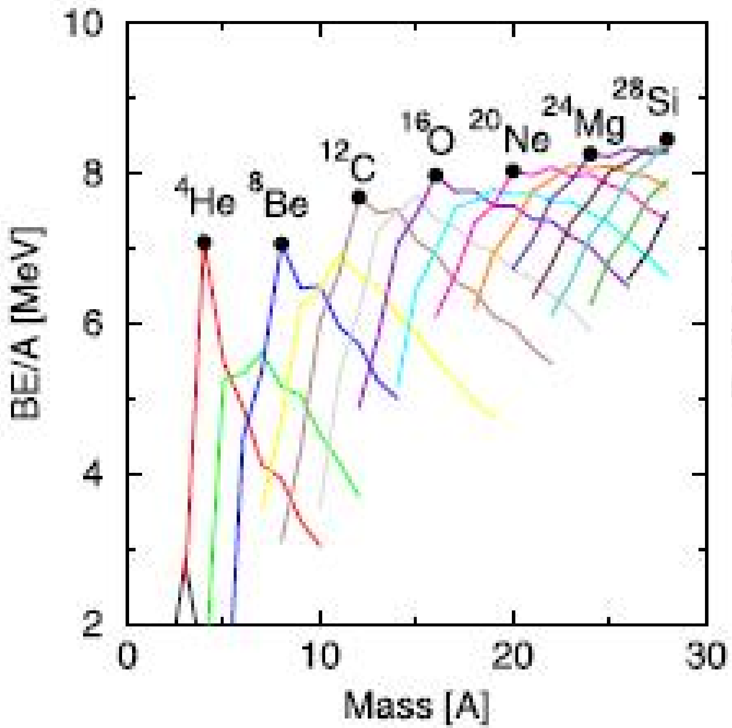
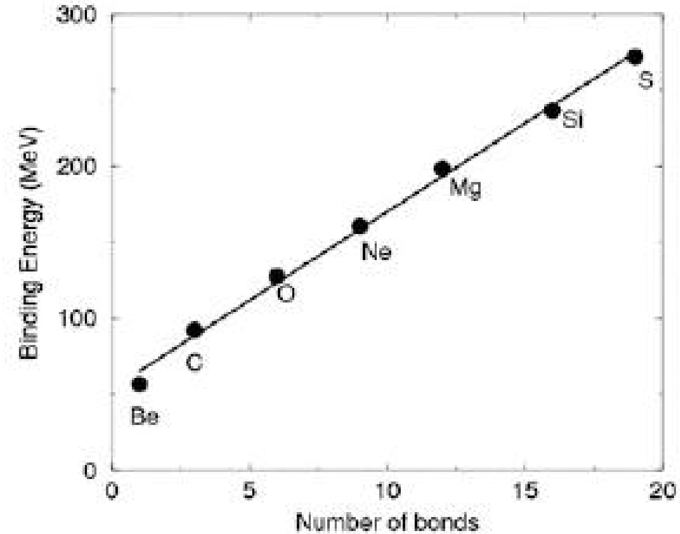
Algebraic Treatment of Alpha-Cluster Nuclei

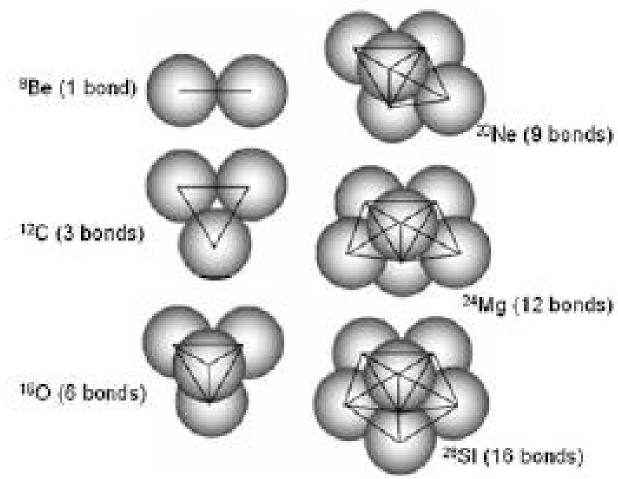
- Introduction
- Algebraic Cluster Model (ACM)
- Point group symmetry
- Applications: ^{12}C
- Cluster Shell Model (CSM)
- Double point group symmetry
- Applications: ^{13}C
- Summary and conclusions

Instituto de Ciencias Nucleares UNAM

Alpha-Cluster Nuclei







Recent Theoretical Work

- Alpha-cluster model (Wheeler 1937, Brink 1966, Robson 1978)

- AMD (Kanada-Enyo, PTP, 2007)
- FMD model (Chernykh et al, PRL, 2007)
- BEC-like cluster model (Funaki et al, PRC, 2009)
- Ab initio no-core shell model (Roth et al, PRL, 2011)
- Lattice EFT (Epelbaum et al, PRL, 2011, 2012)
- No-core symplectic model (Dreyfuss et al, PLB, 2013)
- Algebraic Cluster Model (2000, 2002, 2014, 2017, 2023)
- and many others

- Recent reviews: Freer & Fynbo, PPNP 78, 1 (2014), Freer, Horiuchi, Kanada-En'yo, Lee & Meissner, RMP 90, 035004 (2018)

Few-Body Systems

- Integro-differential methods
- Algebraic Cluster Model (ACM)
- For k dof introduce a SGA of U(k+1)
- Two-body systems: U(4) vibron model (diatomic molecules, mesons)
- Three-body systems: U(7) model (baryons, nuclear clusters, molecules)
- Four-body systems: U(10) model
- n-body systems: U(3n-2) model

ACM for n-Body Systems

3n-3 relative degrees of freedom: Jacobi coordinates

$$\vec{\rho}_k \;=\; \frac{1}{\sqrt{k(k+1)}}\left(\sum_{i=1}^{k}\vec{r}_i - k\vec{r}_{k+1}\right) \;, \qquad k=1,2,\ldots,n-1$$

Introduce n-1 dipole bosons and a scalar boson

$$b^{\dagger}_{k,m} \;,\; s^{\dagger} \;, \qquad k=1,2,\ldots,n-1$$

Number conserving one- and two-body Hamiltonian

$$\begin{aligned} H \;=\; & \epsilon_0\, s^{\dagger}\tilde{s} - \epsilon_1 \sum_k b^{\dagger}_k\cdot\tilde{b}_k + u_0\, s^{\dagger}s^{\dagger}\,\tilde{s}\tilde{s} \\ & -u_1 \sum_k s^{\dagger}b^{\dagger}_k\cdot\tilde{b}_k\tilde{s} + v_0 \sum_k \left[b^{\dagger}_k\cdot b^{\dagger}_k\,\tilde{s}\tilde{s} + \text{h.c.}\right] \\ & + \sum_L \sum_{ijkl} v^{(L)}_{ijkl}\,[b^{\dagger}_i\times b^{\dagger}_j]^{(L)}\cdot[\tilde{b}_k\times\tilde{b}_l]^{(L)} \;, \end{aligned}$$

Mixing between oscillator shells

Wave Functions

- Labeled by $\left|[N], \alpha, L_t^P\right\rangle$
- Total number of bosons: N
- Angular momentum: L
- Parity: P
- Identical clusters: Permutation symmetry: t

Three-Body Clusters

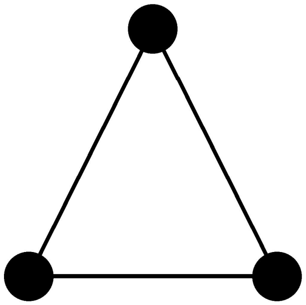
Permutation symmetry

$$
\begin{aligned}
H &= \xi_1 \, (R^2 \, s^\dagger s^\dagger - b_1^\dagger \cdot b_1^\dagger - b_2^\dagger \cdot b_2^\dagger) \, (\text{h.c.}) \\
&\quad + \xi_2 \left[(b_1^\dagger \cdot b_1^\dagger - b_2^\dagger \cdot b_2^\dagger) \, (\text{h.c.}) + 4 (b_1^\dagger \cdot b_2^\dagger)(\text{h.c.}) \right] \\
&\quad + \kappa \, \vec{L} \cdot \vec{L}
\end{aligned}
$$

