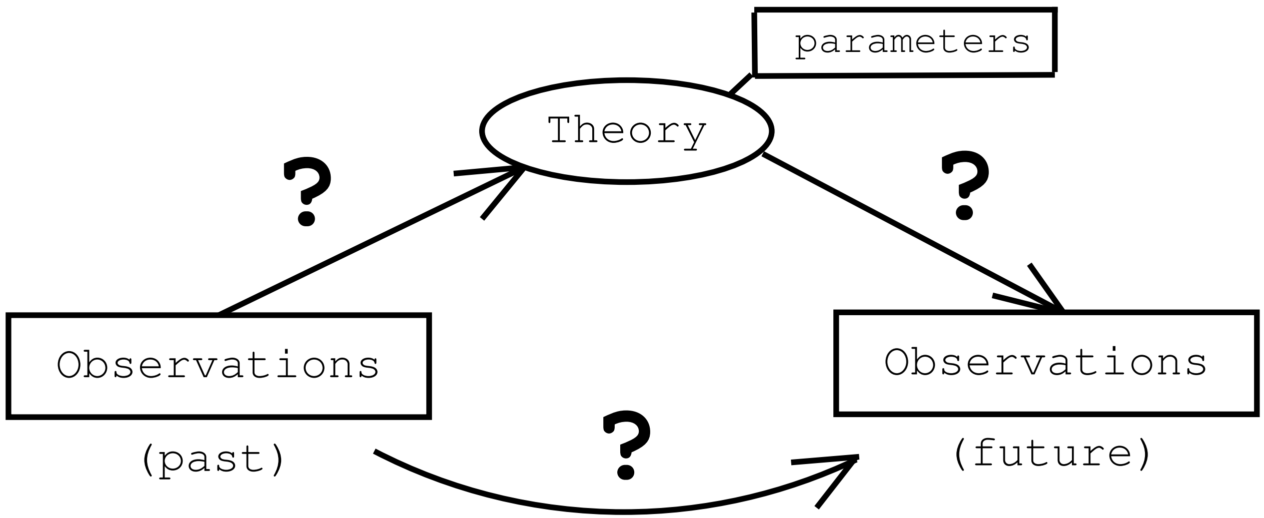

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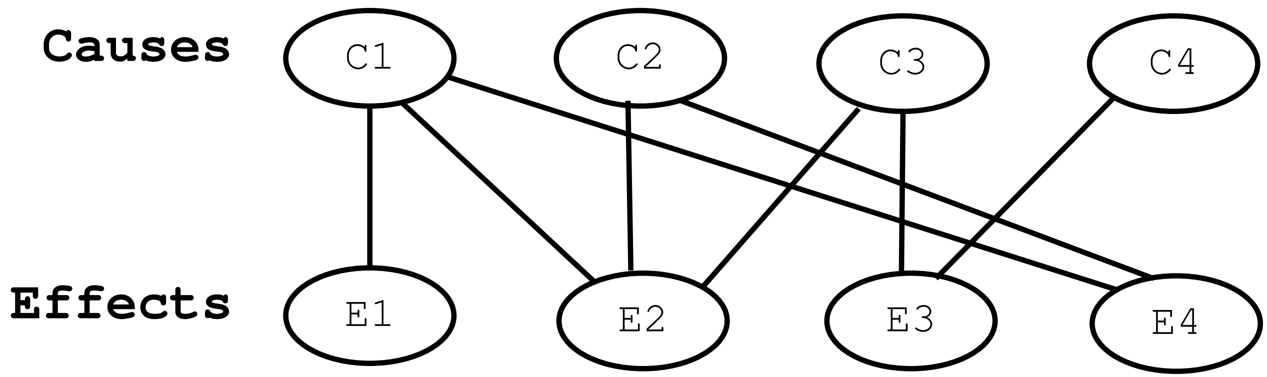
$\Longrightarrow$ **Uncertainty about causal connections**

CAUSE $\Longleftrightarrow$ **EFFECT**

Causes → effects

The same *apparent* cause might produce several,different effects



Given an observed effect, we are not sure about the exact cause that has produced it.

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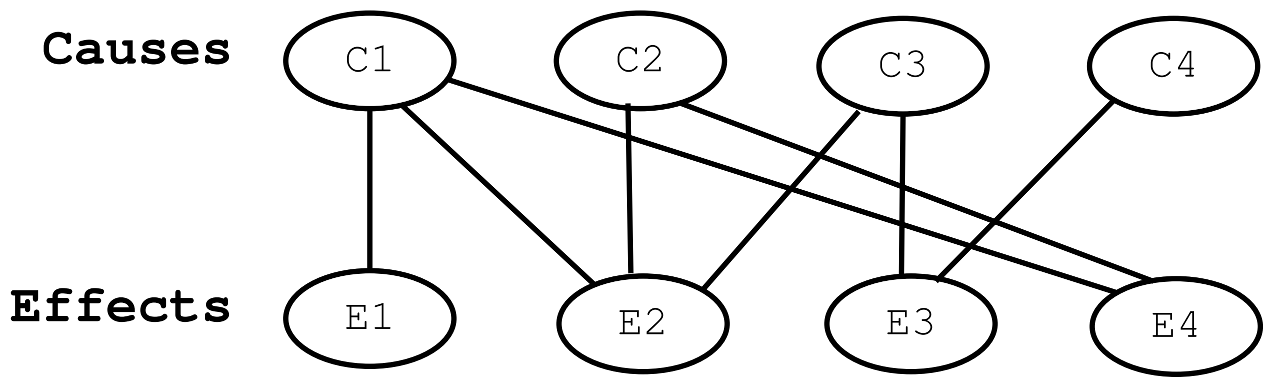

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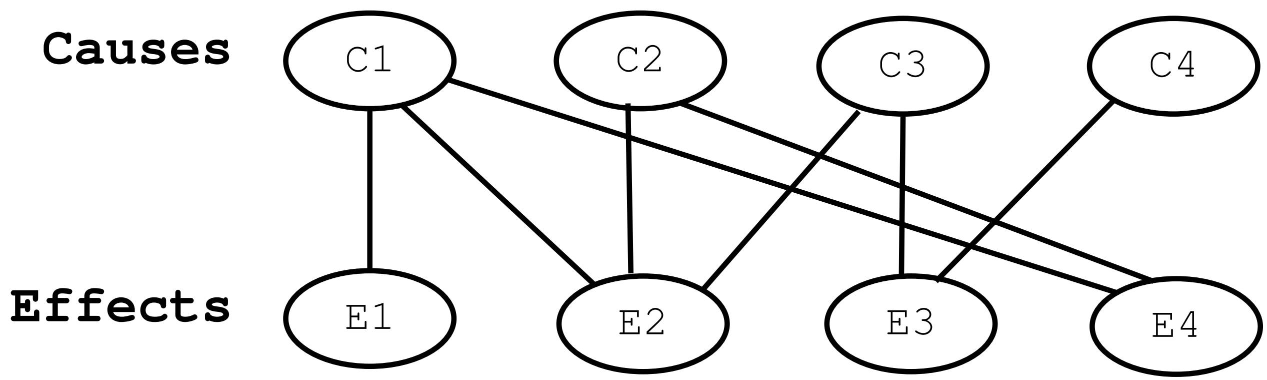

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$$\mathbf{E_2} \Rightarrow \{C_1,\, C_2,\, C_3\}?$$

The "essential problem" of the Sciences

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Why physics students are not taught how to tackle this kind of problems?

Uncertainty and probability

We, as physicists, consider absolutely natural and meaningful statements of the following kind

- $P(-10 < \epsilon'/\epsilon \times 10^4 < 50) >> P(\epsilon'/\epsilon \times 10^4 > 100)$
- $P(172 \le m_{top}/\text{GeV} \le 174) \approx 70\%$
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[The fact that for several people in this audience this criticism is misterious is a clear indication of the confusion concerning this matter]

Doing Science in conditions of uncertainty

The constant status of uncertainty does not prevent us from doing Science (in the sense of Natural Science and not just Mathematics)

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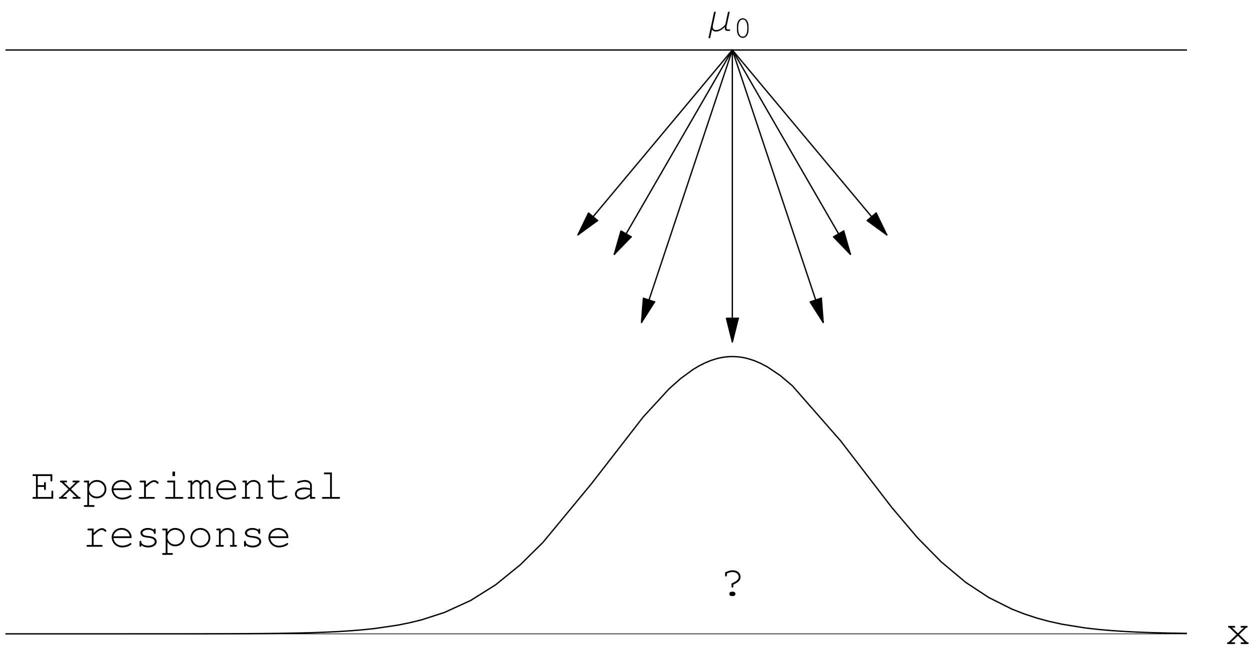
Indeed

