Odd-mass Nb Isotopes as a Region of Intertwined Quantum Phase Transitions

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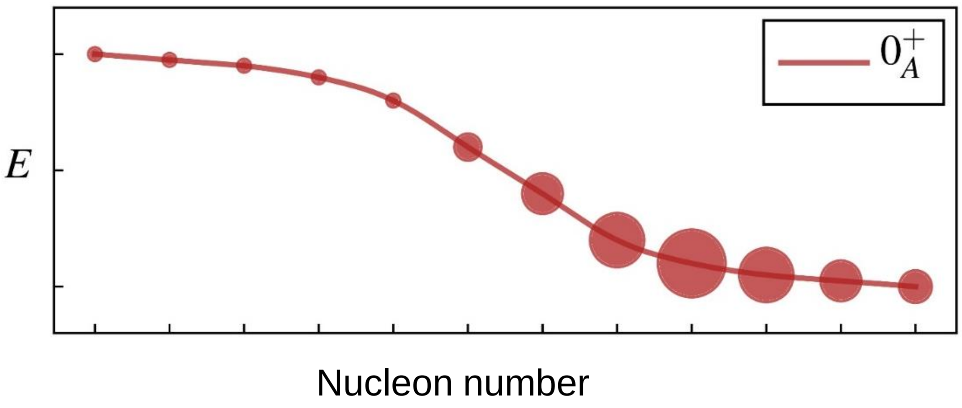
Quantum Phase Transitions (QPTs)

- Type I: Single configuration
  (shape-phase transitions)

$$\hat{H} = (1-\xi)\hat{H}_1 + \xi\hat{H}_2$$

neutron number 90 region: Nd-Sm-Gd



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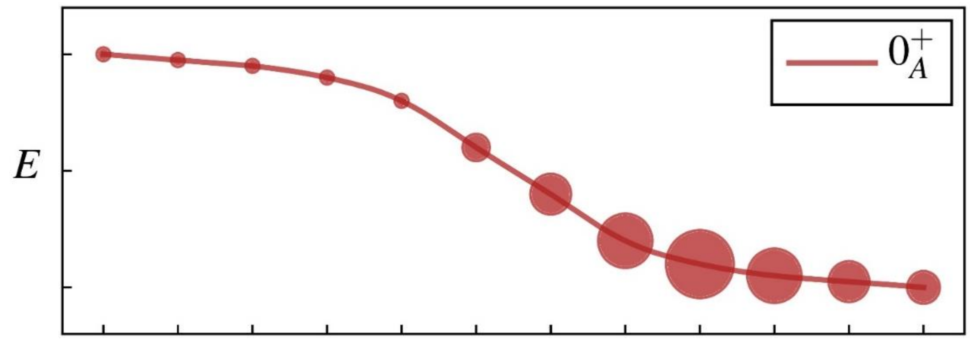
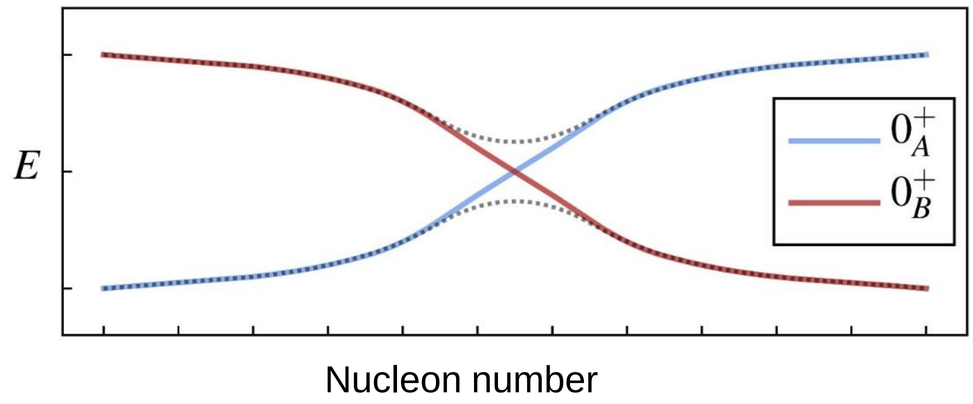


- Type II: Two configurations A, B
  (coexistence normal-intruder states)

$$\hat{H} = \begin{bmatrix} \hat{H}_A(\xi_A) & \hat{W}(\omega) \\ \hat{W}(\omega) & \hat{H}_B(\xi_B) \end{bmatrix}$$

nuclei near shell-closure: Pb-Hg



In most cases, the separate Type I QPTs are masked by the strong mixing between the two configurations

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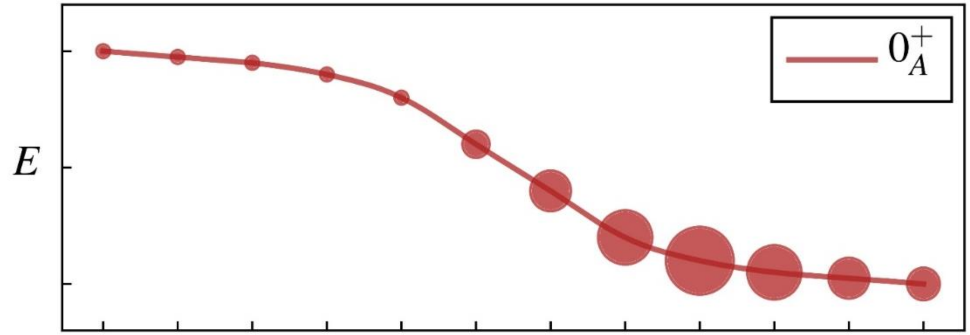

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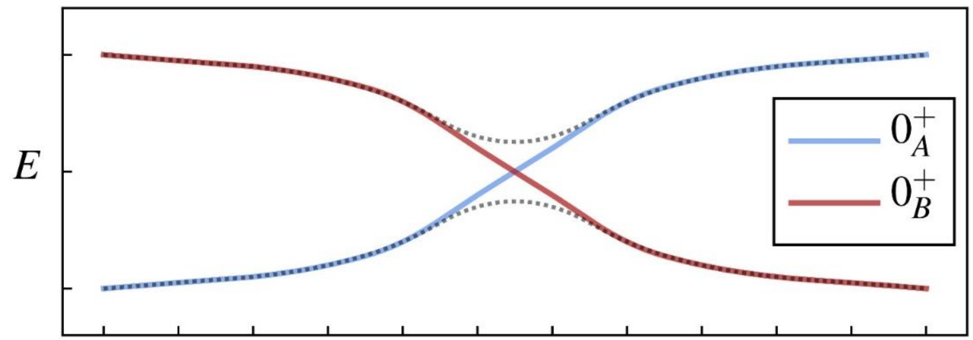
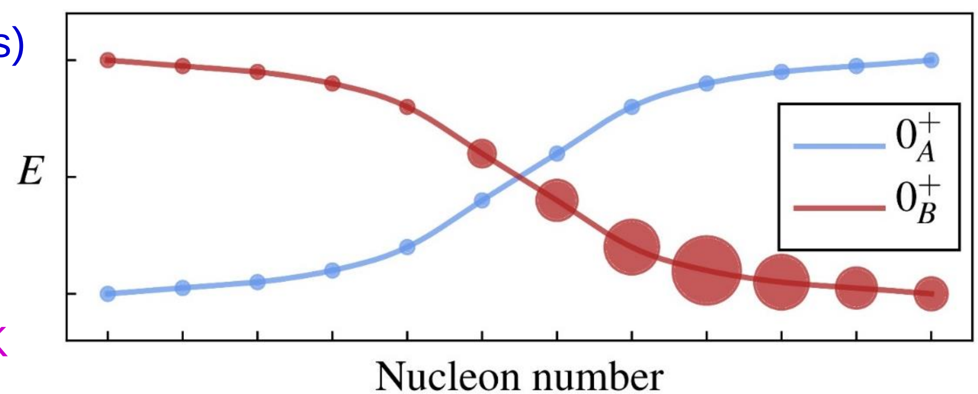


- Intertwined quantum phase transitions (IQPTs)

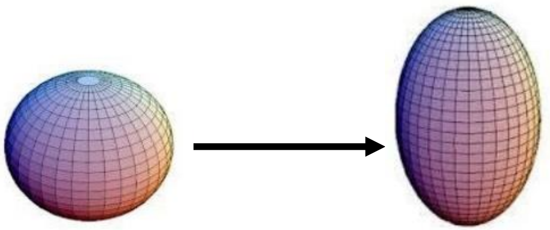
Type II QPT and Type I QPT coexist
configuration crossing accompanied by
pronounced individual shape-evolutions

neutron numbr 60 region: Zr, Nb THIS TALK



Type I QPT (shape-phase transition)

- IBM: Dynamical symmetries ↔ phases
- QPT: H(ξ) interpolates between different DS limits



G1 G2

$$H(\xi) = \xi\, H_{G1} + (1-\xi)\, H_{G2}$$

G_i = U(5), SU(3), SO(6) ↔ phases [spherical, deformed: axial, γ-unstable]