Classical limit: potential energy surface

Equilibrium shape: two perpendicular
Jacobi coordinates of equal length

Equilateral triangle:
Oblate top



Rotation-Vibration Spectrum

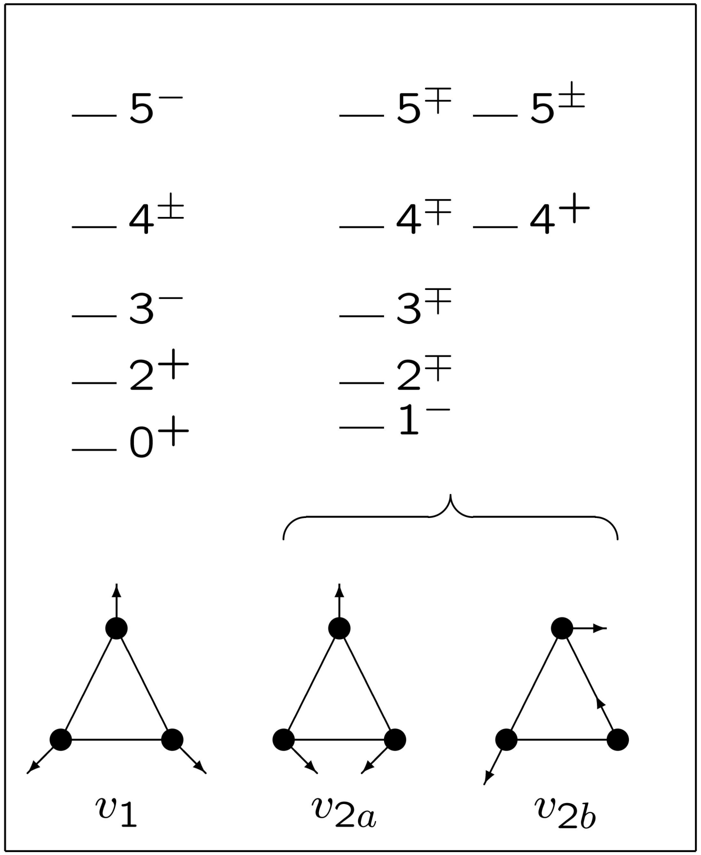
$$E = E_{\text{vib}} + E_{\text{rot}}$$

$$E_{\text{vib}} \approx \omega_1(\nu_1 + \frac{1}{2}) + \omega_2(\nu_2 + 1)$$

$$\omega_1 = 4NR^2\xi_1$$

$$\omega_2 = \frac{4NR^2}{1+R^2}\xi_2$$

$$E_{\text{rot}} = \kappa L(L+1)$$

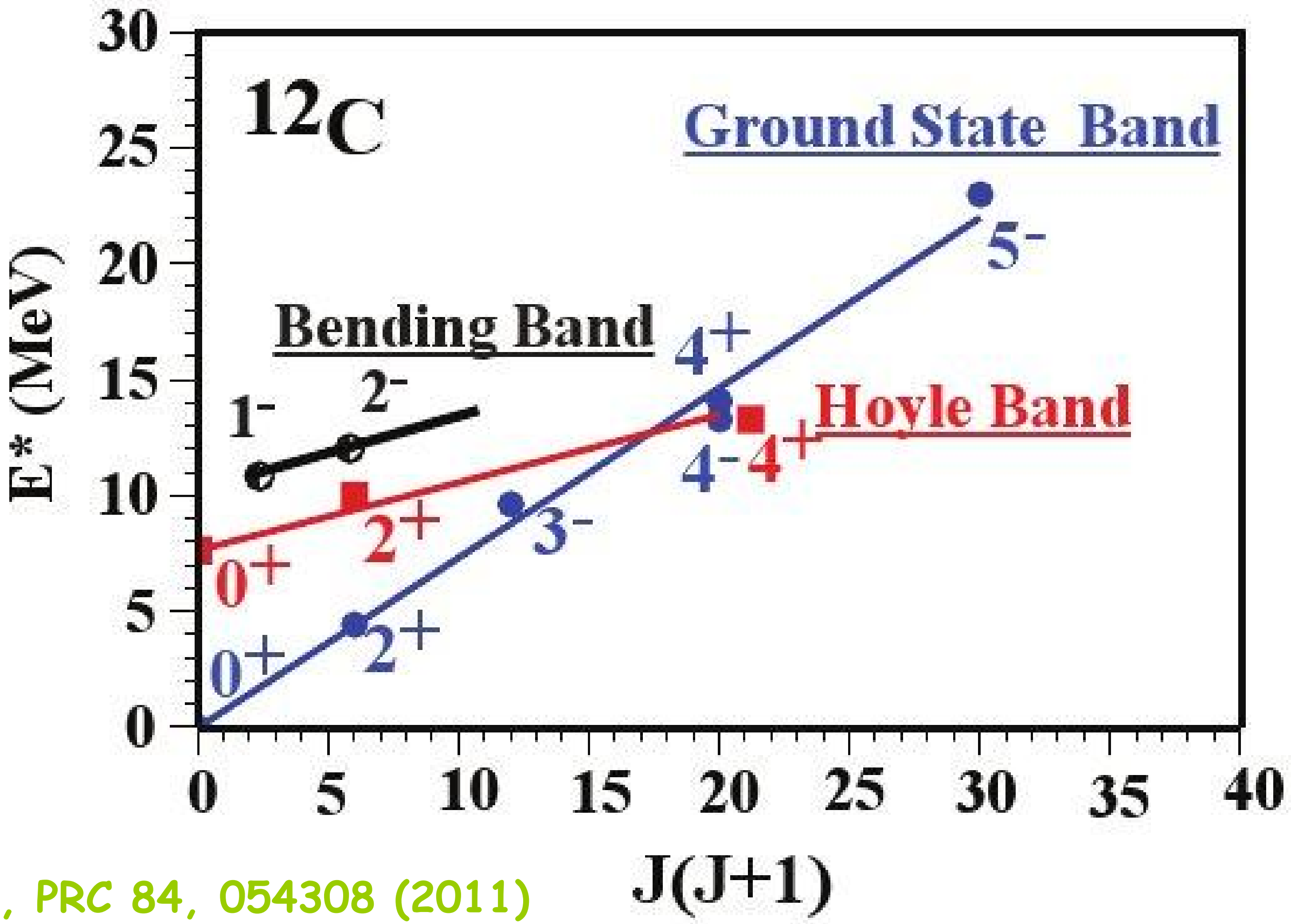


Rotational sequence

$$L^P = 0^+, 2^+, 3^-, 4^\pm, 5^-, \ldots$$

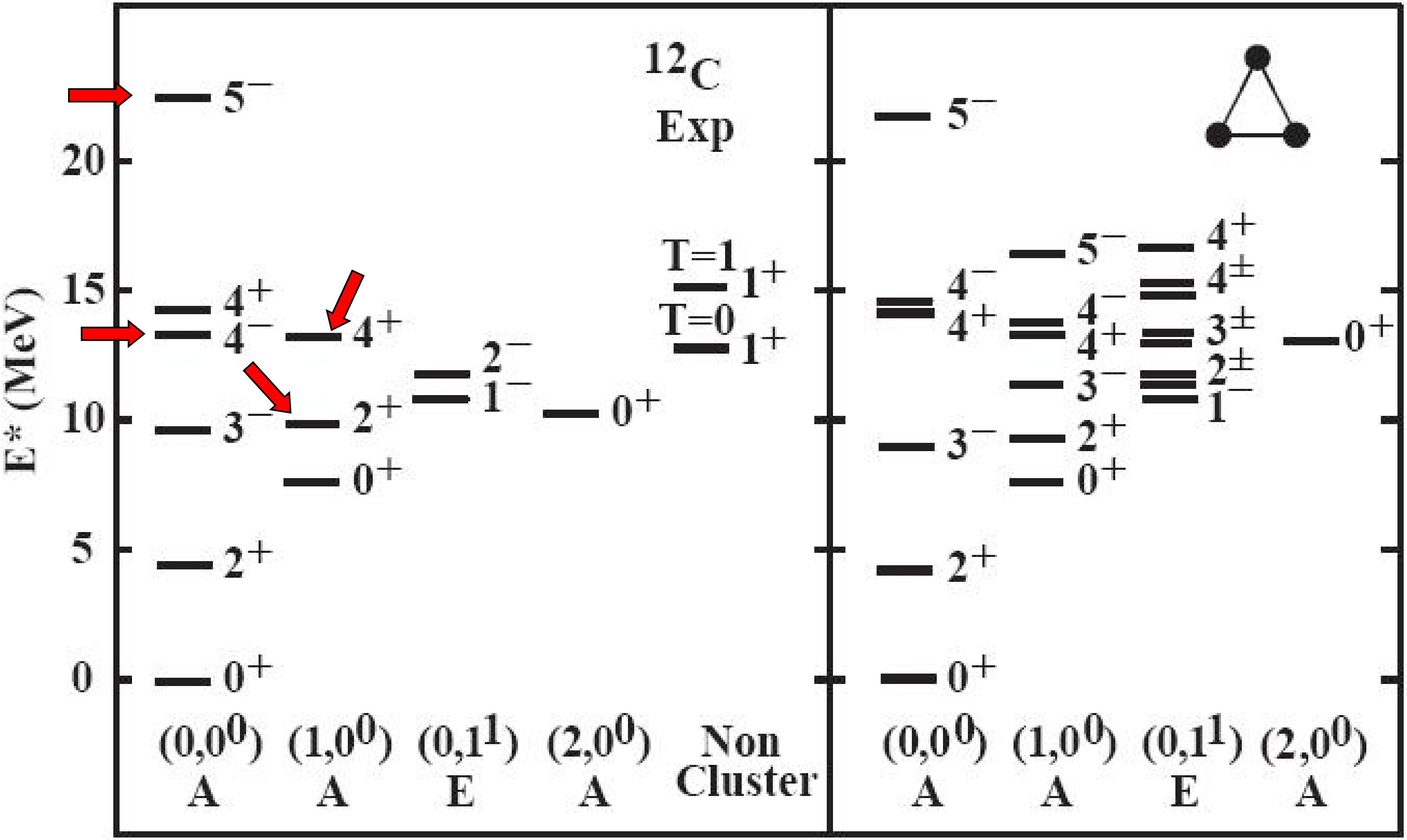
Fingerprint of triangular symmetry (D_{3h})