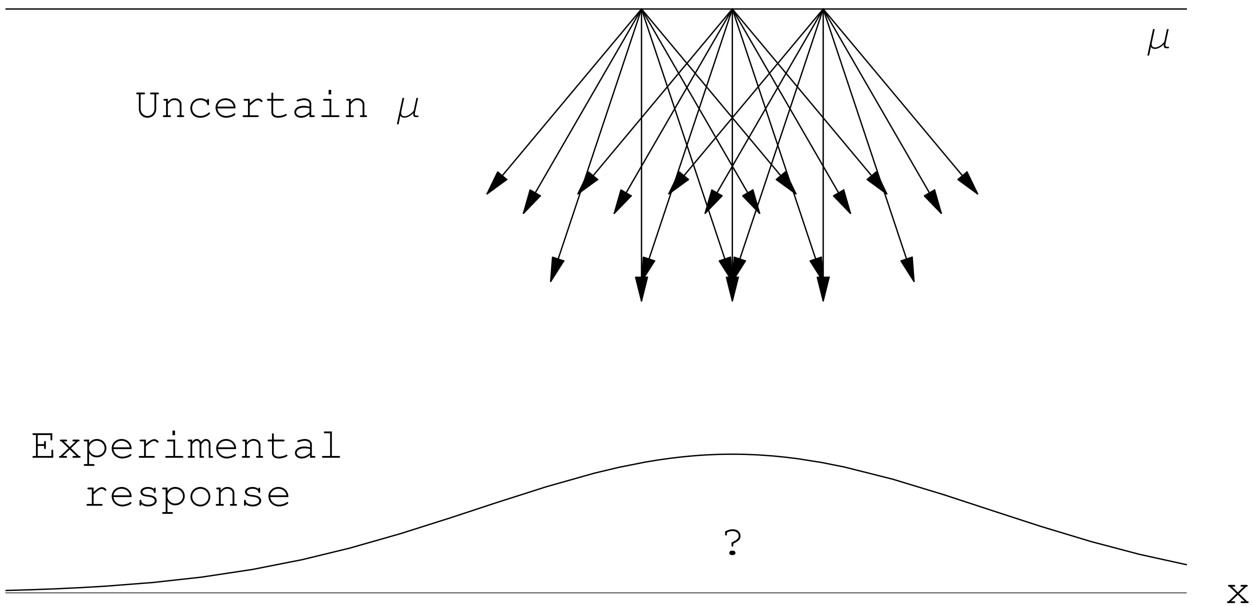
"It is scientific only to say what is more likely and what is less likely" (Feynman)

From 'true value' to observations



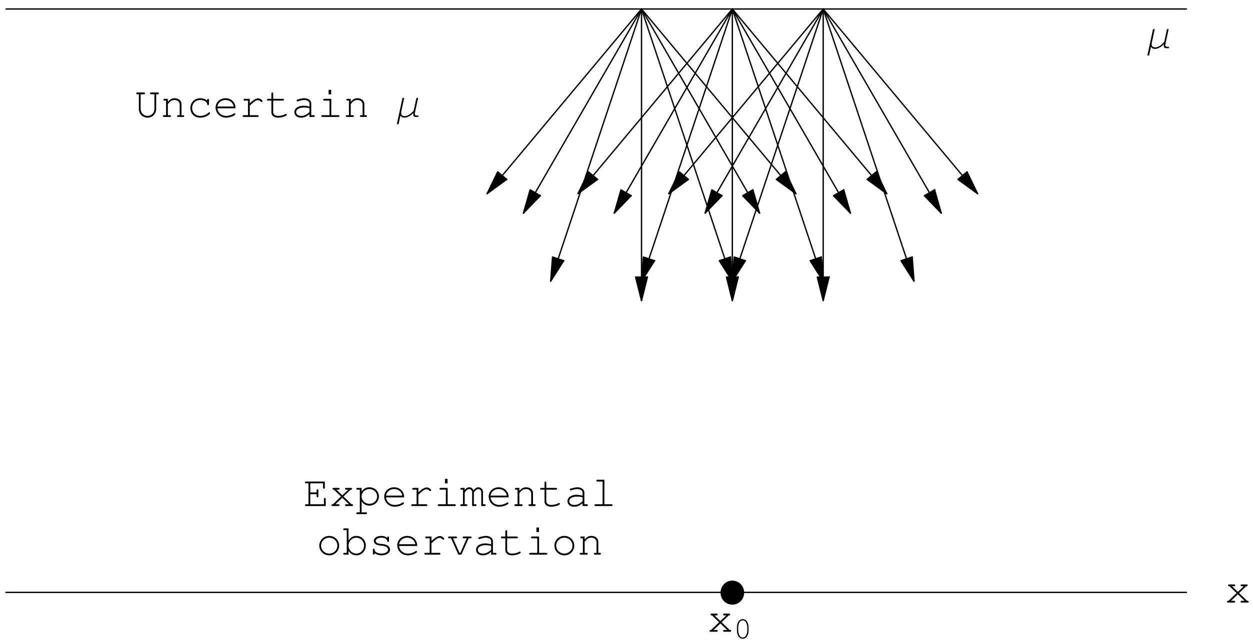
Given μ (exactly known) we are uncertain about x

From ‘true value’ to observations



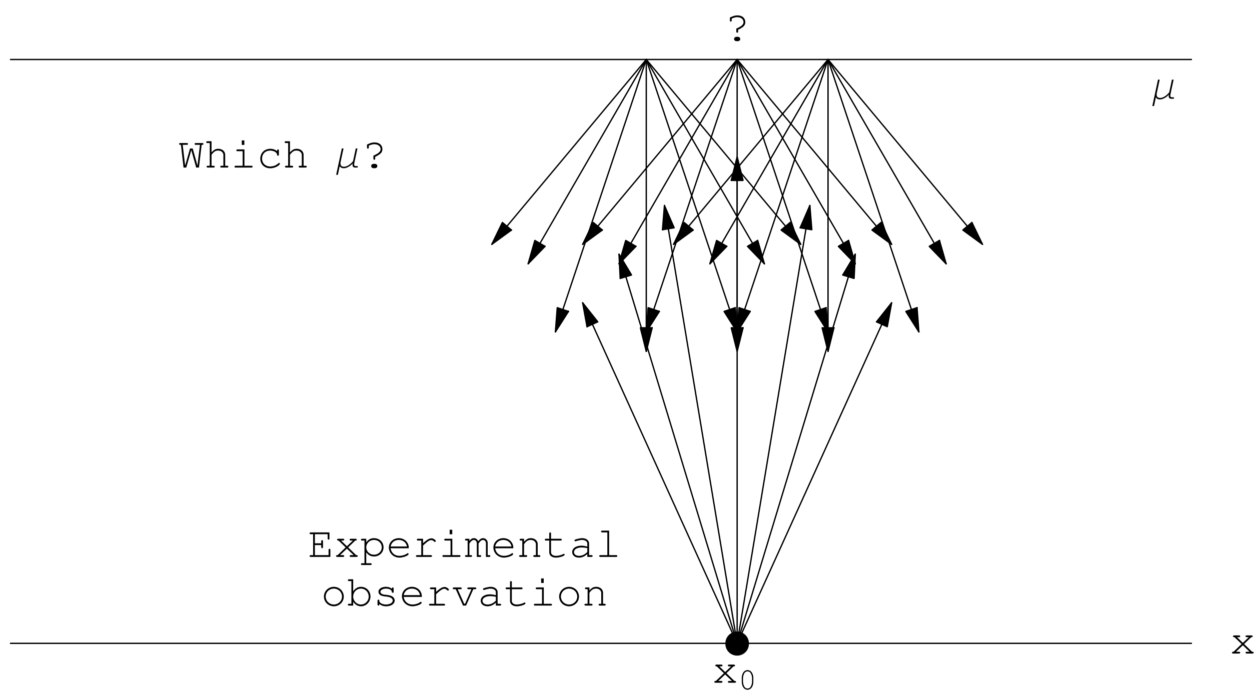
Uncertainty about μ makes us more uncertain about x