- Landau potential $E_N(\beta,\gamma;\xi) = \langle \beta,\gamma; N|\hat{H}|\beta,\gamma;N\rangle$

$$|\beta,\gamma;N\rangle = (N!)^{-1/2}(b_c^\dagger)^N|0\rangle \; .$$

- Order parameter $\dfrac{\langle \hat{n}_d\rangle_{0_1^+}}{N} \approx \dfrac{\beta_{\rm eq}^2}{1+\beta_{\rm eq}^2}$

$$b_c^\dagger = \tfrac{1}{\sqrt{2\beta+1}}[\beta\cos\gamma d_0^\dagger + \tfrac{1}{\sqrt{2}}\beta\sin\gamma(d_2^\dagger + d_{-2}^\dagger) + s^\dagger]$$

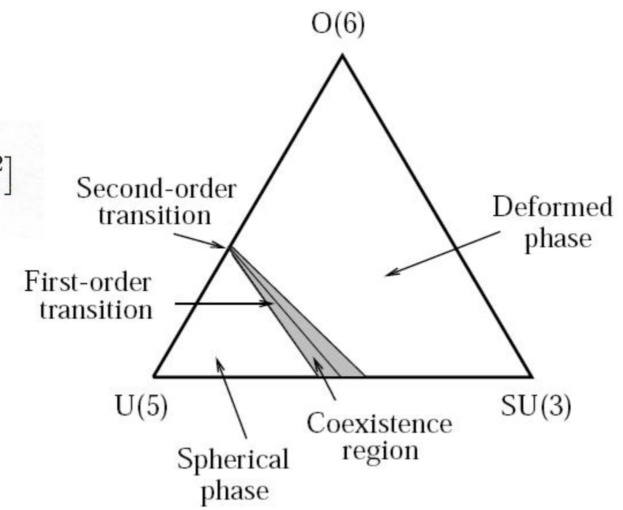
$$\hat{H}(\epsilon_d,\kappa,\chi) = \epsilon_d\,\hat{n}_d + \kappa\,\hat{Q}_\chi\cdot\hat{Q}_\chi$$

$$\hat{Q}_\chi = d^\dagger s + s^\dagger\tilde{d} + \chi(d^\dagger\times\tilde{d})^{(2)}$$

$$E_N(\beta,\gamma;\epsilon_d,\kappa,\chi) = 5\kappa\,N + \frac{N\beta^2}{1+\beta^2}\left[\epsilon_d + \kappa(\chi^2-4)\right] + \frac{N(N-1)\beta^2}{(1+\beta^2)^2}\kappa\left[4 - 4\bar{\chi}\beta\,\Gamma + \bar{\chi}^2\beta^2\right]$$

$$\bar{\chi} = \sqrt{\tfrac{2}{7}}\chi \qquad \Gamma = \cos 3\gamma$$

U(5):	$\kappa = 0$	U(5)-SU(3)	1st order
SU(3):	$(\epsilon_d = 0, \chi = -\sqrt{7}/2)$	U(5)-SO6)	2nd order
SO(6):	$(\epsilon_d = 0, \chi = 0)$	SU(3)-SO(6)	crossover



Type II QPT (coexistence near shell closure)

- Multiparticle-multihole intruder excitations across shell gaps
- Interacting boson model with configuration mixing (IBM-CM) [Duval, Barrett, PLB 81]

0p-0h, 2p-2h, 4p-4h,… → [N] ⊕ [N+2] ⊕ [N+4]… normal ⊕ intruder states

- Hamiltonian

$$\hat{H} = \begin{bmatrix} \hat{H}_A(\xi_A) & \hat{W}(\omega) \\ \hat{W}(\omega) & \hat{H}_B(\xi_B) \end{bmatrix}$$

$$\hat{H}_A(\xi_A) = \epsilon_d^A \, \hat{n}_d + \kappa_A \, \hat{Q}_\chi \cdot \hat{Q}_\chi$$

$$\hat{Q}_\chi = d^\dagger s + s^\dagger \tilde{d} + \chi (d^\dagger \times \tilde{d})^{(2)}$$

$$\hat{H}_B(\xi_B) = \epsilon_d^B \, \hat{n}_d + \kappa_B \, \hat{Q}_\chi \cdot \hat{Q}_\chi + \kappa'_B \, \hat{L} \cdot \hat{L} + \Delta_B$$

$$\hat{W}(\omega) = \omega \, [\, (d^\dagger d^\dagger)^{(0)} + (s^\dagger)^2 \, + \mathrm{H.c.}]$$

- Wave functions

$$|\Psi; L\rangle = a|\Psi_A; [N], L\rangle + b|\Psi_B; [N+2], L\rangle$$

normal ↓ intruder ↓

- Geometry

$$E(\beta,\gamma) = \begin{bmatrix} E_A(\beta,\gamma;\xi_A) & \Omega(\beta,\gamma;\omega) \\ \Omega(\beta,\gamma;\omega) & E_B(\beta,\gamma;\xi_B) \end{bmatrix}$$

Matrix coherent states $|\beta,\gamma;N\rangle$, $|\beta,\gamma; N+2\rangle$

$E_\pm(\beta,\gamma)$ Eigen-potentials

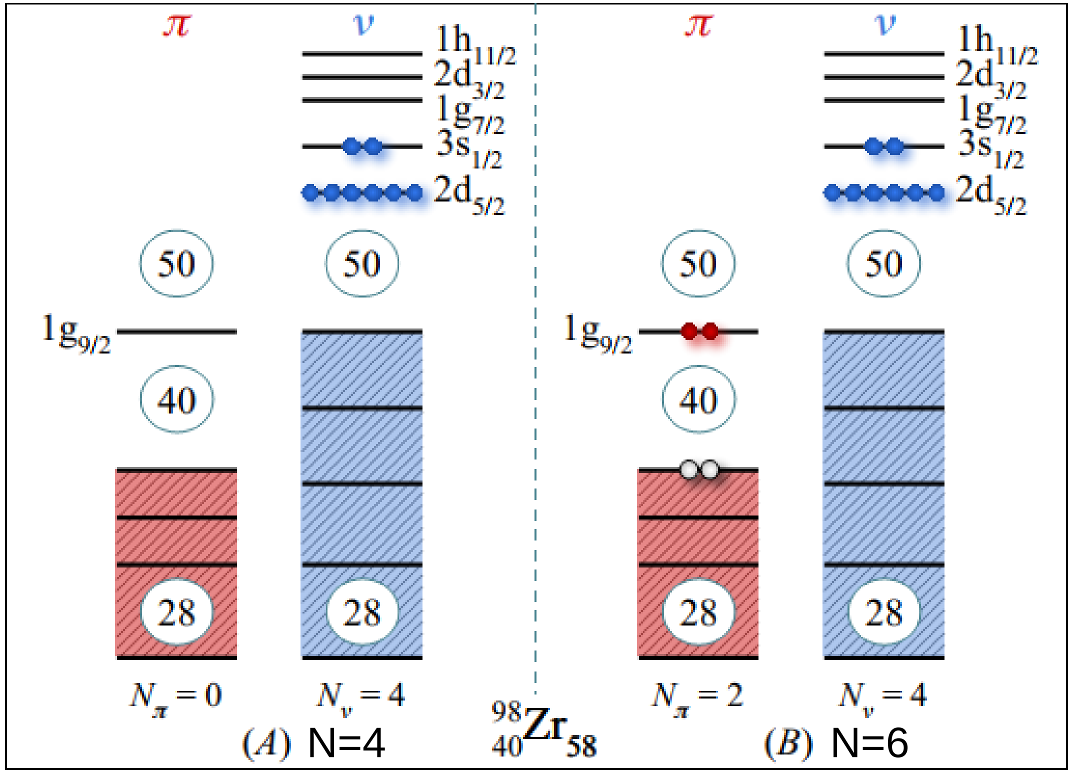
- Order parameters

$$\langle \hat{n}_d \rangle_A \qquad \langle \hat{n}_d \rangle_B \qquad \langle \hat{n}_d \rangle_{0_1^+} = a^2 \, \langle \hat{n}_d \rangle_A + b^2 \, \langle \hat{n}_d \rangle_B$$

IBM-CM in the Zr chain

$_{40}$Zr isotopes

Positive parity states



Normal (A) configuration Z=40 subshell closure [N]
Intruder (B) configuration two-proton excitation (2p-2h states) [N+2]

$$|\Psi; L\rangle = a|\Psi_A; [N], L\rangle + b|\Psi_B; [N+2], L\rangle ,$$

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QPTs in odd-mass Nuclei

- Interacting boson-fermion model (IBFM)

$$\hat{H}(\xi) = \hat{H}_{\rm b} + \hat{H}_{\rm f} + \hat{V}_{\rm bf}$$

$$\hat{H}_{\rm b}(\epsilon_d, \kappa, \chi) = \epsilon_d\, \hat{n}_d + \kappa\, \hat{Q}_\chi \cdot \hat{Q}_\chi$$

$$\hat{H}_{\rm f}(\epsilon_j) = \epsilon_j\, \hat{n}_j$$

$$\hat{V}_{\rm bf}(\chi, A, \Gamma, \Lambda) = A\, \hat{n}_d\, \hat{n}_j + \Gamma\, \hat{Q}_\chi \cdot (a_j^\dagger\, \tilde{a}_j)^{(2)} + \Lambda\sqrt{2j+1} : [(d^\dagger\, \tilde{a}_j)^{(j)} \times (\tilde{d}\, a_j^\dagger)^{(j)}]^{(0)} :$$