Bijker & Iachello, AP 298, 334 (2002)



Itoh et al, PRC 84, 054308 (2011)
Freer et al, PRC 86, 034320 (2012)
Zimmerman et al, PRL 110, 152502 (2013)

Marín-Lámbarri et al,
PRL 113, 012502 (2014)



Marín-Lámbarri, Bijker et al, PRL 113, 012502 (2014)

Electric Transitions

$$\rho(\vec{r}) = \frac{Ze}{k}\left(\frac{\alpha}{\pi}\right)^{3/2} \sum_{i=1}^{k} e^{-\alpha(\vec{r}-\vec{r}_i)^2}$$

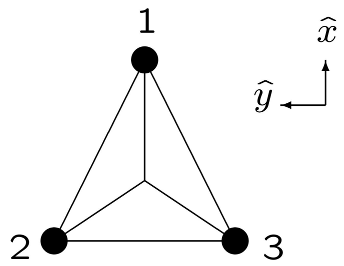
$$\vec{r}_1 = (\beta, \frac{\pi}{2}, 0)$$

$$\vec{r}_2 = (\beta, \frac{\pi}{2}, \frac{2\pi}{3})$$

$$\vec{r}_2 = (\beta, \frac{\pi}{2}, \frac{4\pi}{3})$$

$$\alpha = 0.56 \ 1/\text{fm}^2$$

$$\beta = 1.82 \ \text{fm}$$



$$\mathcal{F}_L(q) = c_L \, j_L(q\beta) \, e^{-q^2/4\alpha}$$

$$\langle r^2 \rangle^{1/2} = \sqrt{\beta^2 + 3/2\alpha}$$

$$B(EL; 0^+ \rightarrow L^P) = \frac{(Ze)^2 \, c_L^2 \, \beta^{2L}}{4\pi}$$

$$c_L^2 = \frac{2L+1}{3}\left[1 + 2P_L(-\tfrac{1}{2})\right]$$

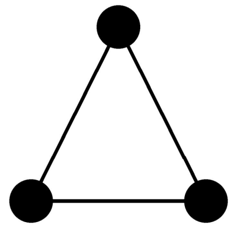
$$Q_{2^+} = +\frac{2}{7} Ze \, \beta^2$$

		ACM	Exp.	
^{12}C	$B(E2; 2_1^+ \rightarrow 0_1^+)$	7.8	7.63 ± 0.19	e^2fm^4
	$B(E3; 3_1^- \rightarrow 0_1^+)$	65	104 ± 14	e^2fm^6
	$B(E4; 4_1^+ \rightarrow 0_1^+)$	48		e^2fm^8
	$Q_{2_1^+}$	5.7	5.97 ± 0.30	$e$fm^2
	$\langle r^2 \rangle^{1/2}$	2.448	2.468 ± 0.012	fm

Bijker & Iachello, AP 298, 334 (2002)

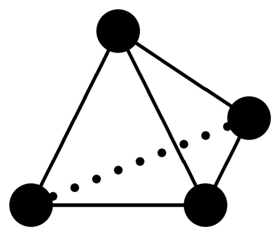
Z_2 Dumbbell ^{8}Be

NPA 973, 1 (2018)



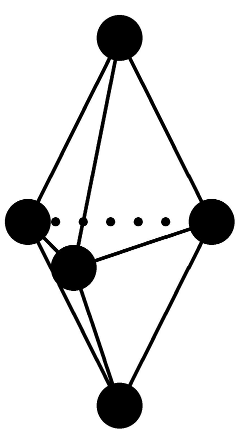
D_{3h} Triangle ^{12}C

PRC 61, 067305 (2000)
Ann Phys 298, 334 (2002)
PRL 113, 012502 (2014)



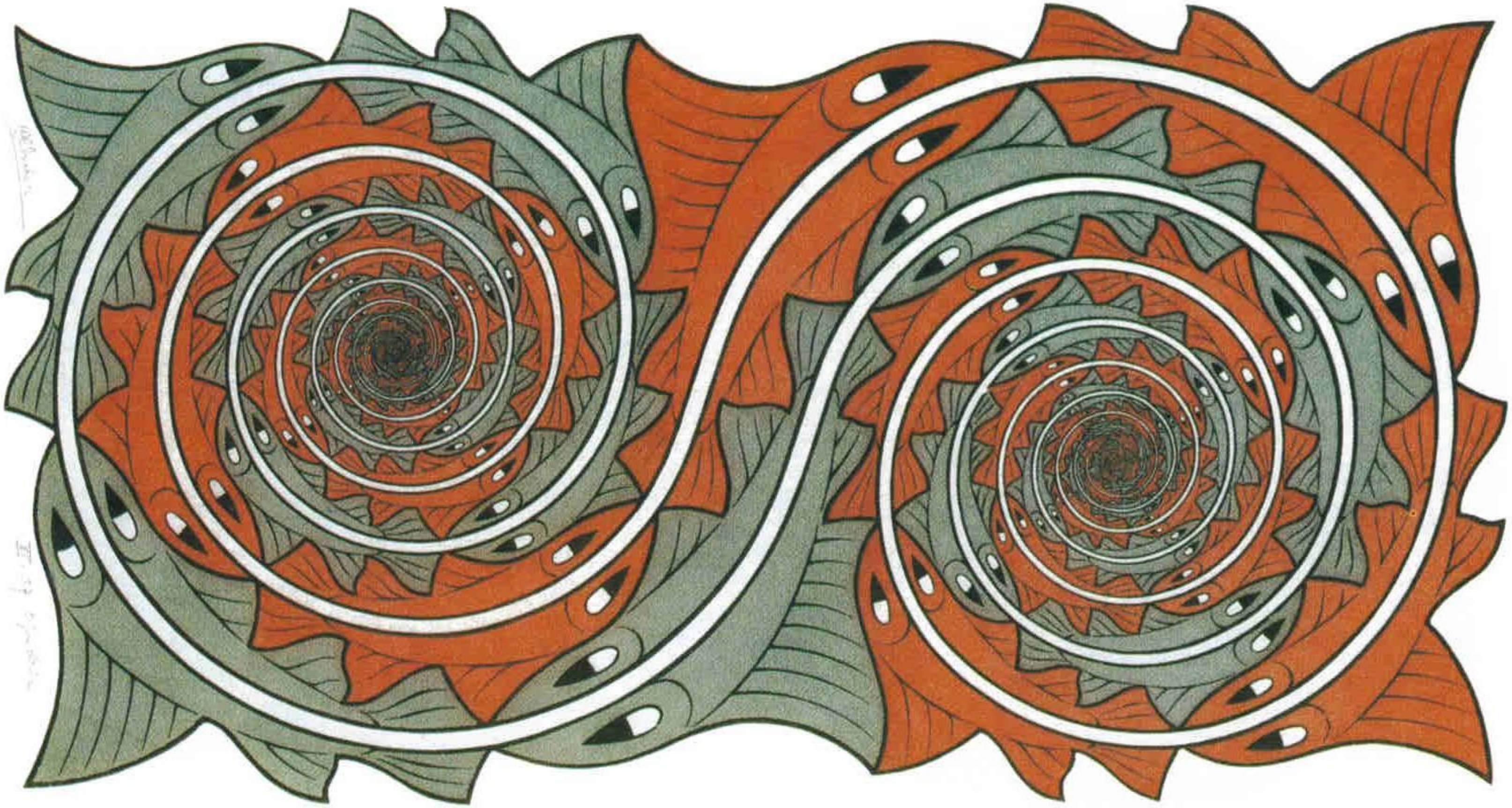
T_d Tetrahedron ^{16}O

PRL 112, 152501 (2014)
NPA 957, 154 (2017)