. . . and back: Inferring a true value



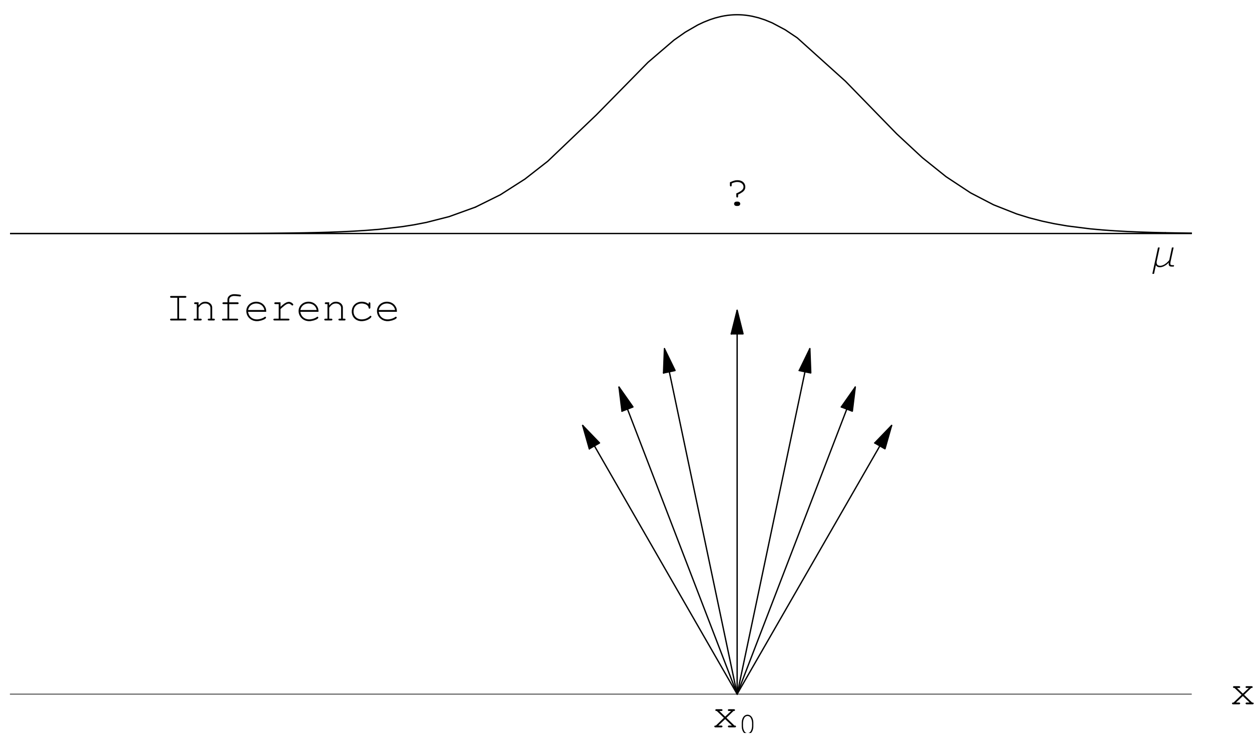
The observed data is certain: $\rightarrow$ ‘true value’ uncertain.

. . . and back: Inferring a true value



Where does the observed value of x comes from?

...and back: Inferring a true value



We are now uncertain about μ, given x.

A very simple experiment

Let’s make an experiment

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- Here
- Now

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For simplicity

- μ can assume only six possibilities:

$$\mathbf{0, 1, \ldots, 5}$$

- x is binary:

$$\mathbf{0, 1}$$

$$[\,(1, 2);\ \text{Black/White};\ \text{Yes/Not};\ \ldots\,]$$

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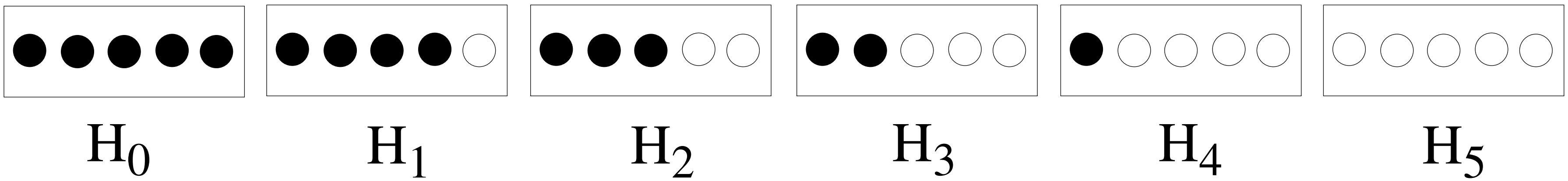
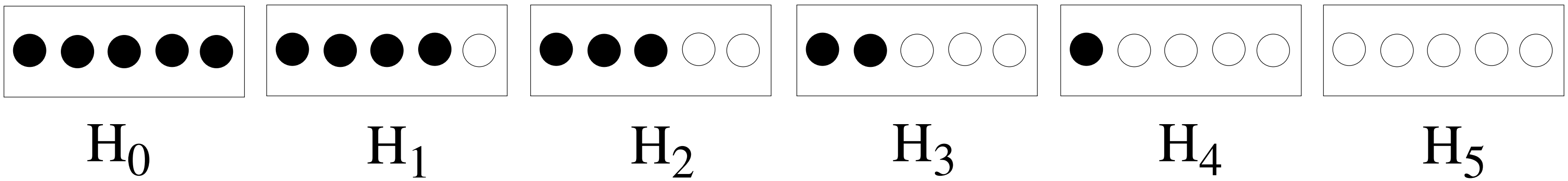
$\Rightarrow$ Later we shall make μ continous.

Which box? Which ball?



Let us take randomly one of the boxes.

Which box? Which ball?



Let us take randomly one of the boxes.

We are in a state of uncertainty concerning several *events*, the most important of which correspond to the following questions:

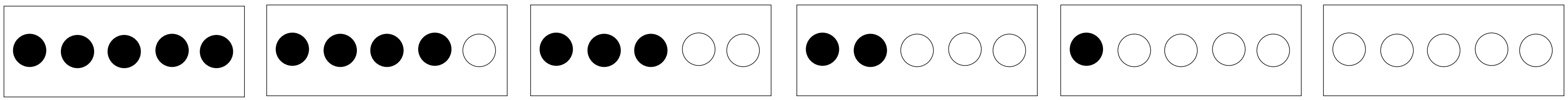
(a) Which box have we chosen, $H_0, H_1, \ldots, H_5$?

(b) If we extract randomly a ball from the chosen box, will we observe a white ($E_W \equiv E_1$) or black ($E_B \equiv E_2$) ball?

Our certainties:

$$\begin{aligned} \cup_{j=0}^{5} H_j &= \Omega \\ \cup_{i=1}^{2} E_i &= \Omega\,. \end{aligned}$$

Which box? Which ball?



H_0 H_1 H_2 H_3 H_4 H_5

Let us take randomly one of the boxes.

- What happens after we have extracted one ball and looked its color?
  - Intuitively feel *how to* roughly *change* our opinion about
    - the possible cause
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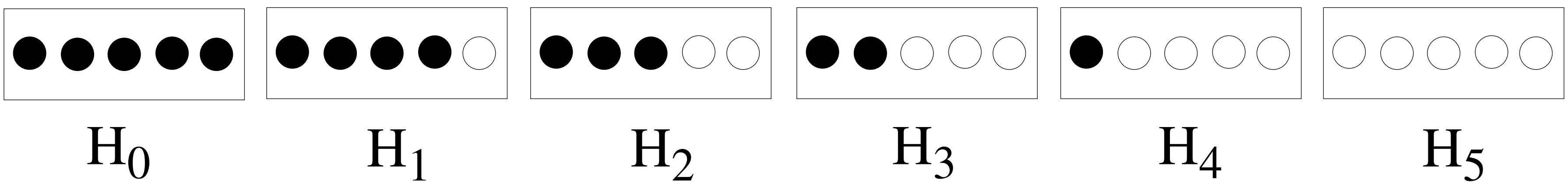

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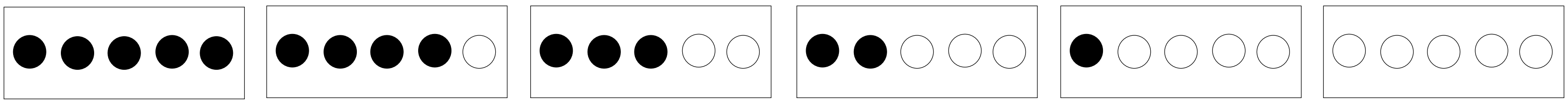

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  - Can we do it *quantitatively*, in an ‘objective way’?
- And after a sequence of extractions?

The toy inferential experiment

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This toy experiment is conceptually very close to what we do in Physics

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The rule of the game is that we are not allowed to watch inside the box! (As we cannot whatch a real movie showing the beginning of the Universe and compare it with our speculations.)

Where <u>is</u> probability?

We all agree that the experimental results change

- the probabilities of the box compositions;
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Where is the probability?

Certainly not *in* the box!

Subjective nature of probability

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Probability depends on the status of information of the *subject* who evaluates it.

Probability is always conditional probability

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$\Rightarrow$ Three box game

(Box with white ball wins)

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$\Rightarrow$ How much we believe something

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$\rightarrow$ 'Degree of belief' $\leftarrow$

Beliefs and 'coherent' bets

Remarks:

- Subjective does not mean arbitrary!

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$\rightarrow P(3477 \leq M_{Sun}/M_{Sat} \leq 3547 \,|\, I(\text{Laplace})) = 99.99\%$

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- It does not imply one has to be 95% confident on something!
- If you do so you are going to make a bad bet!

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For more on the subject
see `http://arxiv.org/abs/1112.3620`
and references therein.

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  They all convey unambiguously the same confidence on something.

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- You might agree or disagree, but at least You know what this person has in his mind. (NOT TRUE with "C.L.'s"!)
- If a person has these beliefs and he/she has the chance to win a rich prize bound to one of these events, he/she is indifferent to the choice.

Unifying role of subjective probability

- Wide range of applicability
- Probability statements all have the same meaning no matter to what they refer and how the number has been evaluated.
  - $P(\text{Rain next Saturday in Paris}) = 68\%$
  - $P(\text{Germany wins European soccer cup}) = 68\%$
  - $P(M_H \leq 130\,\text{GeV}) = 68\%$
  - $P(\text{Free neutron decays before 17 s}) = 68\%$
  - $P(\text{White ball from a box with 68W+32B}) = 68\%$

We can talk very naturally about probabilities of true values!

Probability Vs “probability”...