- Interacting boson-fermion model with configuration mixing (IBFM-CM)

$$\hat{H}(\xi_A, \xi_B, \omega) = \begin{bmatrix} \hat{H}_A(\xi_A) & \hat{W}(\omega) \\ \hat{W}(\omega) & \hat{H}_B(\xi_B) \end{bmatrix}$$

$$\hat{H}_A(\xi_A) = \hat{H}(\xi_A)$$

$$\hat{H}_B(\xi_B) = \hat{H}(\xi_B) + \kappa'_B\, \hat{L} \cdot \hat{L} + \Delta_B$$

$$\hat{W}(\omega) = \omega\,[\,(d^\dagger d^\dagger)^{(0)} + (s^\dagger)^2 \ + \text{H.c.}]$$

A (normal) and B (intruder) configurations

$$|\Psi; J\rangle = \sum_{\alpha,L} C^{(N,J)}_{\alpha,L,j} |\Psi_{\rm A}; [N], \alpha, L; j; J\rangle + \sum_{\alpha,L} C^{(N+2,J)}_{L,j} |\Psi_{\rm B}; [N+2], \alpha, L; j; J\rangle$$

↓ normal ↓ intruder

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IBFM-CM in the Nb chain

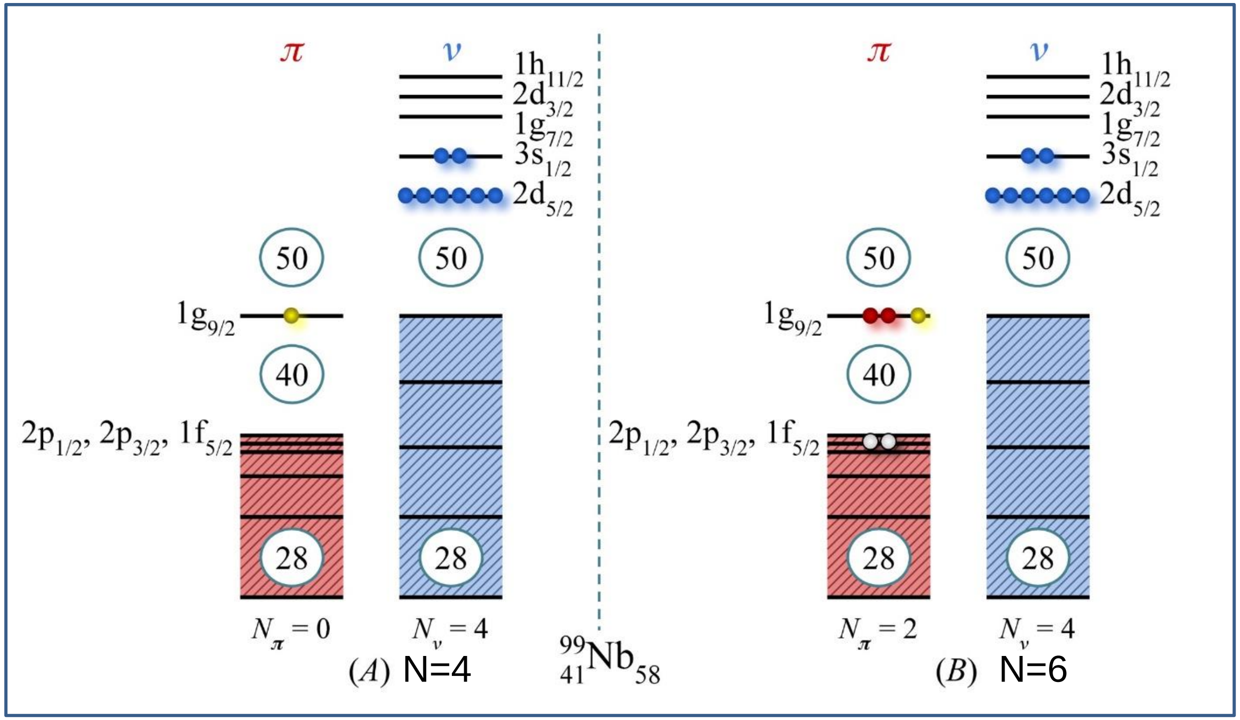
$_{41}$Nb isotopes

Positive parity states

$\left({}_{40}\text{Zr core} \right)$ + single-j

$\left([N] \oplus [N+2]\right) \otimes \pi(1g_{9/2})$

↓ normal ↓ intruder

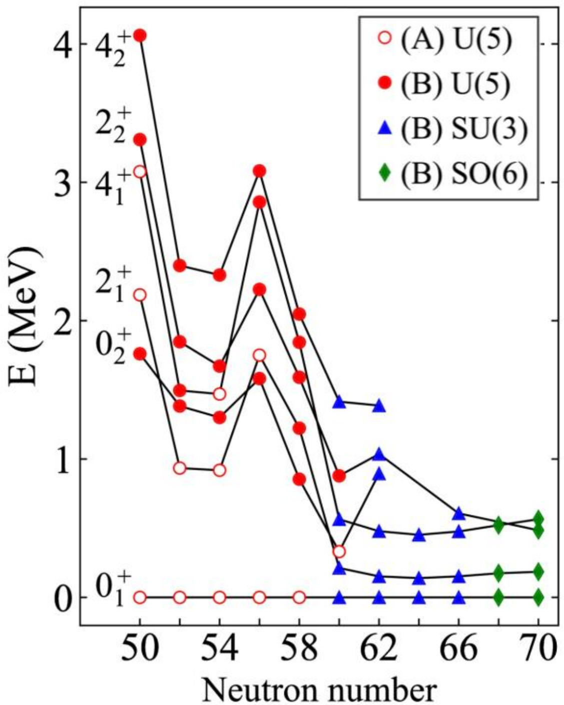


$$|\Psi; J\rangle = \sum_{\alpha,L} C_{\alpha,L,j}^{(N,J)} |\Psi_{\rm A}; [N], \alpha, L; j; J\rangle + \sum_{\alpha,L} C_{L,j}^{(N+2,J)} |\Psi_{\rm B}; [N+2], \alpha, L; j; J\rangle$$

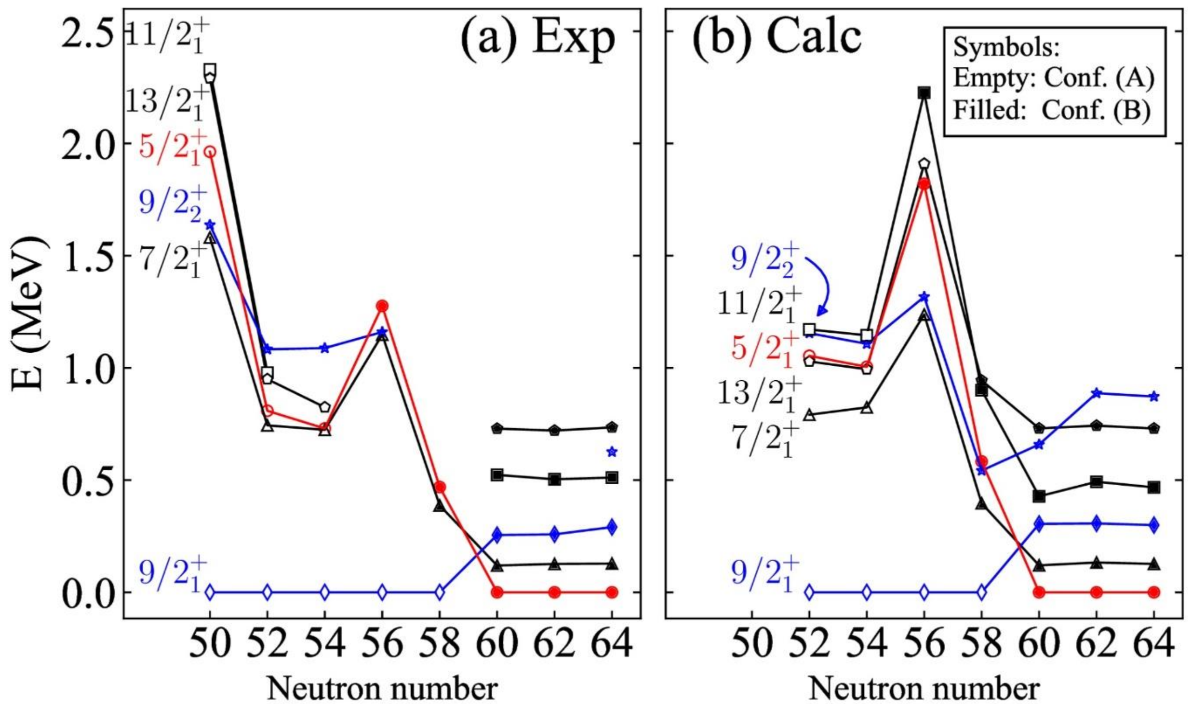
$$a^2 = \sum_{\alpha,L} |C_{\alpha,L,j}^{(N,J)}|^2 \quad , \quad b^2 = \sum_{\alpha,L} |C_{\alpha,L,j}^{(N+2,J)}|^2 \qquad a^2 + b^2 = 1$$