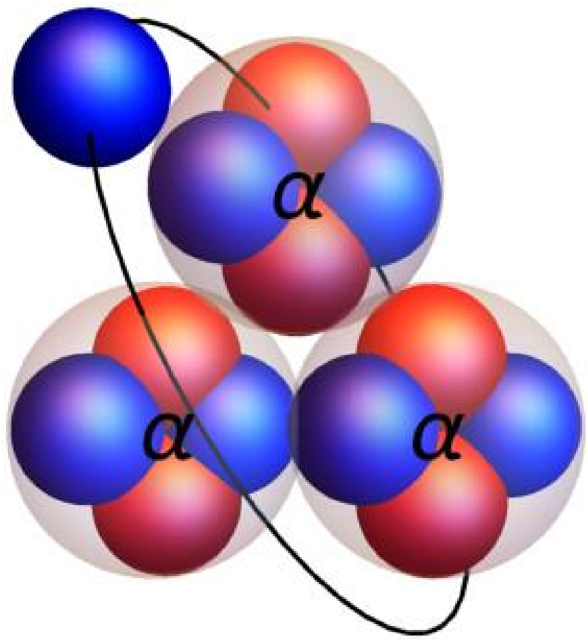
D_{3h} Bi-pyramid ^{20}Ne

NPA 1006, 122077 (2021)



Odd Cluster Nuclei

- What are the signatures of α-clustering in odd-mass nuclei?
- Cluster Shell Model (CSM)
- Splitting of sp levels in cluster potentials (analogous to Nilsson model)
- Double point groups



Cluster Shell Model

Cluster density

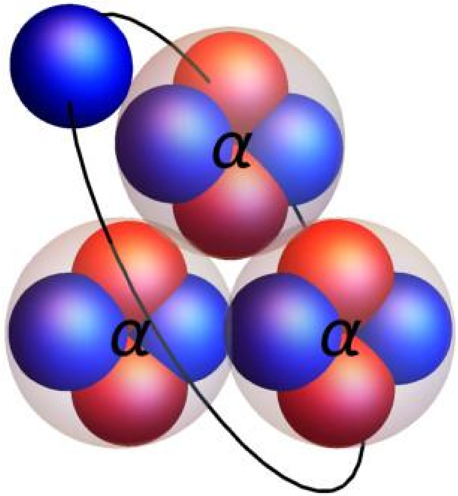
$$\rho(\vec{r}) = \frac{Ze}{k}\left(\frac{\alpha}{\pi}\right)^{3/2}\sum_{i=1}^{k}\exp\left[-\alpha\left(\vec{r}-\vec{r}_i\right)^2\right]$$

$$= \frac{Ze}{k}\left(\frac{\alpha}{\pi}\right)^{3/2} e^{-\alpha(r^2+\beta^2)} 4\pi \sum_{\lambda\mu} i_\lambda(2\alpha\beta r) Y_{\lambda\mu}(\theta,\phi) \sum_{i=1}^{k} Y^*_{\lambda\mu}(\theta_i,\phi_i)$$

$$\vec{r_i} = (\beta,\theta_i,\phi_i)$$

Cluster potential

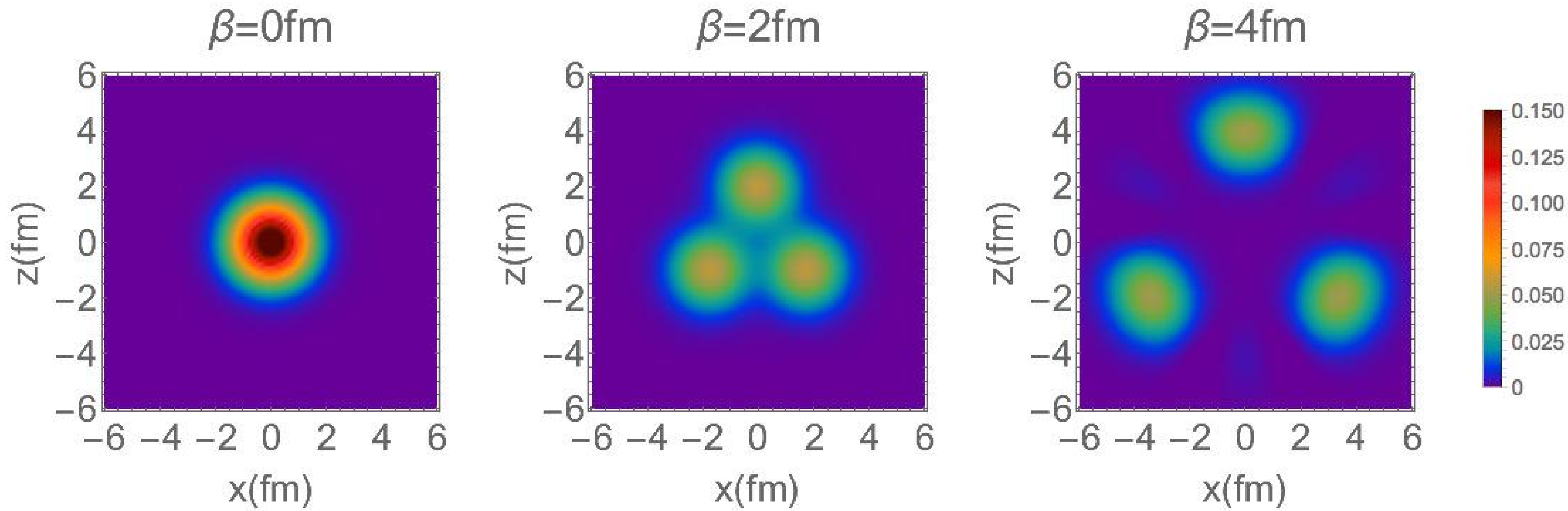
$$H = \sum_i \frac{\vec{p}_i^{\,2}}{2m} + V(\vec{r}) + V_{\mathrm{so}}(\vec{r}) + V_{\mathrm{C}}(\vec{r})$$



Della Rocca, Bijker & Iachello
NPA 966, 158 (2017)

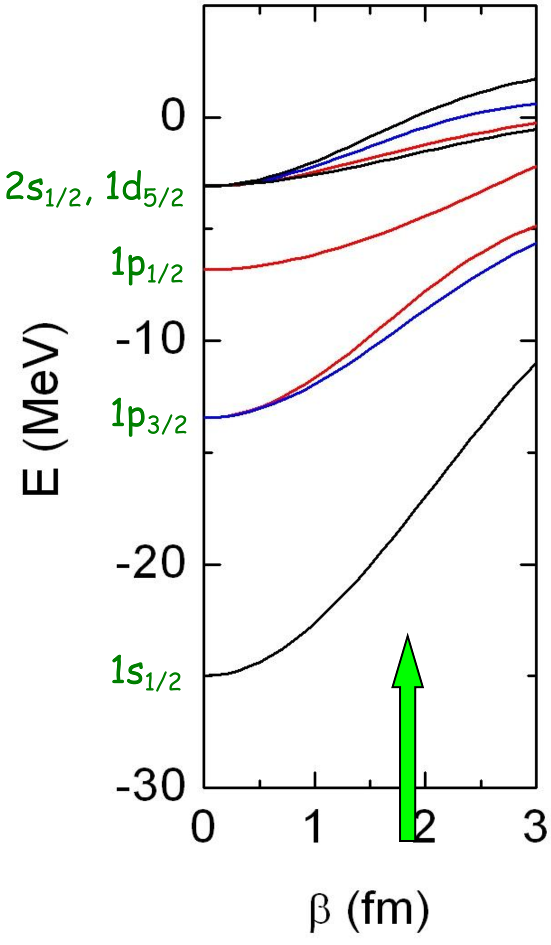
Adrian Horacio Santana Valdés
M.Sc. Thesis, UNAM (2018)

