Inflation and the Theory of Cosmological Perturbations

Exercises

First Session

1. **Friedmann equations**

   Using the FLRW metric and Einstein equations, find the Friedmann equations.

2. **Matter-Radiation Transition**

   Calculate the epoch of matter-radiation equality as a function of redshift.

3. **Flatness Problem**

   Consider an FLRW model dominated by a perfect fluid with equation of state $P = w\rho$, for $w = \text{const}$. Show that

   $$\frac{\mathrm{d}\Omega}{\mathrm{d}N} = (1 + 3w)\Omega(\Omega - 1)$$

   and discuss the stability of the solution $\Omega = 1$.

4. **Horizon Problem**

   The (comoving) particle horizon at the time t equals the amount of conformal time τ between $t_i = 0$ and t. In a flat universe filled with matter, radiation and dark energy, we have

   $$H^2 = H_0^2 \left[\Omega_r a^{-4} + \Omega_m a^{-3}\right] \tag{0.1}$$

   where H_0 is the Hubble constant and Ω_i are the densities today (in units of the critical density) ignoring the contribution from dark energy.

   Show that

   $$\tau = \frac{2}{\sqrt{\Omega_m H_0^2}} \left[\sqrt{a + a_{\mathrm{eq}}} - \sqrt{a_{\mathrm{eq}}}\right] \tag{0.2}$$

   What is the conformal time today? What is it at recombination? You may leave your answer in terms of the Hubble constant.

   Calculate the angle subtended by the horizon at recombination.

5. **E-folds and reheating temperature**

   Using entropy conservation and assuming instantaneous reheating at temperature $T_{\rm rh}$, show that we need a number of e-folds of

$$N = 53 + \log\left[\frac{T_{\rm rh}}{10^{10}\text{GeV}}\right] + \frac{1}{2}\log|\Omega_i - 1| \tag{0.3}$$

   to solve the flatness problem.

6. **Photons in deSitter**

   Show that photons are not produced during inflation.

Second Session

1. **Ultra Slow-Roll**

   Discuss a phase of flattening in the inflation potential (ultra-slow roll): compute slow-roll parameters during this phase and show that the power spectrum of curvature perturbations is enhanced.

2. **Power-law Inflation**

   Consider a model of inflation with potential $V(\phi) = \alpha\phi^n$. Compute slow-roll parameters for these models and the number of e-folds N_* corresponding to the time passing between the moment a given mode exits the horizon and the end of inflation. Then, compute the field excursion as a function of N_* and n.

3. **Power-law Inflation II**

   Using the power-law model from the previous exercise, compute the spectral index n_s and the tensor-to-scalar ratio r as a function of N_* and n. For values of $50 < N_* < 60$, check for which values of n the power-law models are allowed by observations, taking $n_s^{\rm obs} = 0.965$ and $r^{\rm obs} < 0.036$ as current constraints.

4. **Higgs Inflation**

   Considering the Higgs potential

$$V(\phi) = \lambda(\phi^2 - v^2)^2 \tag{0.4}$$

   where $v = 246$ GeV, show for which field ranges we could have viable inflation and compute slow-roll parameters in these regimes. Then, verify whether Higgs inflation is allowed by observations, calculating the power spectrum as a function of N_* and $m_{\rm H}$ and using the observed value of $P_\zeta \simeq 2 \cdot 10^{-9}$. Is the current value of the Higgs mass

$m_H = 125$ GeV in agreement with these constraints? To calculate λ, remember that $m_H^2 = V''(\phi = v) = 8\lambda v^2$.

5. **Higgs Inflation II**

   Consider the following non-minimal coupling of the Higgs to gravity

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{M_p^2}{2} f(\phi) R + \frac{1}{2} g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi - \frac{\lambda}{4} \phi^4 \right] \tag{0.5}$$

   where $f(\phi) = 1 + \xi\phi^2/M_p^2$ and assume to be in the regime $\phi \gg v$. With the redefinitions $\tilde{g}_{\mu\nu} = f(\phi) g_{\mu\nu}$ and $\Phi/M_p = \sqrt{3}/2 \log\left[f(\phi)\right]$ we get an action equivalent to the minimal coupling,

$$S = \int \mathrm{d}^4x \sqrt{-\tilde{g}} \left[\frac{M_p^2}{2} \tilde{R} + \frac{1}{2} \tilde{g}_{\mu\nu} \partial^\mu \phi \partial^\nu \Phi + V(\Phi) \right] \tag{0.6}$$

   where

$$V(\Phi) \approx \frac{\lambda M_p^4}{4\xi^2} \left[1 - 2e^{-\sqrt{\frac{2}{3}} \frac{\Phi}{M_p}} \right]. \tag{0.7}$$

   Show that with this potential and $\Phi \gg M_p$ we get viable inflation by computing slow-roll parameters. Then compute predictions for n_s and r using $50 < N_* < 60$. Using constraints from the previous example, can you establish whether this model is allowed by observations? What is the corresponding value of ξ that agrees with the measured power spectrum of curvature perturbations $P_\zeta \approx 2 \cdot 10^{-9}$?. Assume no running of λ from the vacuum to $\Phi \gg M_p^2$.

References

1. Inflation and Cosmological Perturbations, Antonio Riotto, https://arxiv.org/pdf/hep-ph/0210162
2. Cosmology Master's Degree course by Daniel Baumann at University of Amsterdam
3. Cosmology Master's Degree course by James Ferguson and Daniel Baumann at University of Cambridge