↓ normal ↓ intruder

Zr



Nb



n=50-56: config. (A) spherical (seniority-like) $R^{(A)}_{4/2}$ ~ 1.6

config. (B) weakly-deformed $R^{(B)}_{4/2}$ ~ 2.3

rise in energy at n=56 due to ν(2d5/2) subshell clousure

From n=58: pronounced drop in energy for states of config. (B)

n=60: two configurations exchange role ⇒ Type II QPT

config. (B) at critical point of U(5)-SU(3) Type I QPT

g.s. changes from $9/2^+$ to $5/2^+$

n>60: config. (B) strongly deformed [SU(3)] ^{104}Zr: $R^{(B)}_{4/2}$ = 3.24

K=$5/2^+$ band develops

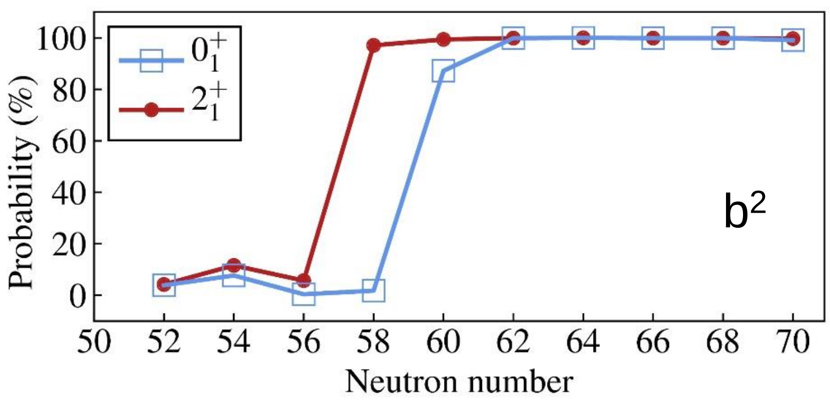
J=$5/2^+$,$7/2^+$, $9/2^+$, $11/2^+$, $13/2^+$

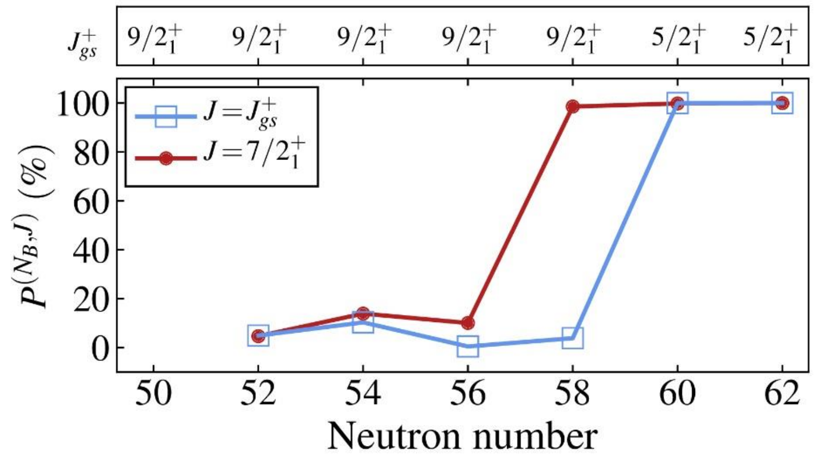
n=66: g.s. becomes γ-unstable (or triaxial) SU(3) → SO(6) crossover

Zr

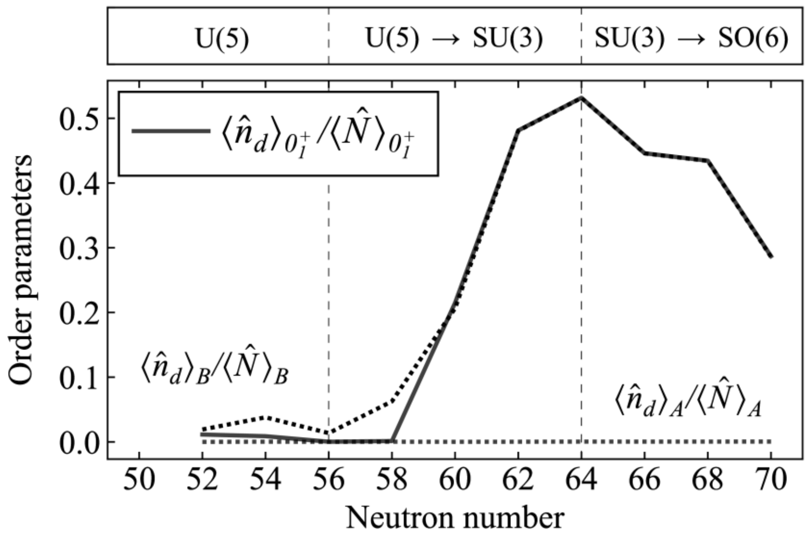
Nb

probablity of intruder component





- Abrupt crossing of the two configurations (Type II QPT)



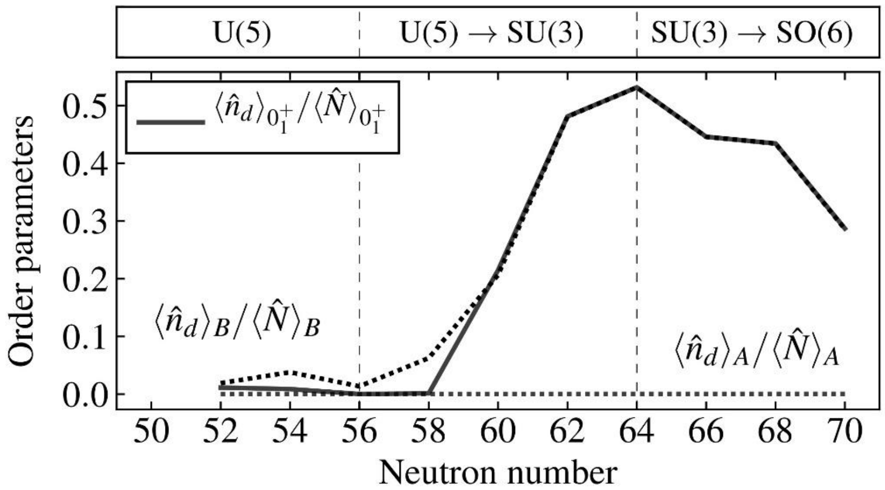
Transition from weak to strong coupling

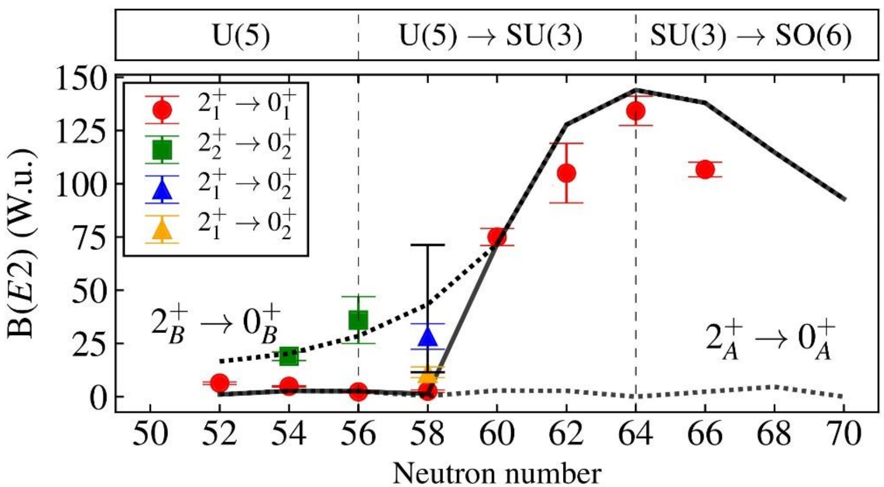
- Gradual spherical to deformed transition (Type I QPT) within the intruder (B) configuration

Coexisting Type I QPT and Type II QPT
$\Rightarrow$ Intertwined QPTs (IQPTs)