Densitities of 3-α cluster



Della Rocca, Bijker & Iachello, NPA 966, 158 (2017)

Triangular Symmetry



Symmetry: $\mathcal{D}'_{3h}$

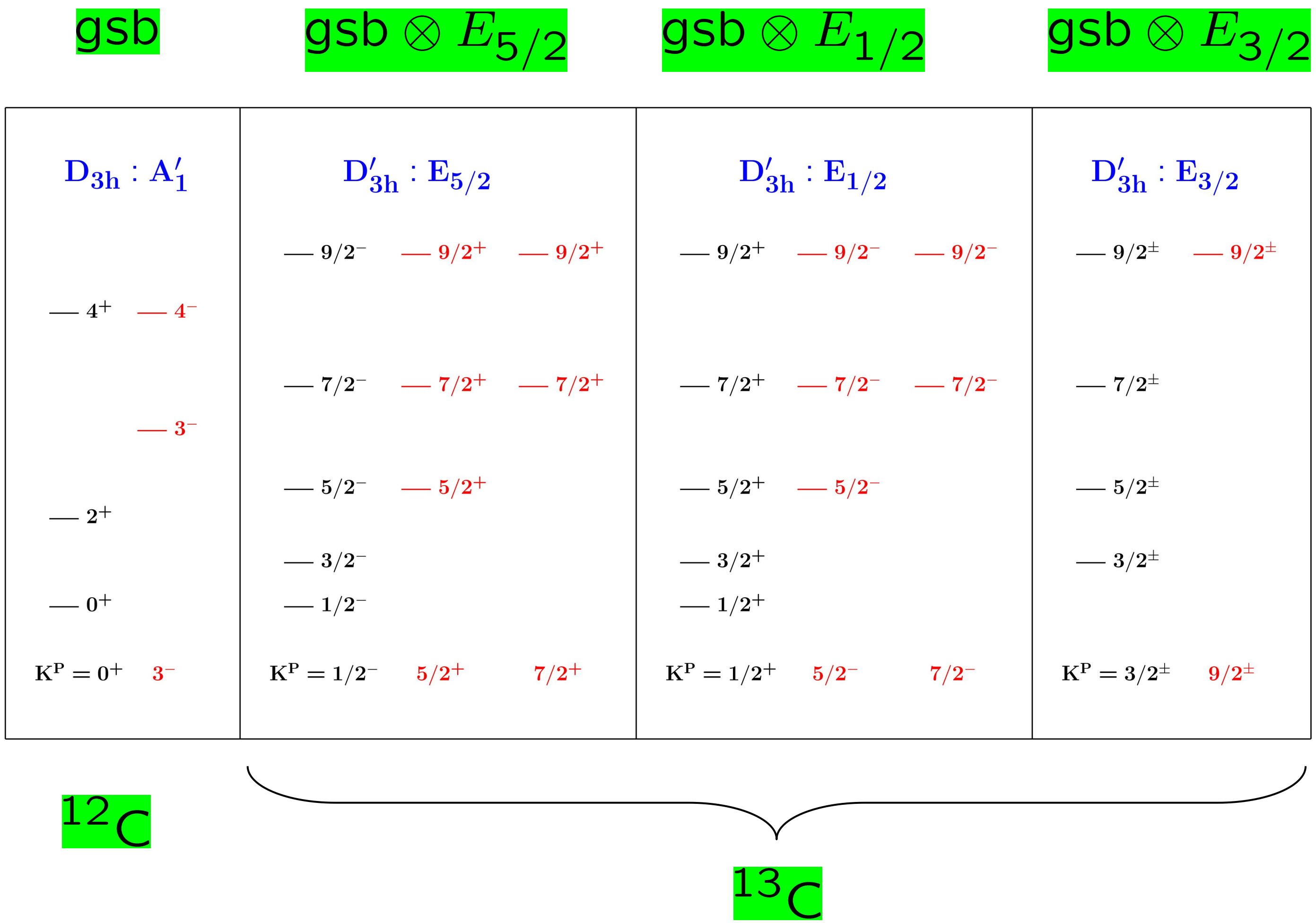
Ω	$E_{1/2}$	$E_{5/2}$	$E_{3/2}$
Deg	2	2	2
$s_{1/2}$	1		
$p_{1/2}$		1	
$p_{3/2}$		1	1
$d_{3/2}$	1		1
$d_{5/2}$	1	1	1

Bijker & Iachello,
PRL 122, 162501 (2019)

Vibrations

	^{12}C	$^{12}C \otimes E_{5/2}$	$^{12}C \otimes E_{1/2}$	$^{12}C \otimes E_{3/2}$
Bend	$\surd$ E'	$E_{3/2}$, $E_{1/2}$	$E_{3/2}$, $E_{5/2}$	$E_{5/2}$, $E_{1/2}$
Hoyle	$\surd$ A'_1	$\surd$ $E_{5/2}$	$E_{1/2}$	$E_{3/2}$
Gsb	$\surd$ A'_1	$\surd$ $E_{5/2}$	$\surd$ $E_{1/2}$	$E_{3/2}$

Rotations



gsb
gsb $\otimes E_{5/2}$
gsb $\otimes E_{1/2}$
gsb $\otimes E_{3/2}$
$D_{3h} : A'_1$
$D'_{3h} : E_{5/2}$
$D'_{3h} : E_{1/2}$
$D'_{3h} : E_{3/2}$
4^+ 4^-
3^-
2^+
0^+
$K^P = 0^+$ 3^-
$9/2^-$ $9/2^+$ $9/2^+$
$7/2^-$ $7/2^+$ $7/2^+$
$5/2^-$ $5/2^+$
$3/2^-$
$1/2^-$
$K^P = 1/2^-$ $5/2^+$ $7/2^+$
$9/2^+$ $9/2^-$ $9/2^-$
$7/2^+$ $7/2^-$ $7/2^-$
$5/2^+$ $5/2^-$
$3/2^+$
$1/2^+$
$K^P = 1/2^+$ $5/2^-$ $7/2^-$
$9/2^\pm$ $9/2^\pm$
$7/2^\pm$
$5/2^\pm$
$3/2^\pm$
$K^P = 3/2^\pm$ $9/2^\pm$
^{12}C
^{13}C

Rotational Energy

$$H_{\text{rot}} = \frac{L_1^2 + L_2^2}{2\mathcal{I}} + \frac{L_3^2}{2\mathcal{I}_3} = \sum_{i=1}^{2} \frac{(J_i - j_i)^2}{2\mathcal{I}} + \frac{(J_3 - j_3)^2}{2\mathcal{I}_3}$$

Wave function

$$|\Omega,\mu;J^P K M\rangle = \frac{1}{\sqrt{2}}\left(1 + \hat{P}\,\mathrm{e}^{i\pi J_2}\,\hat{p}\,\mathrm{e}^{-i\pi j_2}\right)|J^P K M\rangle\,|\Omega,\mu\rangle$$