Errors on ratios of small numbers of events
F. James$^{(*)}$ and M. Roos
Nucl. Phys. **B172** (1980) 475
(`http://ccdb4fs.kek.jp/cgi-bin/img_index?8101205`)

When the result of the measurement of a physical quantity is published as $R=R_0\pm\sigma_0$ without further explanation, it is implied that R is a Gaussian-distributed measurement with mean R_0 and variance σ_0^2. This allows one to calculate various confidence intervals of given "probability", i.e. the "probability" P that the true value of R is within a given interval. P is given by the area under the corresponding part of the Gaussian curve, and is the basis of well-known rules-of-thumb such as "the probability of exceeding two standard deviations is 5%".

$^{(*)}$ Influential CERN 'frequentistic guru' of HEP community

Mathematics of beliefs

The good news:

The basic laws of degrees of belief are the same we get from the inventory of favorable and possible cases, or from events occurred in the past.

[Details skipped. . .]

Basic rules of probability

1. $0 \leq P(A \,|\, I) \leq 1$
2. $P(\Omega \,|\, I) = 1$
3. $P(A \cup B \,|\, I) = P(A \,|\, I) + P(B \,|\, I) \quad [\text{if } P(A \cap B \,|\, I) = \emptyset]$

4. $P(A \cap B \,|\, I) = P(A \,|\, B, I) \cdot P(B \,|\, I) = P(B \,|\, A, I) \cdot P(A \,|\, I)$

Remember that probability is always conditional probability!

I is the background condition (related to information 'I'_s')

$\rightarrow$ usually implicit (we only care of 're-conditioning')

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Remember that probability is always conditional probability!

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Note: 4. does not define conditional probability.
(Probability is always conditional probability!)

Mathematics of beliefs

An even better news:

The fourth basic rule
can be fully exploided!

Mathematics of beliefs

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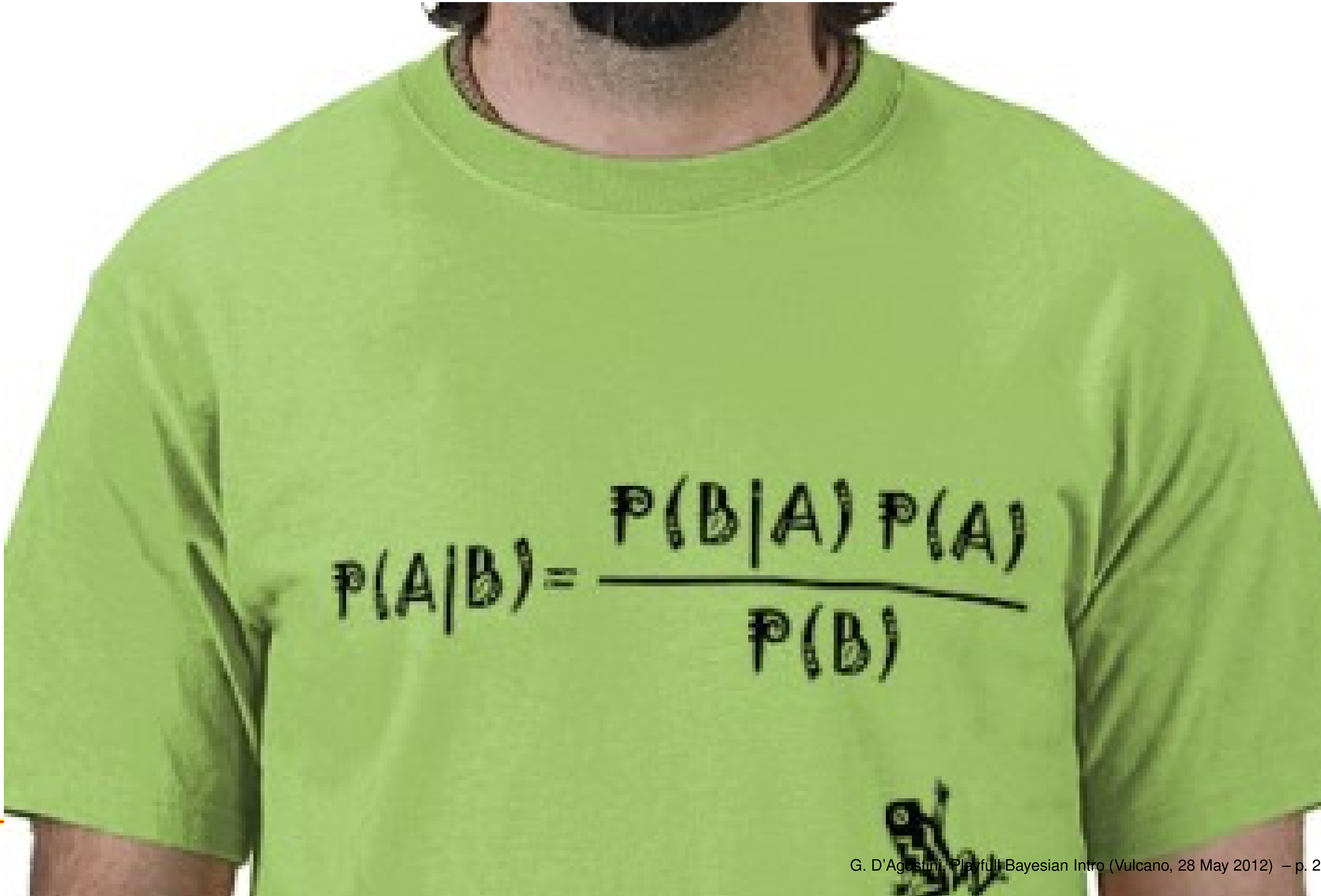
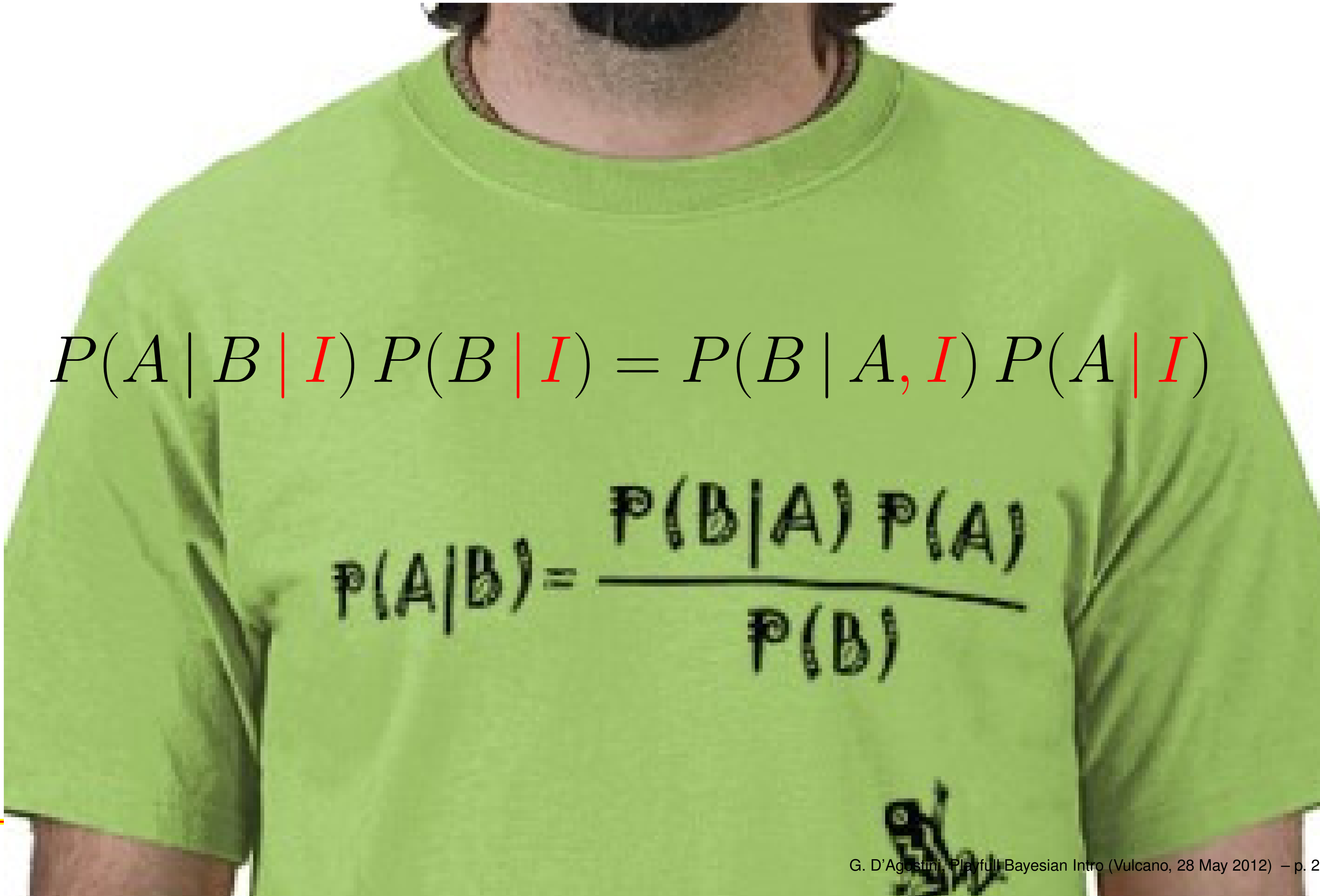
(Liberated by a curious ideology that forbits its use)