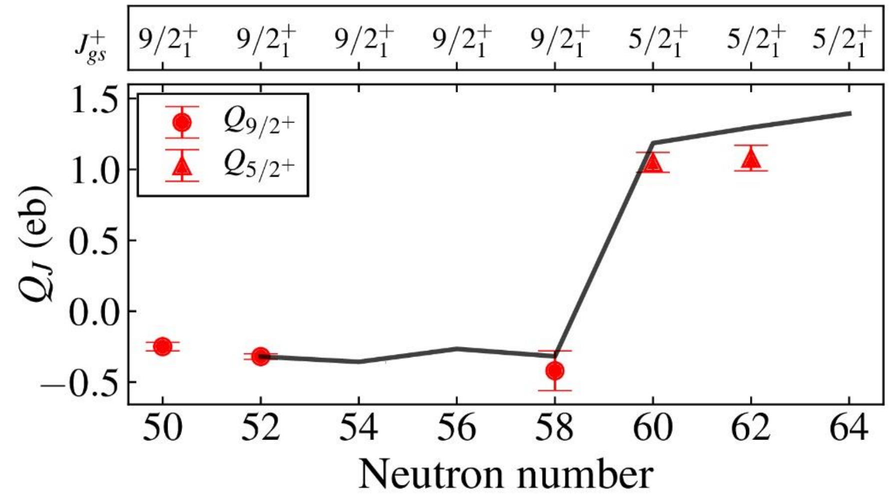
B(E2) values and quadrupole moments

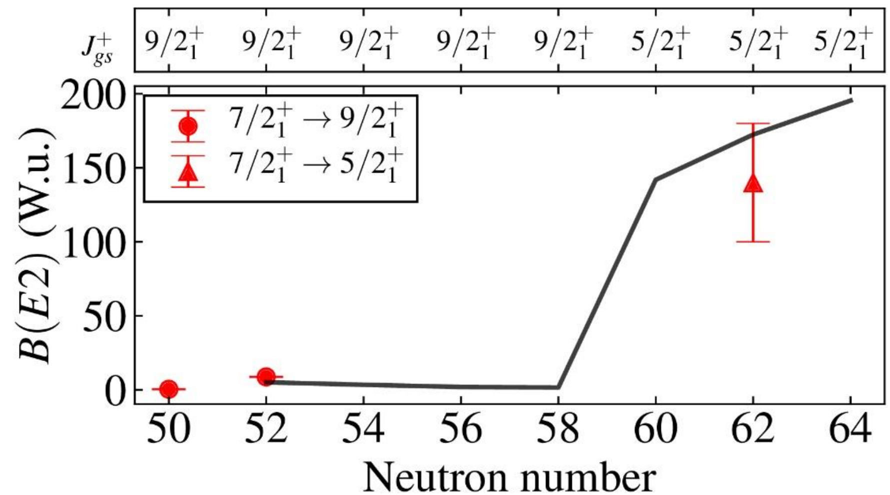
Zr





Nb





$2_A \to 0_A$ coincides with $2_1 \to 0_1$ for n=52-58

$2_B \to 0_B$ coincides with $2_2 \to 0_2$ for n=52-56

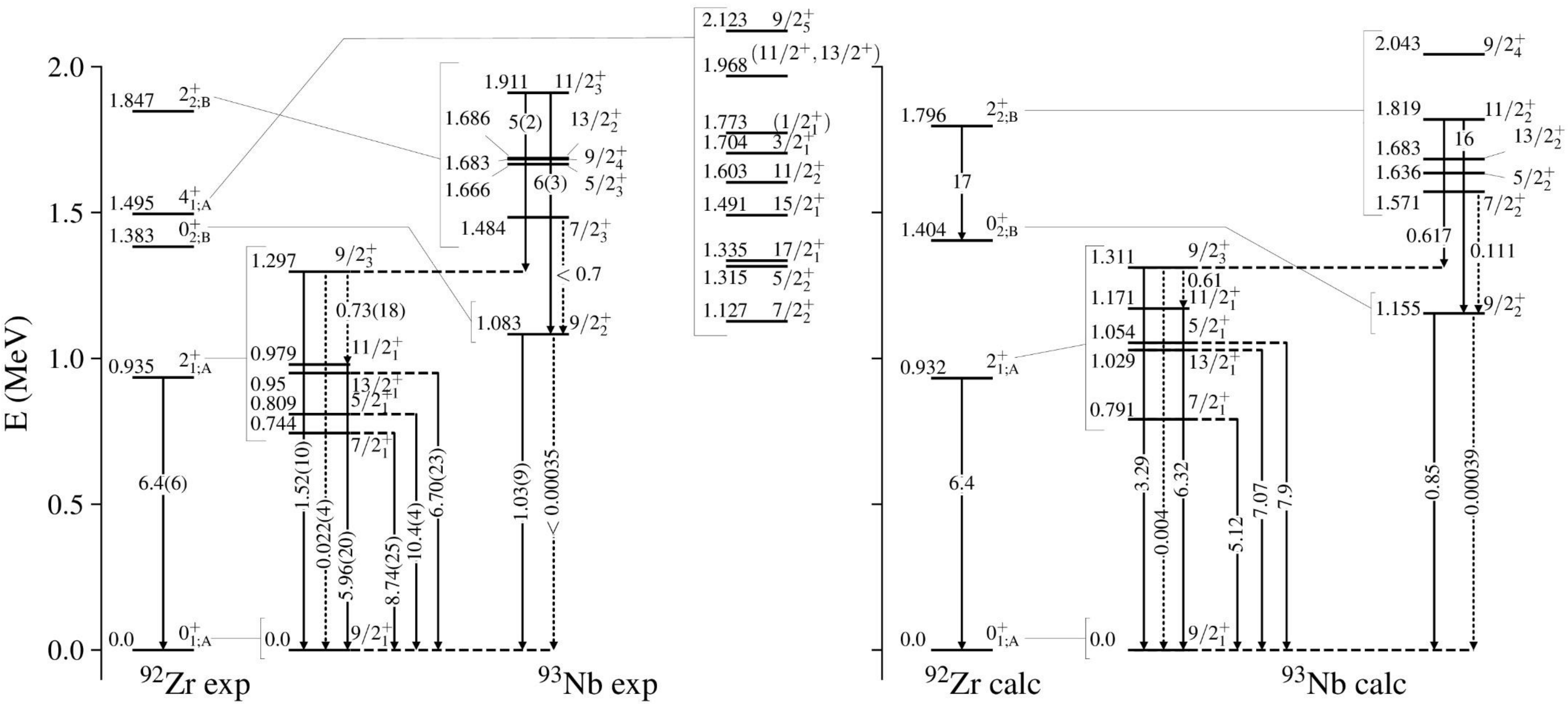
$2_1 \to 0_2$ n=58

$2_1 \to 0_1$ n=60-64

B(E2) and Q(J_{gs}) related to deformation (order param)

n=60: Type II QPT (jump from A to B configuration)

Type I QPT within the intruder B configuration



^{93}Nb Weak coupling (WC)

$[(L=0^+_{1;A}) \otimes \pi(1g_{9/2})]$ J = $9/2^+$

$[(L=2^+_{1;A}) \otimes \pi(1g_{9/2})]$ J = $5/2^+$, $7/2^+$, $9/2^+$, $11/2^+$, $13/2^+$

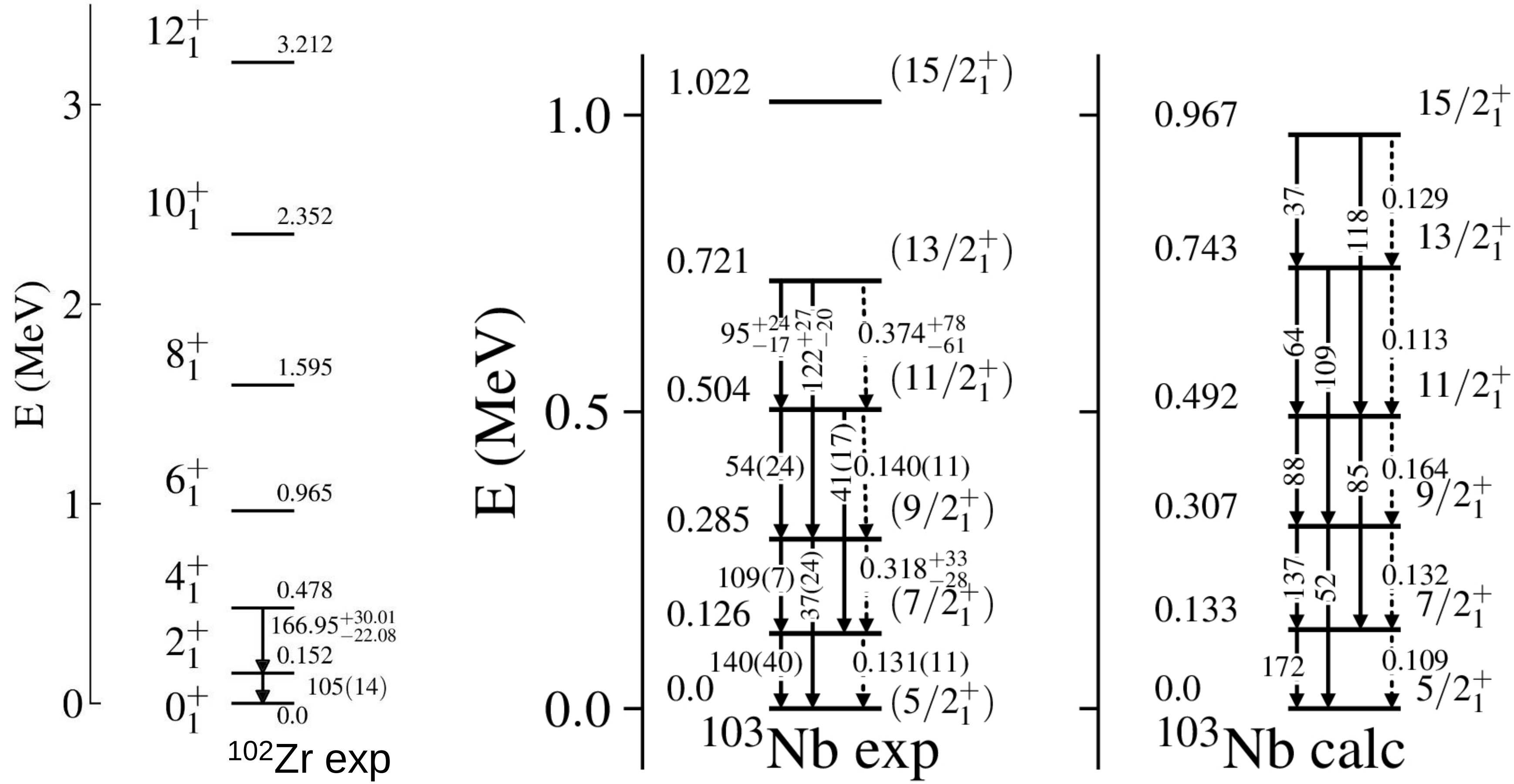
$[(L=4^+_{1;A}) \otimes \pi(1g_{9/2})]$ J = $1/2^+$, $3/2^+$, $5/2^+$, $7/2^+$, $9/2^+$, $11/2^+$, $13/2^+$, $15/2^+$