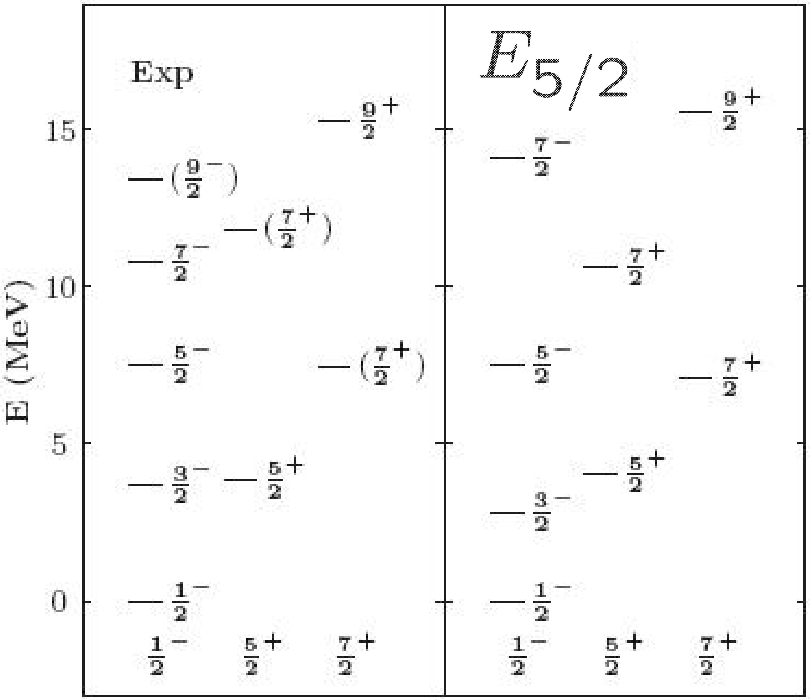
Energies

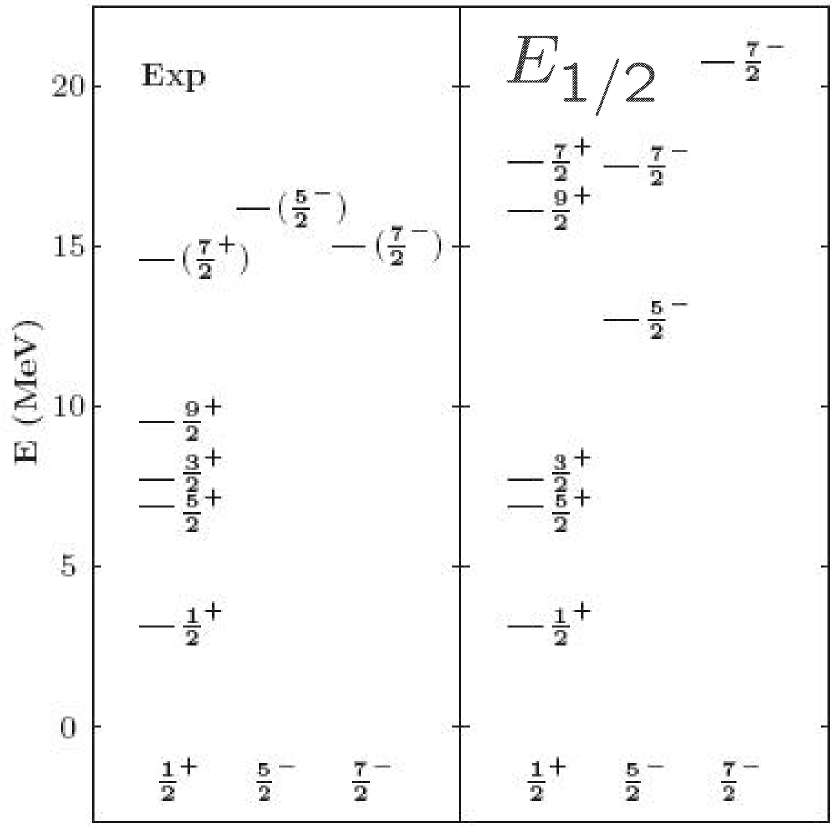
$$E_\Omega(J) \approx \frac{1}{2\mathcal{I}}\left[J(J+1) - 2K^2 + \delta_{K,1/2}\,a_\Omega(-1)^{J+1/2}\left(J+\frac{1}{2}\right)\right]$$

a_Ω Decoupling parameter

Generalized formula for triangular symmetry

Bandas Rotacionales en ^{13}C



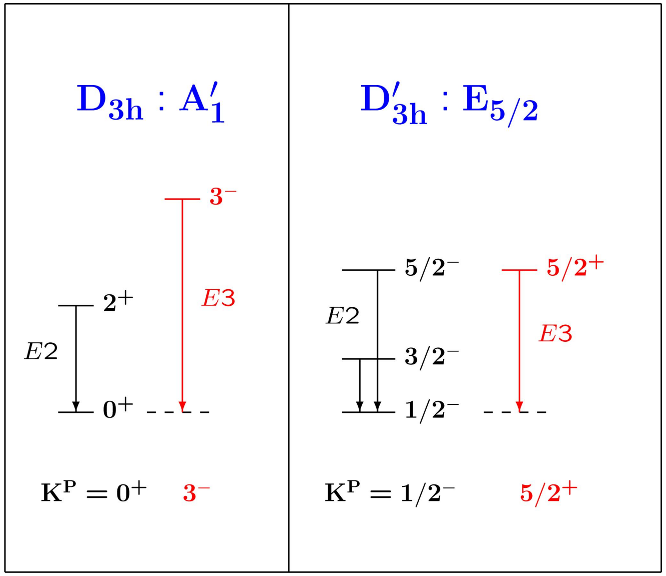


Electric Transitions

$$\frac{B(E3;3_1^- \to 0_1^+)}{B(E2;2_1^+ \to 0_1^+)} = \frac{5}{2}\beta^2$$

Dominated by collective part

$$\begin{aligned} B(E2;3/2_1^- \to 1/2_1^-) &= B(E2;5/2_1^- \to 1/2_1^-) \\ &= B(E2;2_1^+ \to 0_1^+) \\ B(E3;5/2_1^+ \to 1/2_1^-) &= B(E3;3_1^- \to 0_1^+) \\ Q_{5/2_1^-} = \frac{10}{7} Q_{3/2_1^-} &= Q_{2_1^+} \end{aligned}$$

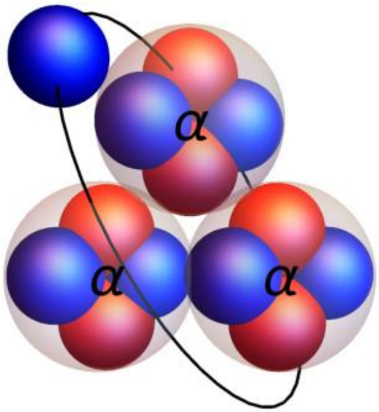


	$B(EL)$	Th	Exp	
^{12}C	$B(E2;2_1^+ \to 0_1^+)$	7.8	7.63 ± 0.19	e^2fm^4
	$B(E3;3_1^- \to 0_1^+)$	65.0	104 ± 14	e^2fm^6
	$Q_{2_1^+}$	5.7	5.97 ± 0.30	$e$fm^2
^{13}C	$B(E2;3/2_1^- \to 1/2_1^-)$	7.8	6.4 ± 1.5	e^2fm^4
	$B(E2;5/2_1^- \to 1/2_1^-)$	7.8	5.6 ± 0.4	e^2fm^4
	$B(E3;5/2_1^+ \to 1/2_1^-)$	65.0	100 ± 40	e^2fm^6
	$Q_{5/2_1^-}$	5.7		$e$fm^2
	$Q_{3/2_1^-}$	4.0		$e$fm^2

Form Factors

Charge distribution

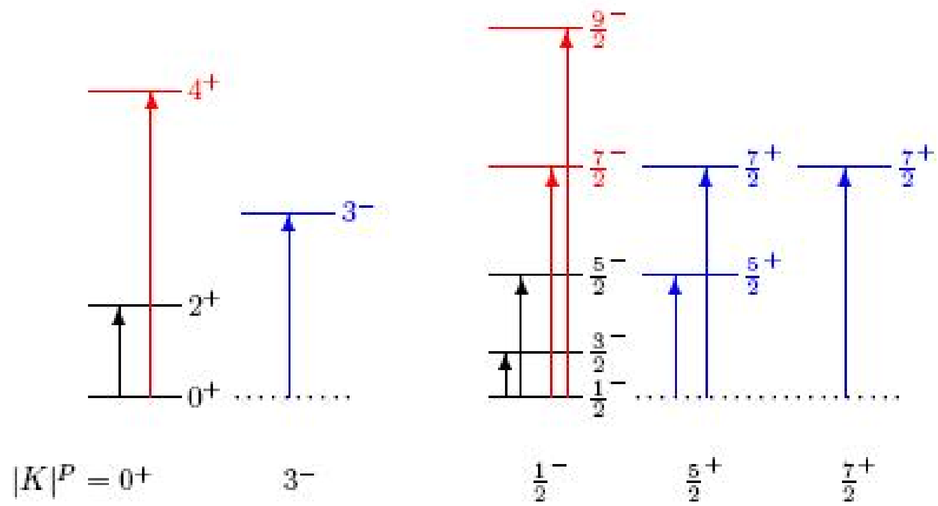
$$\begin{aligned}\rho_{\text{ch}}(\vec{r}) &= \rho^{\text{c}}_{\text{ch}}(\vec{r}) + \rho^{\text{sp}}_{\text{ch}}(\vec{r})\\ \rho^{\text{c}}_{\text{ch}}(\vec{r}) &= \frac{(Ze)_{\text{c}}}{3}\left(\frac{\alpha}{\pi}\right)^{3/2}\sum_{i=1}^{3}\text{e}^{-\alpha(\vec{r}-\vec{r}_i)^2}\\ \rho^{\text{sp}}_{\text{ch}}(\vec{r}) &= \tilde{e}\,\delta(\vec{r}-\vec{r}_{\text{sp}})\end{aligned}$$