A simple, powerful formula

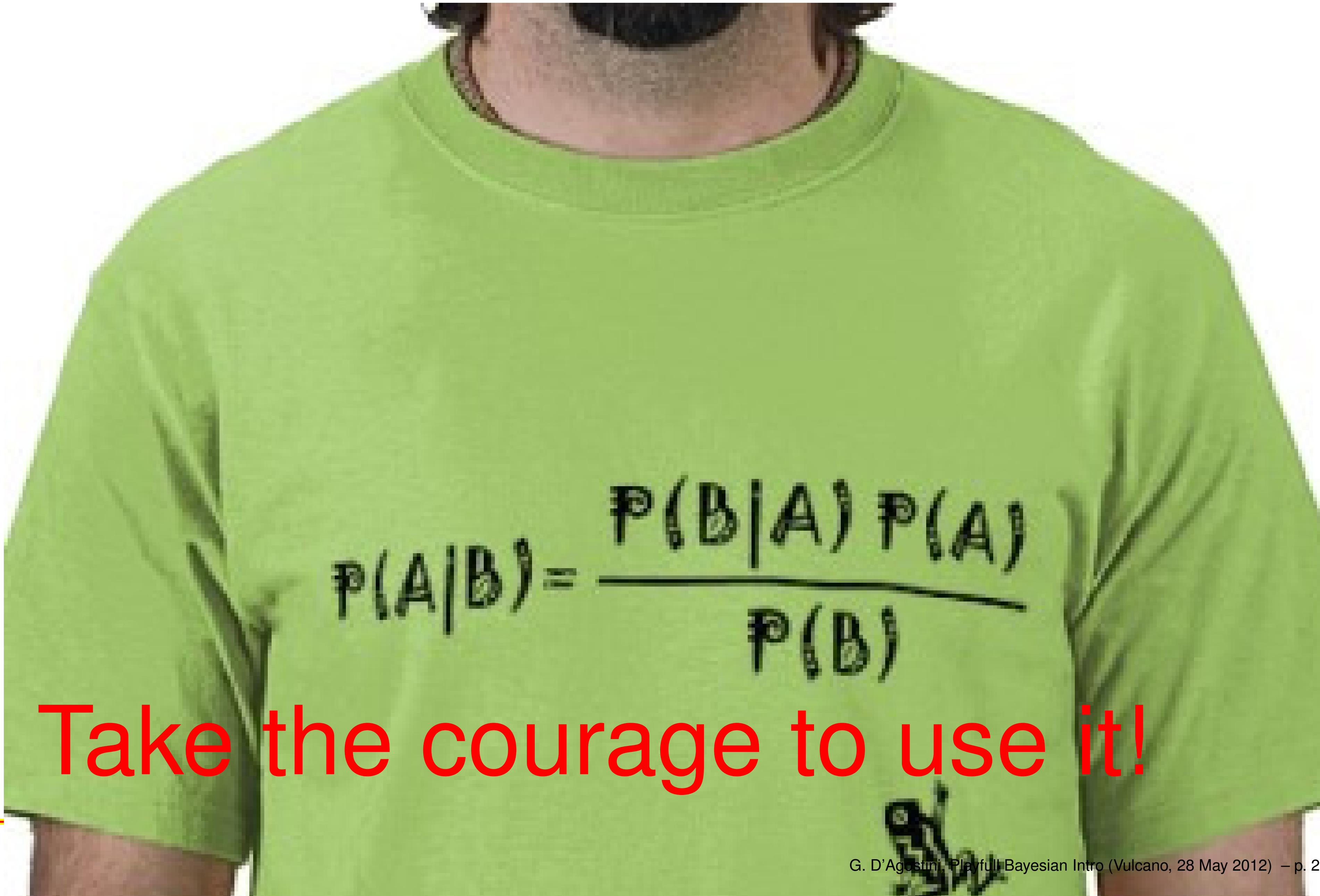


A simple, powerful formula



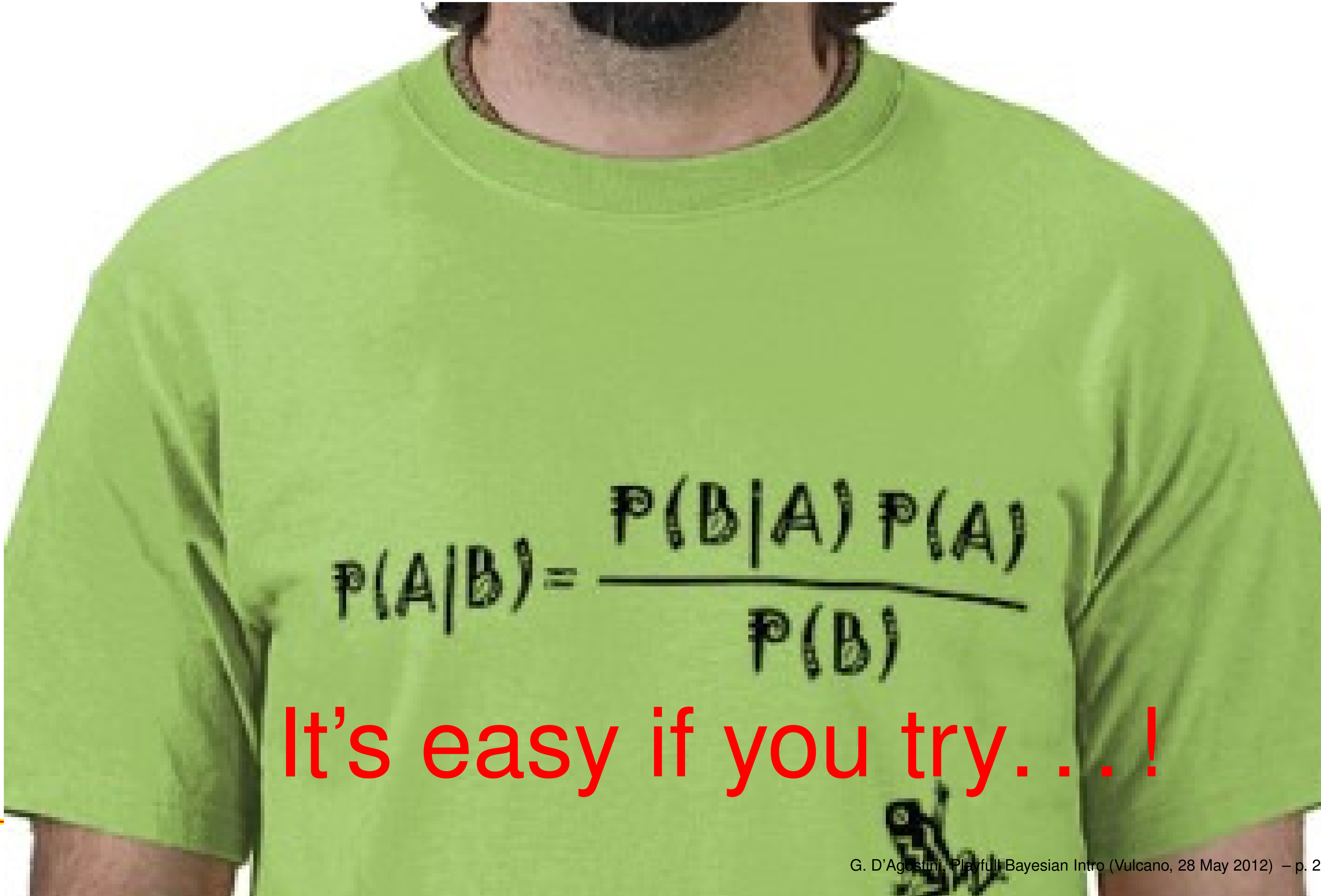
$$P(A \,|\, B \,|\, I)\, P(B \,|\, I) = P(B \,|\, A, I)\, P(A \,|\, I)$$

A simple, powerful formula



Take the courage to use it!

A simple, powerful formula



It's easy if you try. . . !

Laplace's "Bayes Theorem"

"The greater the probability of an observed event given any one of a number of causes to which that event may be attributed, the greater the likelihood of that cause {given that event}.

$$P(C_i \,|\, E) \propto P(E \,|\, C_i)$$

Laplace's "Bayes Theorem"

"The greater the probability of an observed event given any one of a number of causes to which that event may be attributed, the greater the likelihood of that cause {given that event}. The probability of the existence of any one of these causes {given the event} is thus a fraction whose numerator is the probability of the event given the cause, and whose denominator is the sum of similar probabilities, summed over all causes.

$$P(C_i \,|\, E) = \frac{P(E \,|\, C_i)}{\sum_j P(E \,|\, C_j)}$$

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$$P(C_i \,|\, E) = \frac{P(E \,|\, C_i)\, P(C_i)}{\sum_j P(E \,|\, C_j)\, P(C_j)}$$

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"This is the fundamental principle (*) of that branch of the analysis of chance that consists of reasoning *a posteriori* from events to causes"

(*) In his "Philosophical essay" Laplace calls 'principles' the 'fondamental rules'.