normal WC multiplets

$[(L=0^+_{2;B}) \otimes \pi(1g_{9/2})]$ J = $9/2^+$

$[(L=2^+_{2;B}) \otimes \pi(1g_{9/2})]$ J = $5/2^+$, $7/2^+$, $9/2^+$, $11/2^+$, $13/2^+$

Intruder WC multiplets



^{103}Nb Strong coupling 5/2$^+$[422]

[(L=$0^+_{1;B}$, $2^+_{1;B}$, $4^+_{1;B}$, $6^+_{1;B}$ …) ⊗ $\pi(1g_{9/2})$] J

Geometry

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IBFM

$$\hat{H}(\xi) = \hat{H}_{\rm b} + \hat{H}_{\rm f} + \hat{V}_{\rm bf}$$

Matrix $\Omega_N(\beta,\gamma;\xi)$: Basis: $|j,m;\beta,\gamma;N\rangle = |j,m\rangle \otimes |\beta,\gamma;N\rangle$

$$|j,m\rangle = a^\dagger_{jm}|0\rangle$$

IBFM-CM

$$|\beta,\gamma;N\rangle = (N!)^{-1/2}(b^\dagger_c)^N|0\rangle$$

$$\hat{H}(\xi_A,\xi_B,\omega) = \begin{bmatrix} \hat{H}_A(\xi_A) & \hat{W}(\omega) \\ \hat{W}(\omega) & \hat{H}_B(\xi_B) \end{bmatrix}$$

$$b^\dagger_c = \tfrac{1}{\sqrt{2\beta+1}}[\beta\cos\gamma d^\dagger_0 + \tfrac{1}{\sqrt{2}}\beta\sin\gamma(d^\dagger_2 + d^\dagger_{-2}) + s^\dagger]$$

$$\Omega_N(\beta,\gamma;\xi_A,\xi_B,\omega) = \left[\begin{array}{c|c} \Omega_A(\beta,\gamma;\xi_A) & \Omega_{AB}(\beta,\gamma;\omega) \\ \hline \Omega_{AB}(\beta,\gamma;\omega) & \Omega_B(\beta,\gamma;\xi_B) \end{array}\right]$$

$$|j,m_1;\beta,\gamma;N\rangle\ ,\ |j,m_2;\beta,\gamma;N{+}2\rangle$$

$\gamma = 0$ $\quad \Omega_N(\beta;\xi_A,\xi_B,\omega) = \{M_{K=j}(\beta),\ M_{K=j-2}(\beta),\ \ldots,\ M_{K=-(j-1)}(\beta)\}$ block-diagonal

$$M_K(\beta) = \begin{bmatrix} E_{A;K}(\beta) & W(\beta) \\ W(\beta) & E_{B;K}(\beta) \end{bmatrix} .$$

$$|\Psi_{A;K}\rangle \equiv |j,K;\beta;N\rangle\ ,\ |\Psi_{B;K}\rangle \equiv |j,K;\beta;N+2\rangle$$

$$\delta_K(\beta) = E_{B;K}(\beta) - E_{A;K}(\beta)$$

unmixed surfaces

$$R_K(\beta) = \tfrac{\delta_K(\beta)}{2W(\beta)}$$

Eigen-potentials

$$E_{\pm,K}(\beta) = \tfrac{1}{2}\,[\,E_{A;K}(\beta) + E_{B;K}(\beta)\,] \pm |W(\beta)|\sqrt{1+[R_K(\beta)]^2}$$

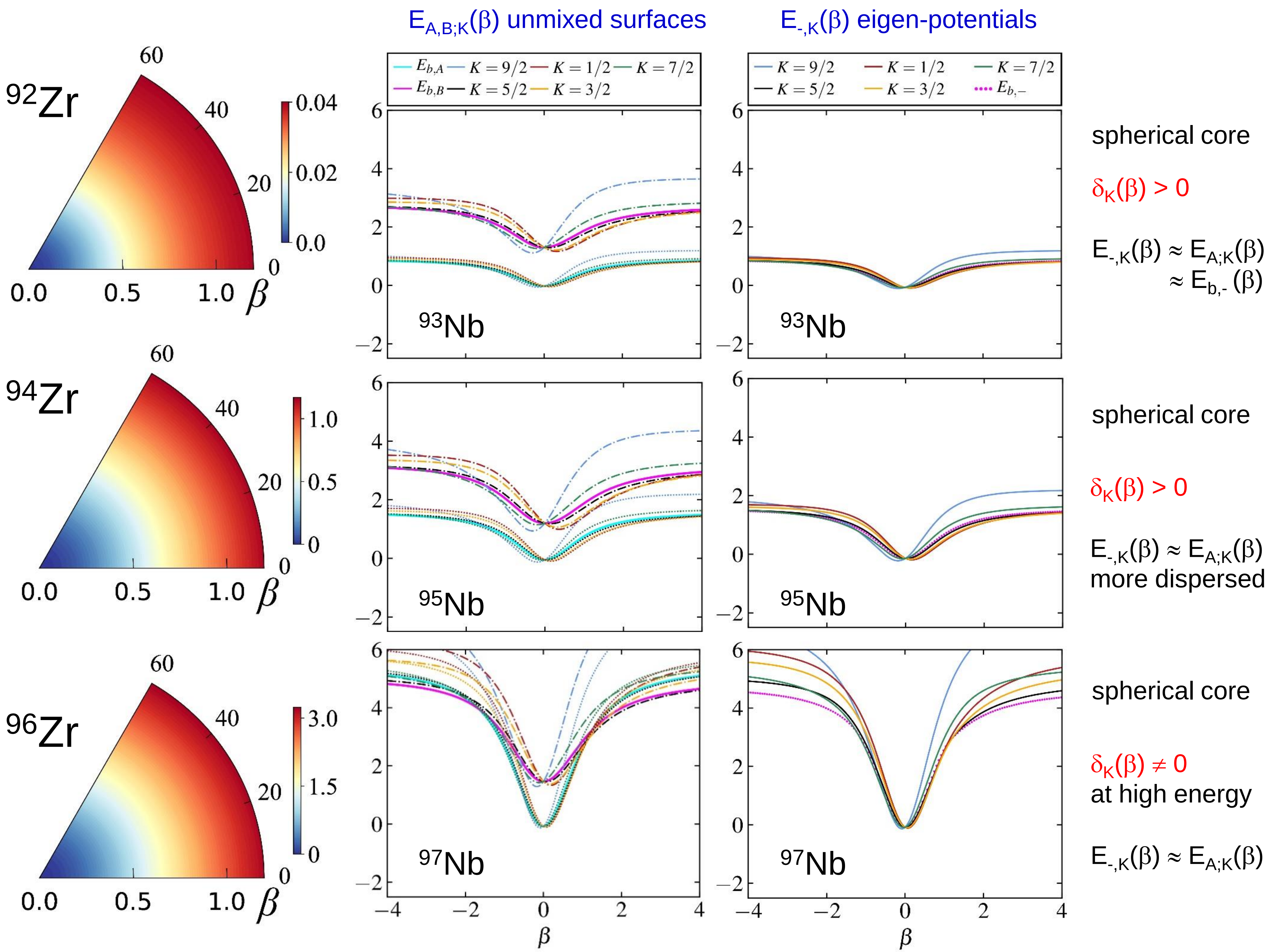
$$\frac{b}{a} = \left[R_K(\beta) \mp \sqrt{1+[R_K(\beta)]^2}\right]$$