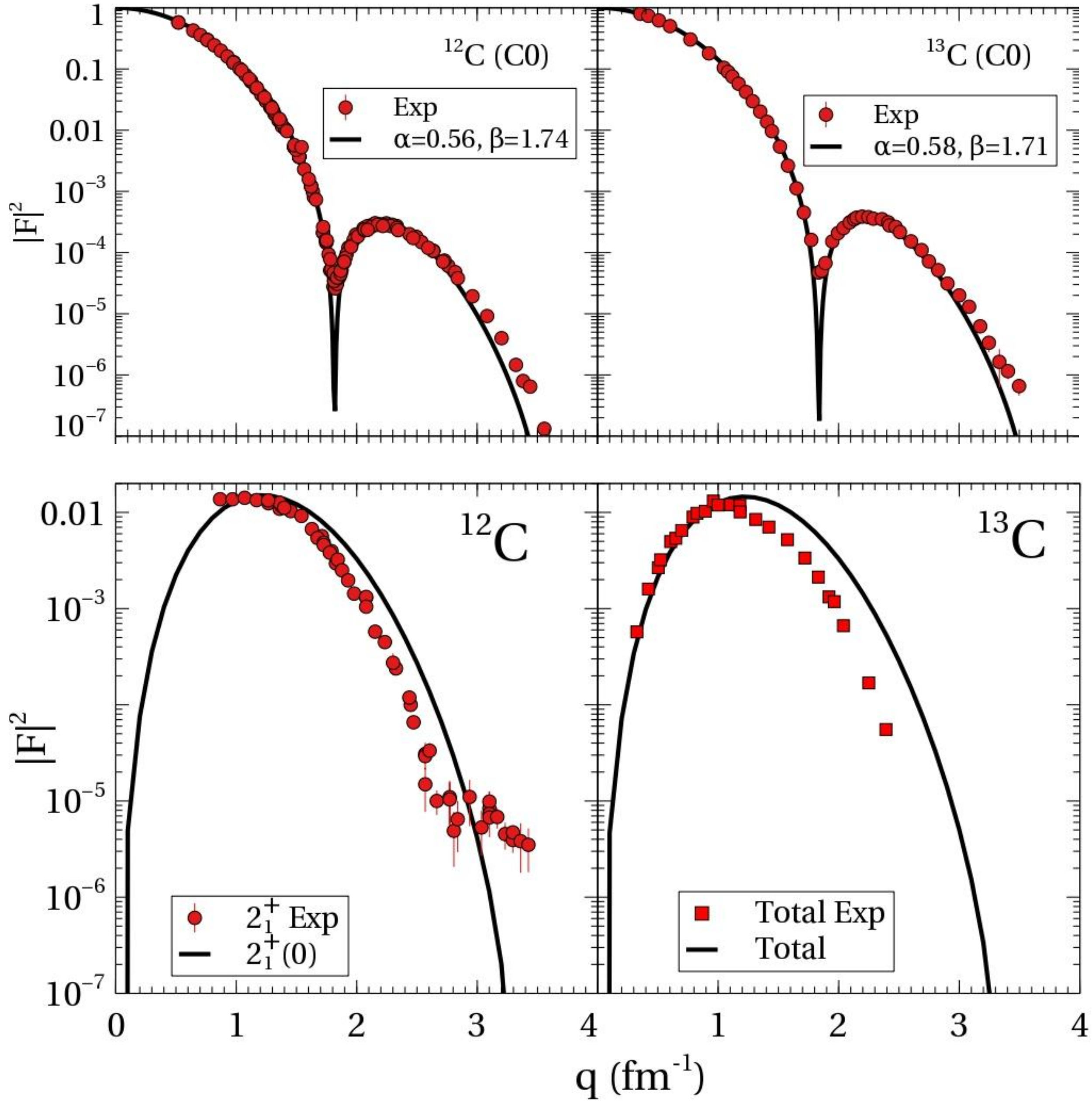
Form factors

$$\left\langle \psi_f \left| \int \rho_{\text{ch}}(\vec{r})\,\text{e}^{i\vec{q}\cdot\vec{r}} d^3r \right| \psi_i \right\rangle$$

Correspondence ^{12}C y ^{13}C



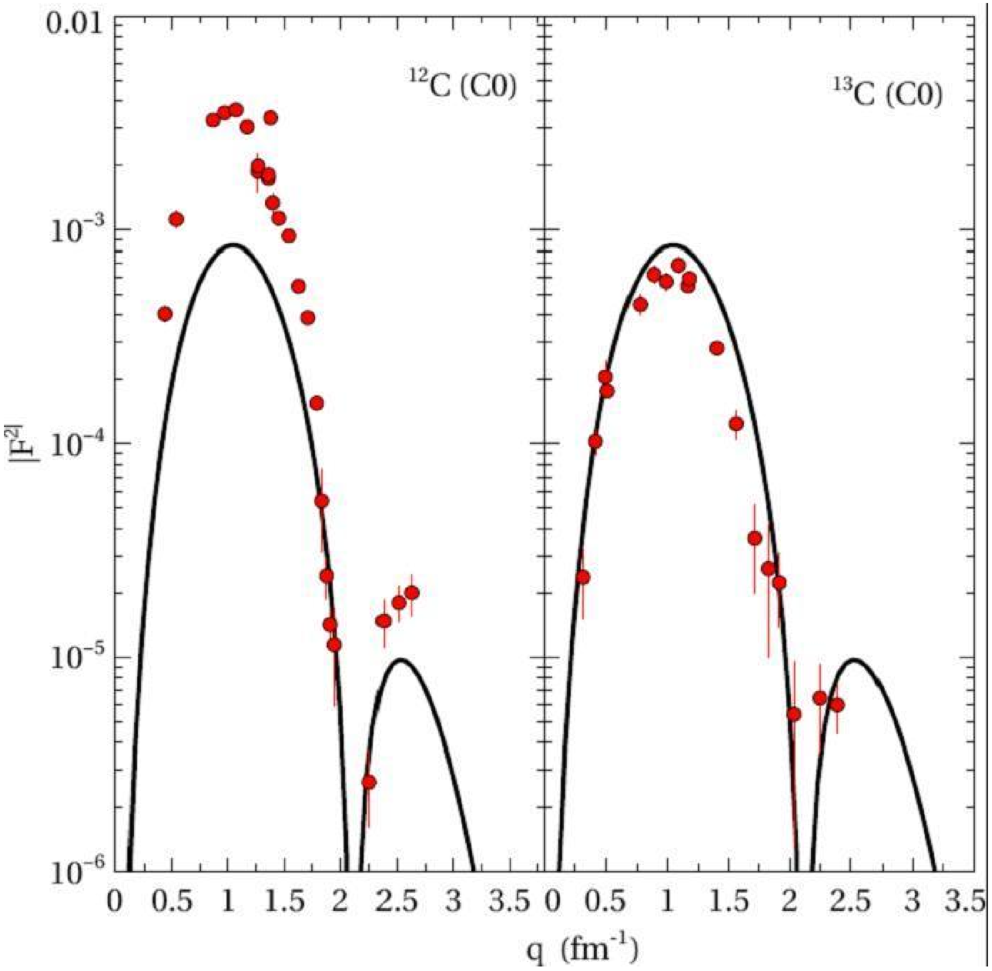
$$\left|F_{C\lambda}(^{12}\mathrm{C}, q; 0^+, 0 \rightarrow L^P, K)\right|^2 = \sum_{I,K'} \left|F_{C\lambda}(^{13}\mathrm{C}, q; \tfrac{1}{2}^-, \tfrac{1}{2} \rightarrow I^{P'}, K')\right|^2_{\mathrm{c}}$$