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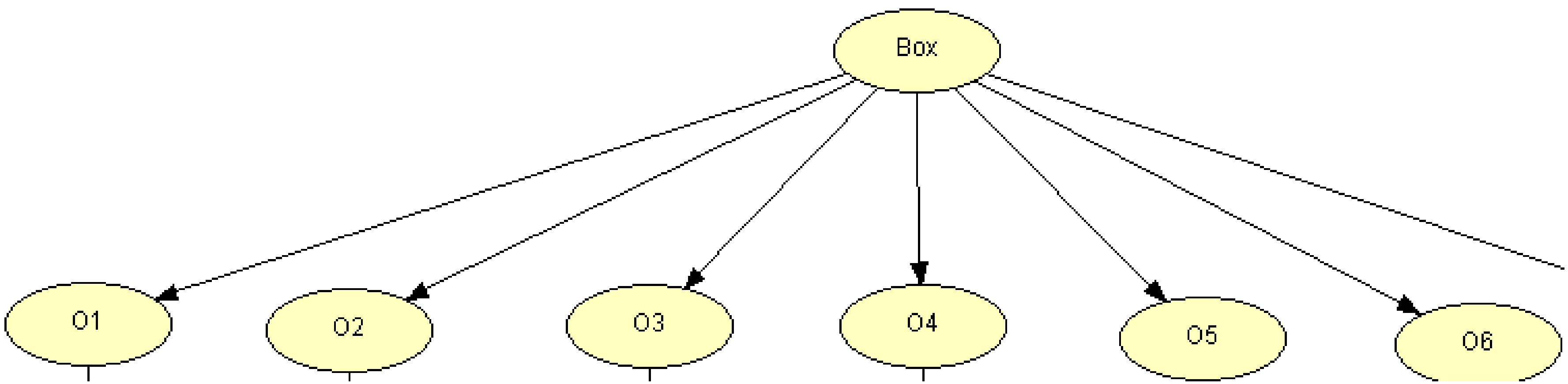
Note: denominator is just a normalization factor.

$$\Rightarrow \qquad P(C_i \,|\, E) \;\propto\; P(E \,|\, C_i)\, P(C_i)$$

Most convenient way to remember Bayes theorem

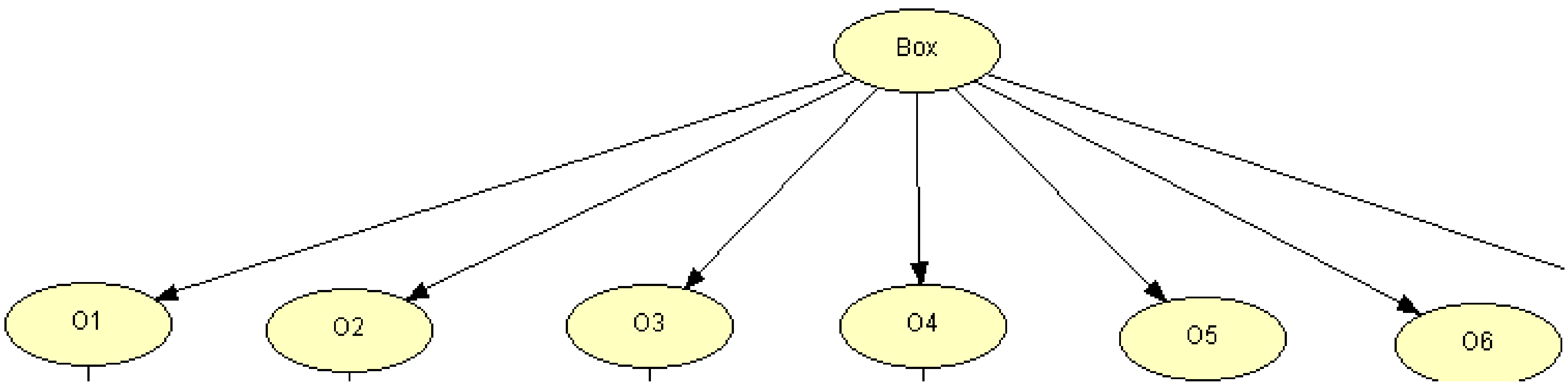
Cause-effect representation

box content → observed color



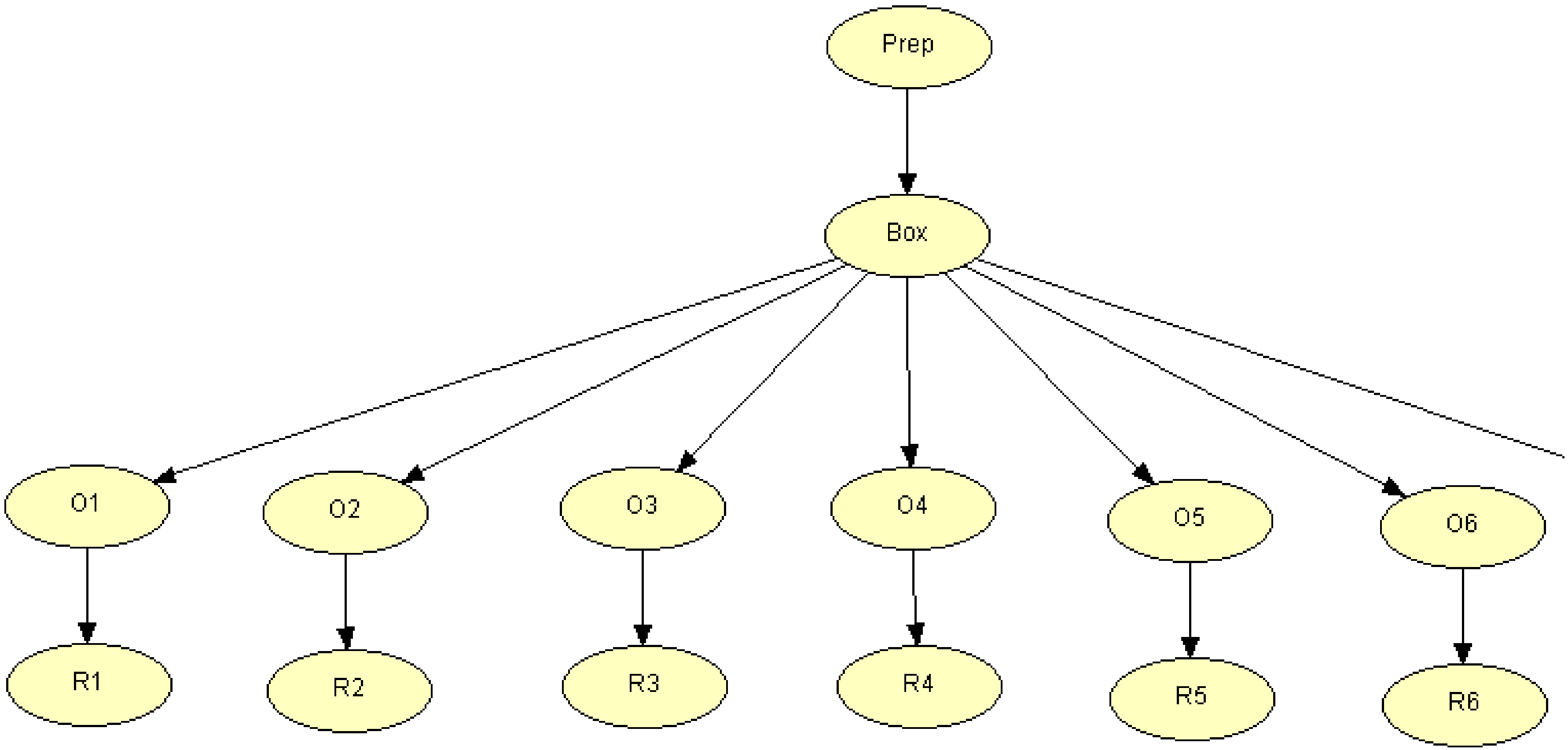
Cause-effect representation

box content $\rightarrow$ observed color



An effect might be the cause of another effect $\Longrightarrow$

A network of causes and effects



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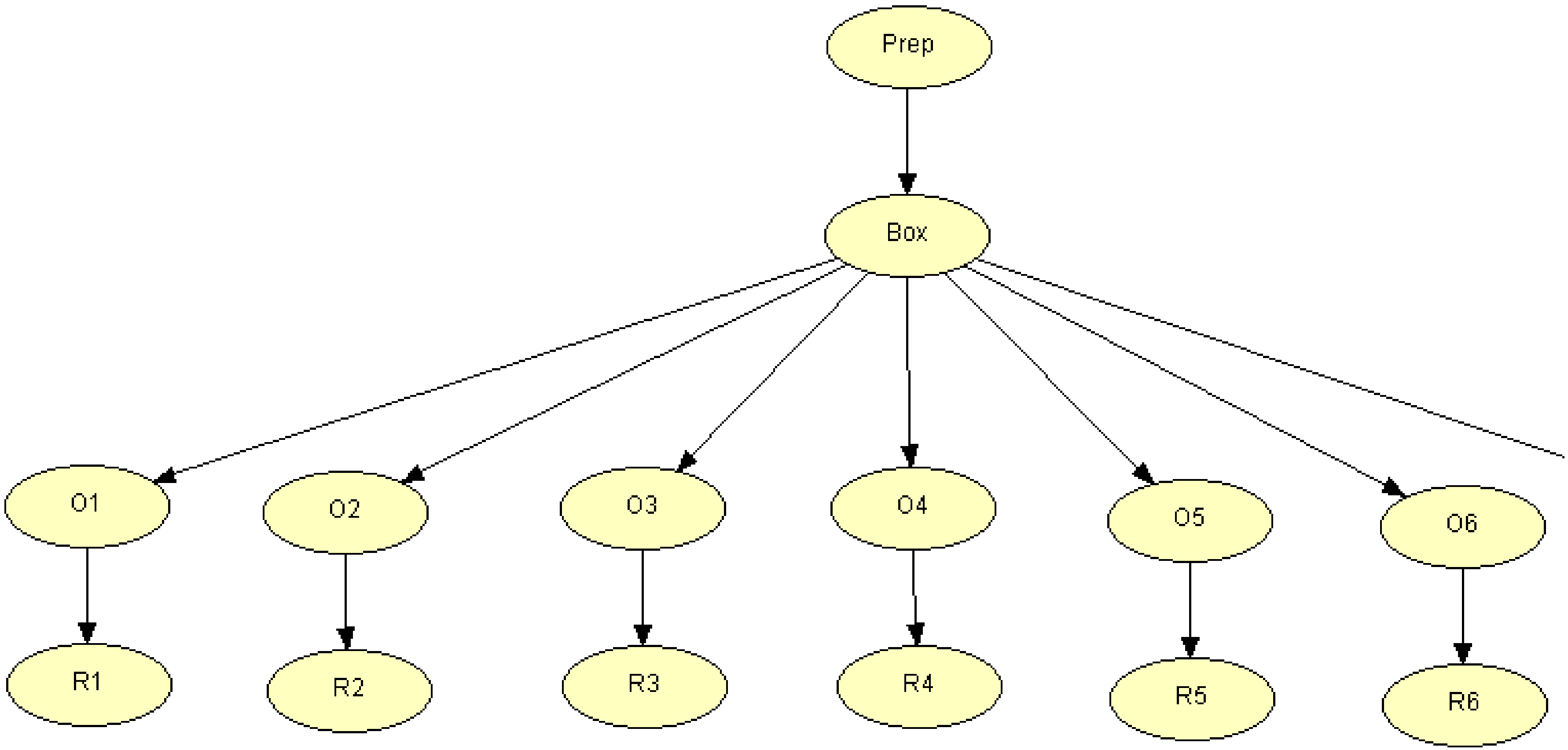

and so on. . . $\Rightarrow$ Physics applications

Inferring ‘proportions’