$$|\Psi_{-,K}(\beta)\rangle = a\,|\Psi_{A;K}\rangle + b\,|\Psi_{B;K}\rangle$$

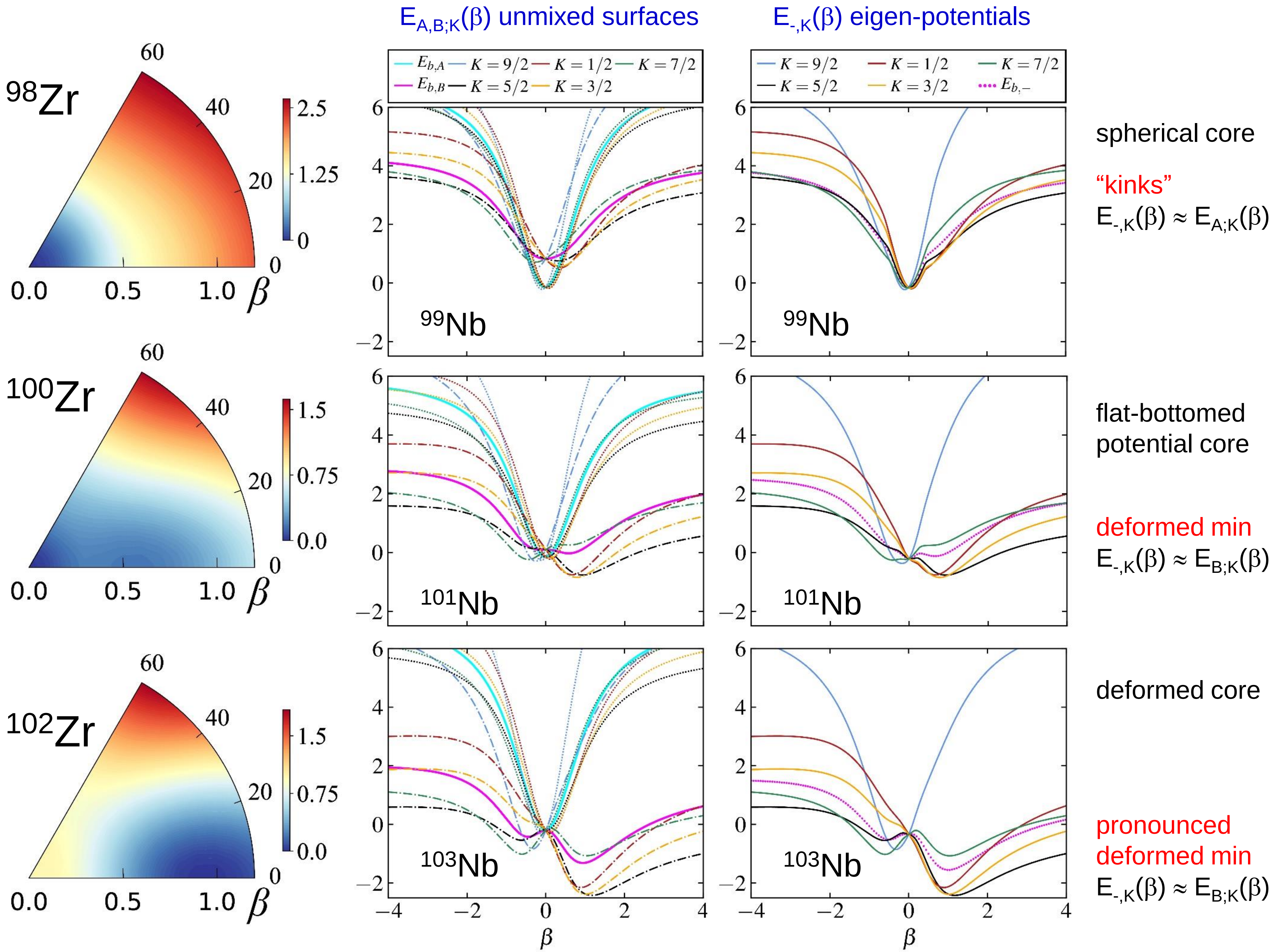
$$a^2 + b^2 = 1$$

$$|\Psi_{+,K}(\beta)\rangle = -b\,|\Psi_{A;K}\rangle + a\,|\Psi_{B;K}\rangle$$

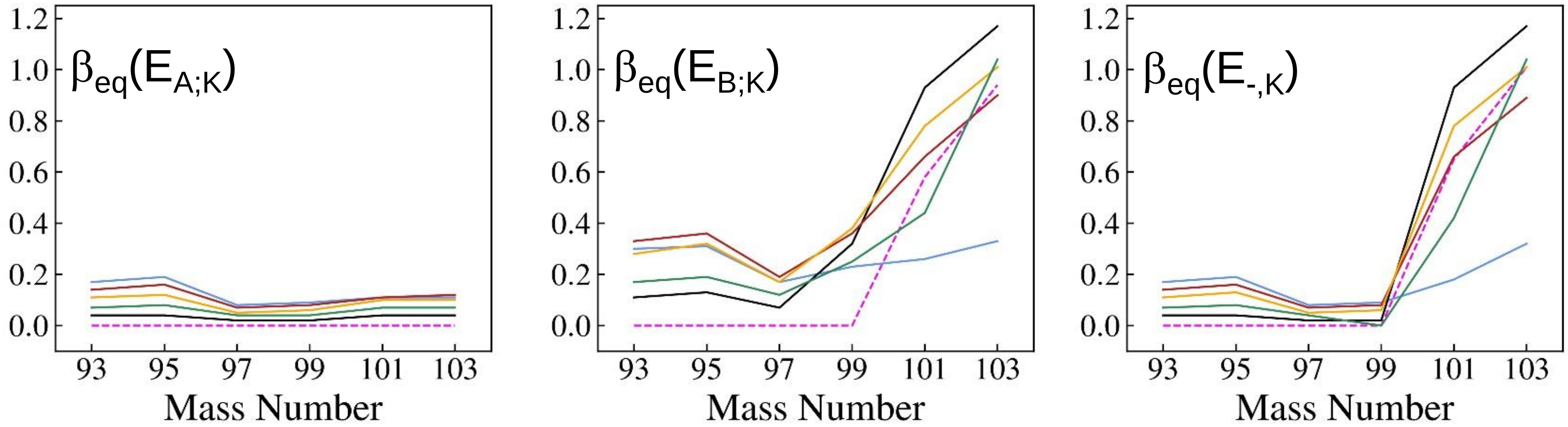
$$b^2 = \frac{1}{1+\left[R_K(\beta) \pm \sqrt{1+[R_K(\beta)]^2}\right]^2}$$



$E_{A,B;K}(\beta)$ unmixed surfaces
$E_{-,K}(\beta)$ eigen-potentials
^{92}Zr
^{94}Zr
^{96}Zr
^{93}Nb
^{95}Nb
^{97}Nb
$E_{b,A}$ $K = 9/2$ $K = 1/2$ $K = 7/2$ $E_{b,B}$ $K = 5/2$ $K = 3/2$
$K = 9/2$ $K = 1/2$ $K = 7/2$ $K = 5/2$ $K = 3/2$ $E_{b,-}$
β
spherical core
$\delta_K(\beta) > 0$
$E_{-,K}(\beta) \approx E_{A;K}(\beta)$
$\approx E_{b,-}(\beta)$
spherical core
$\delta_K(\beta) > 0$
$E_{-,K}(\beta) \approx E_{A;K}(\beta)$
more dispersed
spherical core
$\delta_K(\beta) \neq 0$
at high energy
$E_{-,K}(\beta) \approx E_{A;K}(\beta)$

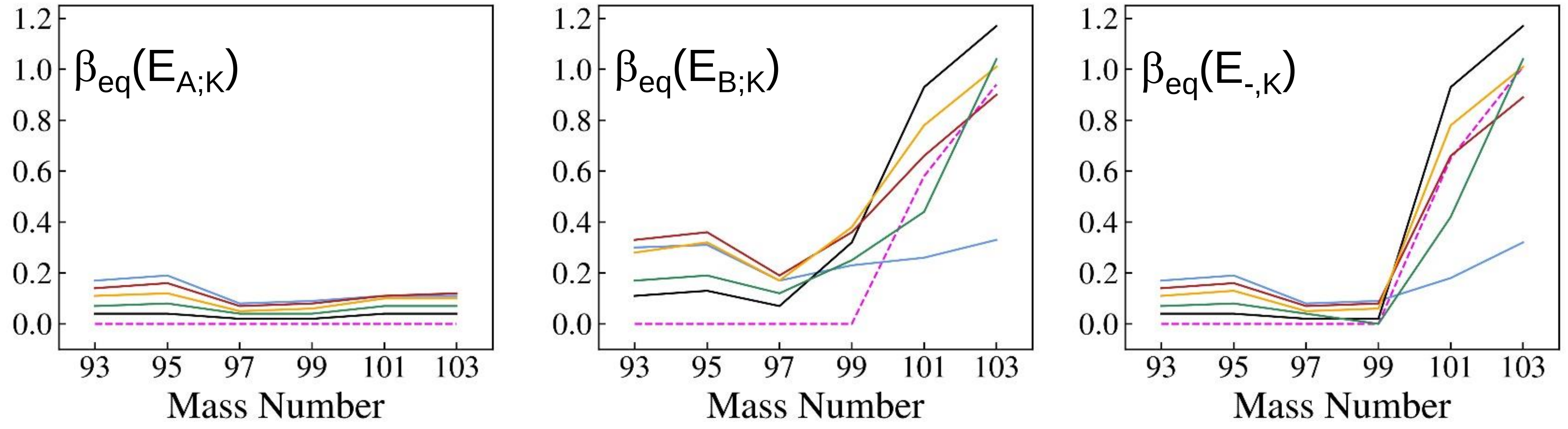


$E_{A,B;K}(\beta)$ unmixed surfaces
$E_{-,K}(\beta)$ eigen-potentials
^{98}Zr
^{100}Zr
^{102}Zr
^{99}Nb
^{101}Nb
^{103}Nb
$E_{b,A}$ $K = 9/2$ $K = 1/2$ $K = 7/2$
$E_{b,B}$ $K = 5/2$ $K = 3/2$
$K = 9/2$ $K = 1/2$ $K = 7/2$
$K = 5/2$ $K = 3/2$ $E_{b,-}$
β
spherical core
"kinks"
$E_{-,K}(\beta) \approx E_{A;K}(\beta)$
flat-bottomed potential core
deformed min
$E_{-,K}(\beta) \approx E_{B;K}(\beta)$
deformed core
pronounced deformed min
$E_{-,K}(\beta) \approx E_{B;K}(\beta)$



- Equilibrium deformations (order parameters)

  - Normal A configuration remains spherical along the Nb chain ($\beta_{eq}(E_{A;K})$ small)
  - Intruder B configuration changes gradually from weakly deformed in ^{93}Nb to strongly deformed in ^{103}Nb ($\beta_{eq}(E_{B;K})$ large)
  - $\beta_{eq}(E_{-,K}) \approx \beta_{eq}(E_{A;K})$ for ^{93,95,97,99}Nb and $\beta_{eq}(E_{-,K}) \approx \beta_{eq}(E_{B;K})$ for ^{101,103}Nb



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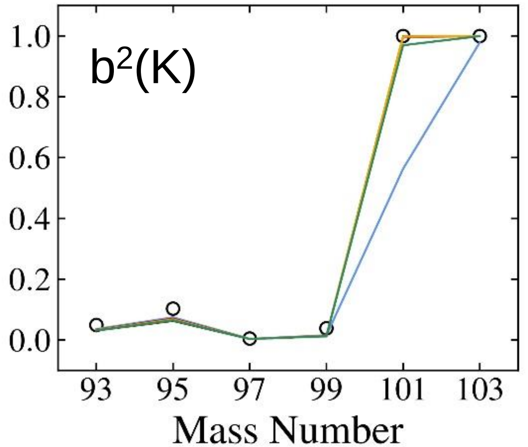
  - $\beta_{eq}(E_{-,K}) \approx \beta_{eq}(E_{A;K})$ for ^{93,95,97,99}Nb and $\beta_{eq}(E_{-,K}) \approx \beta_{eq}(E_{B;K})$ for ^{101,103}Nb

- Probability (b^2) of intruder component in $|\Psi_{-,K}(\beta)\rangle$

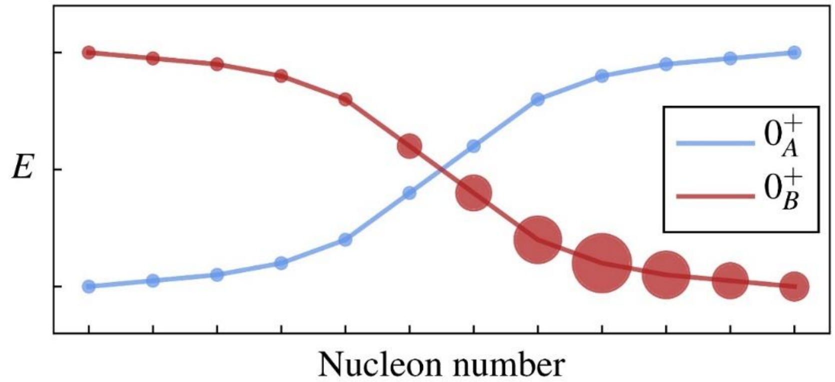
  Rapid change in structure
  from normal configuration in ^{93,95,97,99}Nb (small b^2)
  to intruder configuration in ^{101,103}Nb (large b^2)



- Classical analysis confirms the scenario of intertwined QPTs

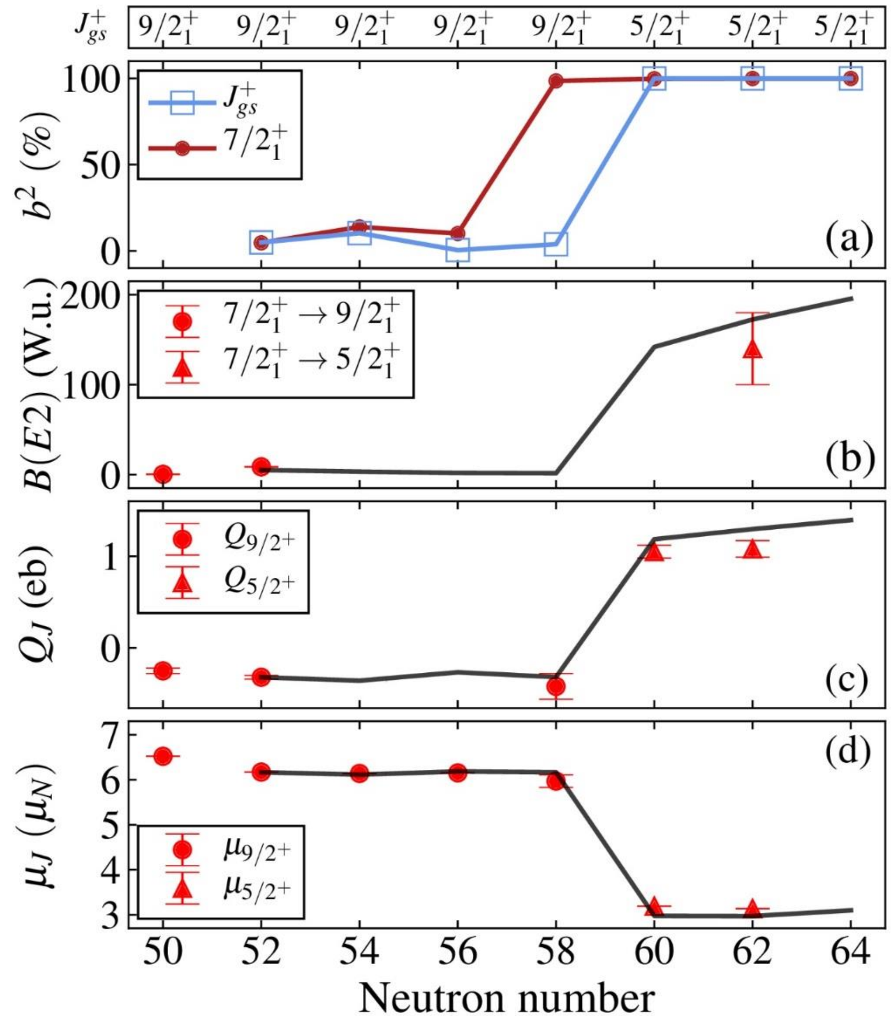
Concluding remarks

- Intertwined Quantum Phase Transitions (IQPTs)



- IBFM-CM framework
  Quantitative description of configuration mixing and related QPTs in odd-mass nuclei

- Nb isotopes
  Zr core + single-j fermion
  (Normal A configuration) $\otimes\ \pi(1g_{9/2})$
  (Intruder B configuration) $\otimes\ \pi(1g_{9/2})$



- Detailed quantum and classical analysis discloses
  a Type II QPT (abrupt crossing of normal and intruder states) accompanied by
  a Type I QPT (gradual shape evolution and transition from weak to strong coupling
  within the Intruder configuration), thus demonstrating IQPTs in odd-mass nuclei

- The observed IQPTs in odd-A Nb isotopes echo the IQPTs
  previously found in the adjacent even-even Zr isotopes

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PRC **105**, 014305 (2022)
PRC **106,** L051304 (2022)
Leviatan, Gavrielov (2025)

Thank you