Let’s turn the toy experiment to a ‘serious’ physics case:

- Inferring H_j is the same as inferring the proportion of white balls:

$$H_j \quad \longleftrightarrow \quad j \quad \longleftrightarrow \quad p = \frac{j}{5}$$

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$$n: \quad 6 \rightarrow \infty$$

$\Rightarrow p$ continous in $[0, 1]$

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- Increase the number of balls

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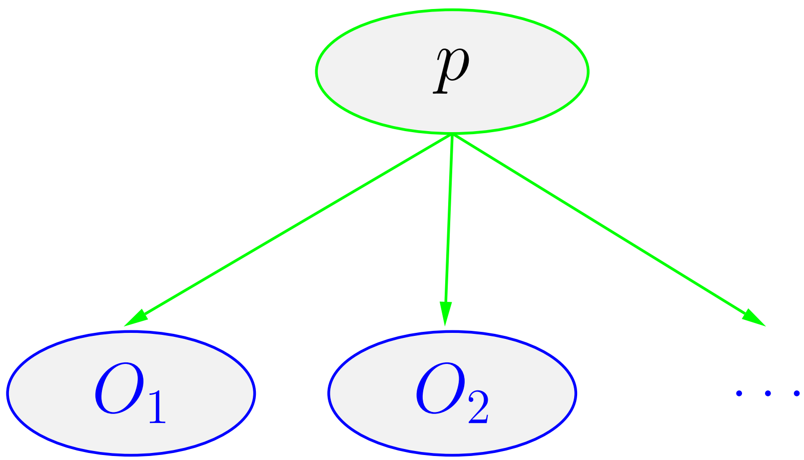
$\Rightarrow$ p continous in $[0, 1]$

- Generalize White/Black $\longrightarrow$ Success/Failure

$\Rightarrow$ efficiencies, branching ratios, ...

Inferring Bernoulli's trial parameter p

Making several independent trials *assuming* the same p



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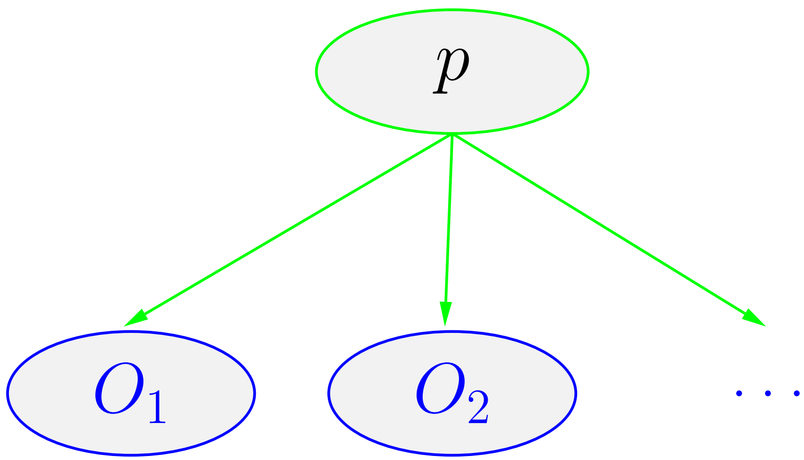

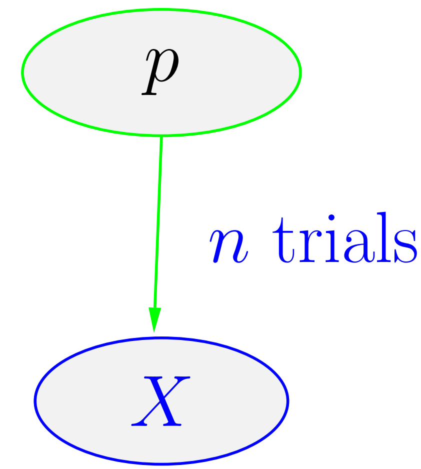

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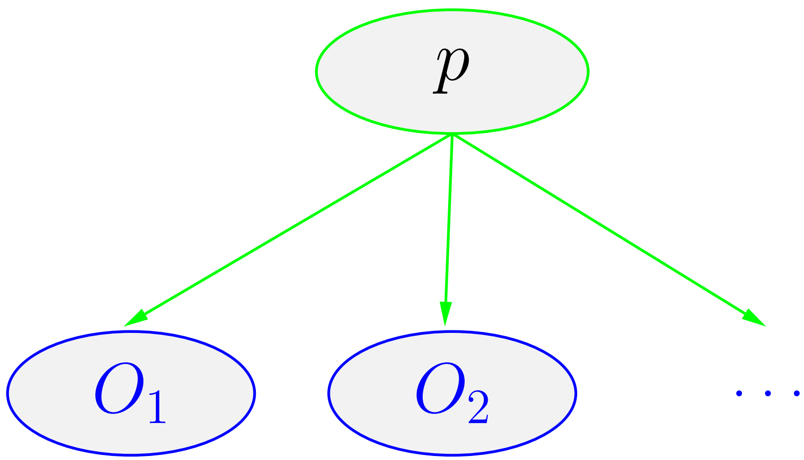
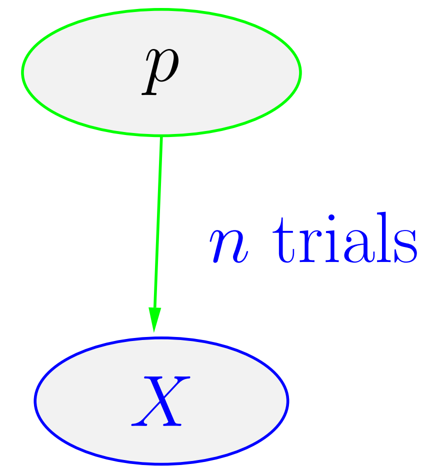


"independent Bernoulli trials"



"binomial distribution"

Inferring Bernoulli's trial parameter p

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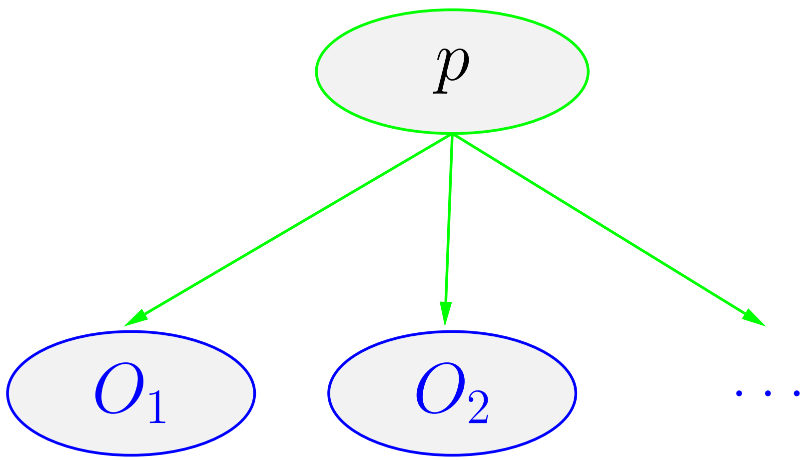

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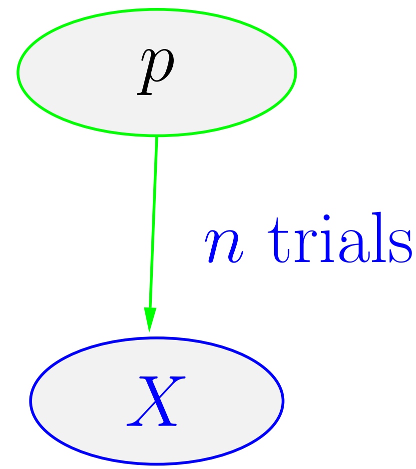
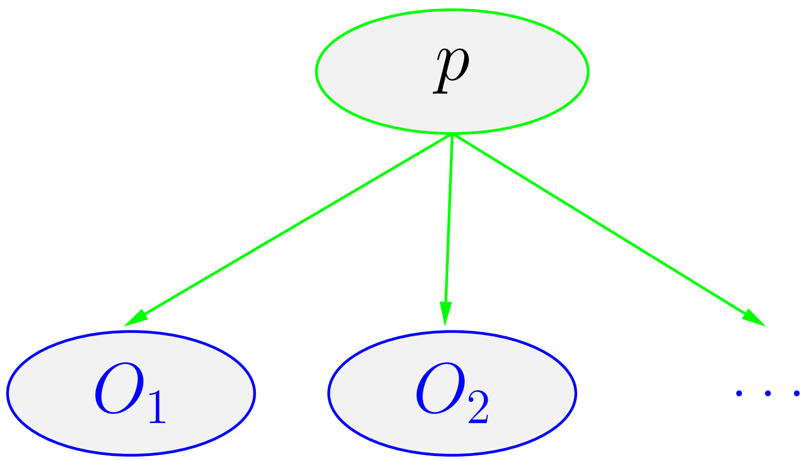


"binomial distribution"

$\Rightarrow$ In the light of the experimental information there will be values of p we shall believe more, and others we shall believe less.

Inferring Bernoulli's trial parameter p

Making several independent trials *assuming* the same p



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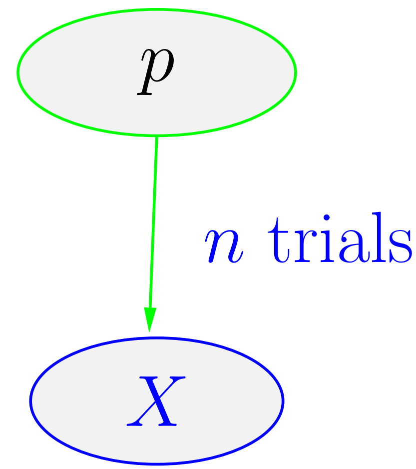

"binomial distribution"

$$P(p_i \,|\, O_1, O_2, \ldots)$$
$$f(p \,|\, O_1, O_2, \ldots)$$

$$P(p_i \,|\, X, n)$$
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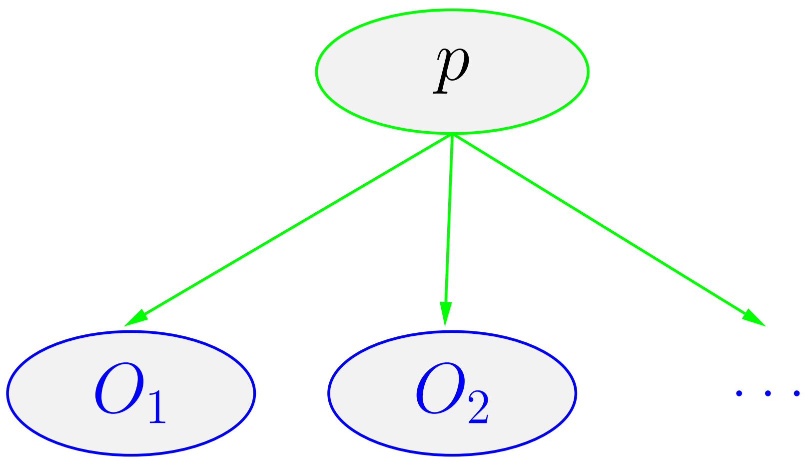
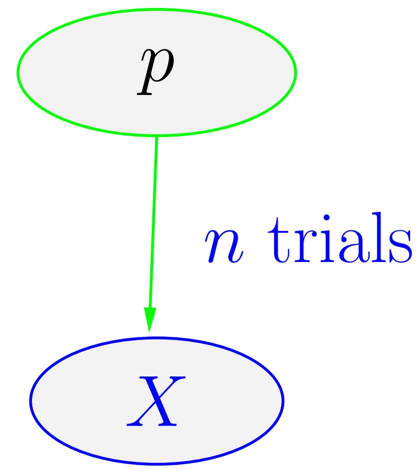


"independent Bernoulli trials"

$$P(p_i \,|\, O_1, O_2, \ldots)$$
$$f(p \,|\, O_1, O_2, \ldots)$$

$$\propto f(O_1, O_2, \ldots \,|\, p) \cdot f_0(p)$$



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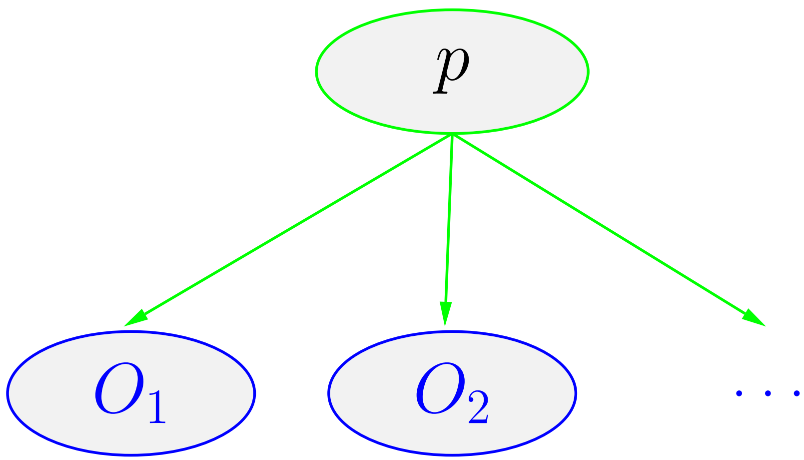

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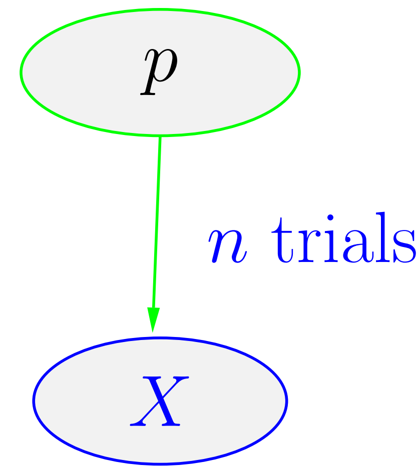


"binomial distribution"

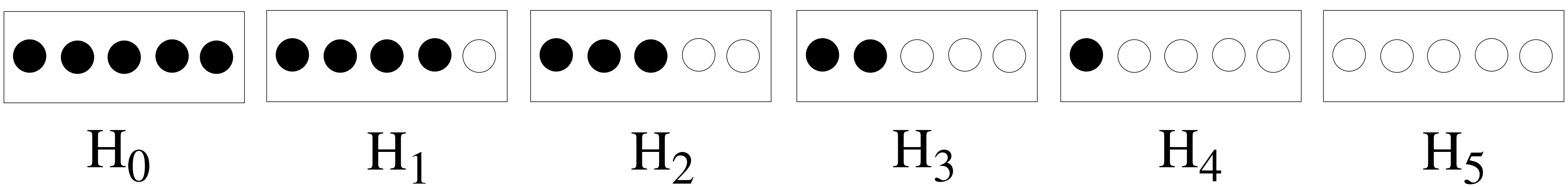
$$P(p_i \,|\, O_1, O_2, \ldots) \qquad\qquad P(p_i \,|\, X, n)$$

$$f(p \,|\, O_1, O_2, \ldots) \qquad\qquad f(p \,|\, X, n)$$

Are the two inferences the same?

(not obvious in principle)

Application to the six box problem



Remind:

- E_1 = White
- E_2 = Black

Collecting the pieces of information we need

Our tool:

$$\boxed{P(H_j \,|\, E_i, I) = \frac{P(E_i \,|\, H_j, I)}{P(E_i \,|\, I)}\, P(H_j \,|\, I)}$$

Collecting the pieces of information we need

Our tool:

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- $P(H_j \,|\, I) = 1/6$

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- $P(E_i \,|\, I) = 1/2$
- $P(E_i \,|\, H_j, \, I)$:

$$\begin{aligned} P(E_1 \,|\, H_j, \, I) &= j/5 \\ P(E_2 \,|\, H_j, \, I) &= (5-j)/5 \end{aligned}$$

Collecting the pieces of information we need

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Our prior belief about H_j

Collecting the pieces of information we need

Our tool:

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Probability of E_i under a well defined hypothesis H_j
It corresponds to the 'response of the apparatus in measurements.
→ likelihood (traditional, rather confusing name!)

Collecting the pieces of information we need

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$$P(H_j \,|\, E_i, I) = \frac{P(E_i \,|\, H_j, I)}{P(E_i \,|\, I)}\, P(H_j \,|\, I)$$

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Probability of E_i taking account all possible H_j
$\rightarrow$ How much we are confident that E_i will occur.

Collecting the pieces of information we need

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Probability of E_i taking account all possible H_j
$\rightarrow$ How much we are confident that E_i will occur.
We can rewrite it as
$P(E_i \,|\, I) = \sum_j P(E_i \,|\, H_j, I) \cdot P(H_j \,|\, I)$

We are ready

Now that we have set up our formalism, let's play a little

- analyse real data
- some simulations

Then

- $H_j \longleftrightarrow j \longleftrightarrow p_j$
- extending p to a continuum:

  $\Rightarrow$ Bayes' billiard

  (prototype for all questions related to efficiencies, branching ratios)

Bayes' billiard

This is the original problem in the theory of chances solved by Thomas Bayes in late '700:

- imagine you roll a ball at random on a billiard;
- you mark the relative position of the ball along the billiard's length (l/L) and remove the ball
- then you roll at random other balls
  - write down if it stopped left or right of the first ball;
  - remove it and go on with n balls.
- Somebody has to guess the position of the first ball knowing only how mane balls stopped left and how many stoppe right

Bayes' billiard and Bernoulli trials

It is easy to recongnize the analogy:

- Left/Right $\rightarrow$ Success/Failure
- if Left $\leftrightarrow$ Success, then
  - $l/L \leftrightarrow p$ of binomial (Bernoulli trials)

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Imagine a sequence $\{S, S, F, S, \ldots\}$ [f_0 is uniform]:

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f(p\,|\,S, S, F) &\propto f(F\,|\,p)\cdot f(p\,|\,S, S) = p^2(1-p)
\end{aligned}
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\ldots & \quad\ \ldots \\
f(p\,|\,\#S,\#F) &\propto p^{\#S}(1-p)^{\#F} = p^{\#S}(1-p)^{(1-\#s)}
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$$

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\ldots & \qquad \ldots \\
f(p\,|\,\#S,\#F) &\propto p^{\#S}(1-p)^{\#F} = p^{\#S}(1-p)^{(1-\#s)} \\
f(p\,|\,x,n) &\propto p^{x}(1-p)^{(n-x)} \qquad\qquad [x=\#S]
\end{aligned}
$$

Conclusions

- The probabilistic framework basically set up by Laplace in his monumental work is healthy and grows up well (browse e.g. Amazon.com)

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- It is very close to the natural way of reasoning of phycisists (as well as of everybody else).
- Its consistent application in small-complex problems was prohibitive many years ago.

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- Mistrust all results that sound as ‘confidence’, ’probability’ etc about physics quantities, if they are obtained by methods that do not contemplate ’beliefs’.