Factor de forma elástico C0

Factor de forma inelástico C2

Analogue of Hoyle state in ^{13}C



^{12}C : $F(0_1^+ \to 0_2^+)$

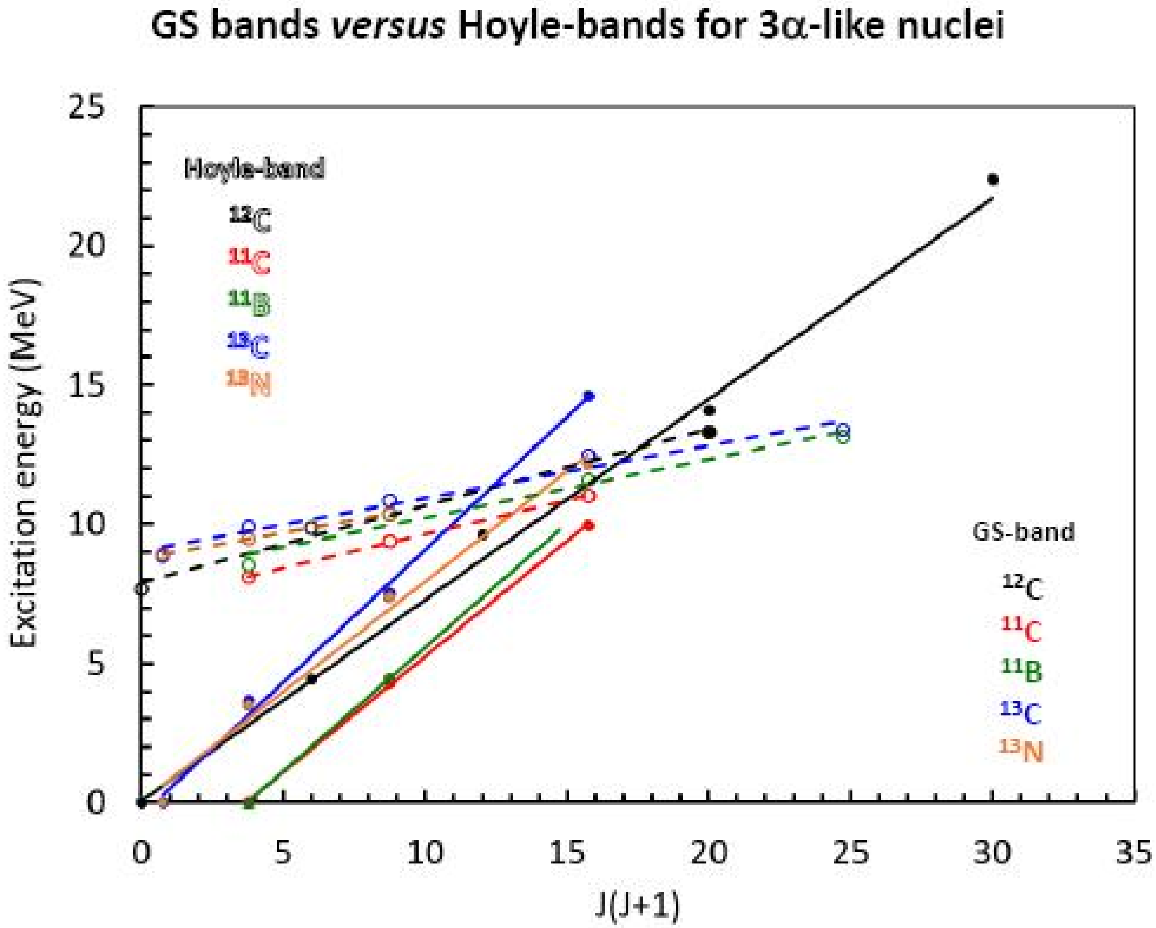
0_2^+ @ 7.65 MeV

^{13}C : $F(1/2_1^- \to 1/2_2^-)$

$1/2_2^-$ @ 8.86 MeV

Santana & Bijker, Phys. Lett. B 843, 138026 (2023)

Energy Systematics

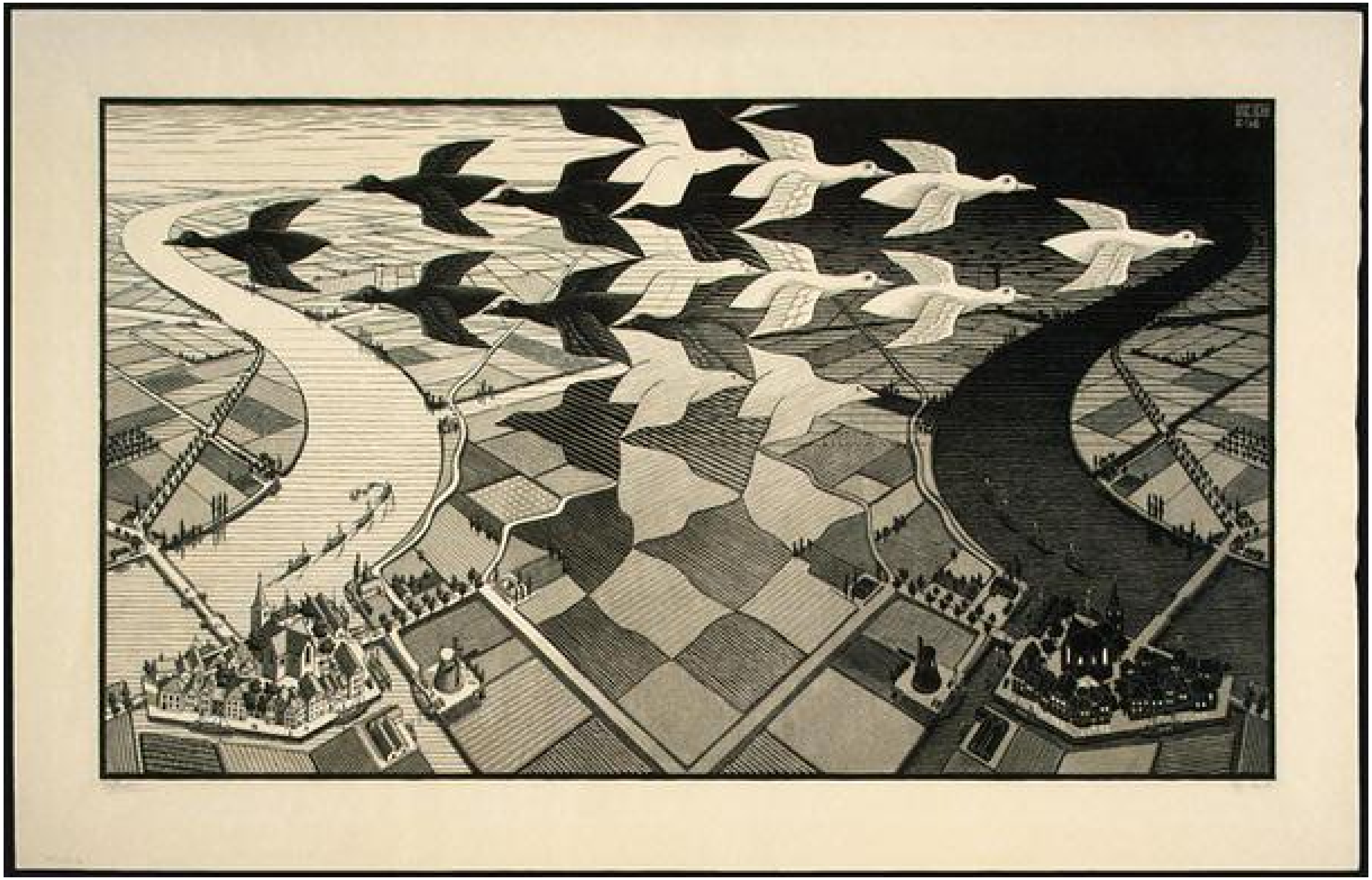


Courtesy
Ivano Lombardo, 2019

Coincides in the identification of the analogue of the Hoyle state in ^{13}C

$4n+1$	Nuclei	References
$n=2$	^{9}Be, ^{9}B	NPA 973, 1 (2018)
$n=3$	^{13}C	PRL 122, 162501 (2019)
		EPJ-ST 229, 2353 (2020)
		PLB 843, 138026 (2023)
	^{13}N	B.Sc. Thesis, Villavicencio (2025)
$n=5$	^{21}Ne, ^{21}Na	NPA 1010, 122193 (2021)

$4n+2$	Nuclei	References
$n=2$	^{10}Be	JPCS 2619, 012006 (2023)
	^{10}B, ^{10}C	in progress, Omar Díaz



Summary and Conclusions I

- Algebraic Cluster Model
- SGA of U(3n-2) for n-body systems
- Discrete and continuous symmetries
- Rotational bands fingerprints of point group symmetries
- Hoyle band: linear, bent or triangular?
- Applications in molecular, nuclear, hadronic physics

Summary and Conclusions II

- Cluster Shell Model: ^{13}C
- Symmetries
- Rotational bands: fingerprints of a triangular configuration of three alpha particles plus a neutron
- Large electric transitions
- Form factor to identify the analogue of Hoyle state in ^{13}C
- Benchmark for microscopic studies of nuclear clustering

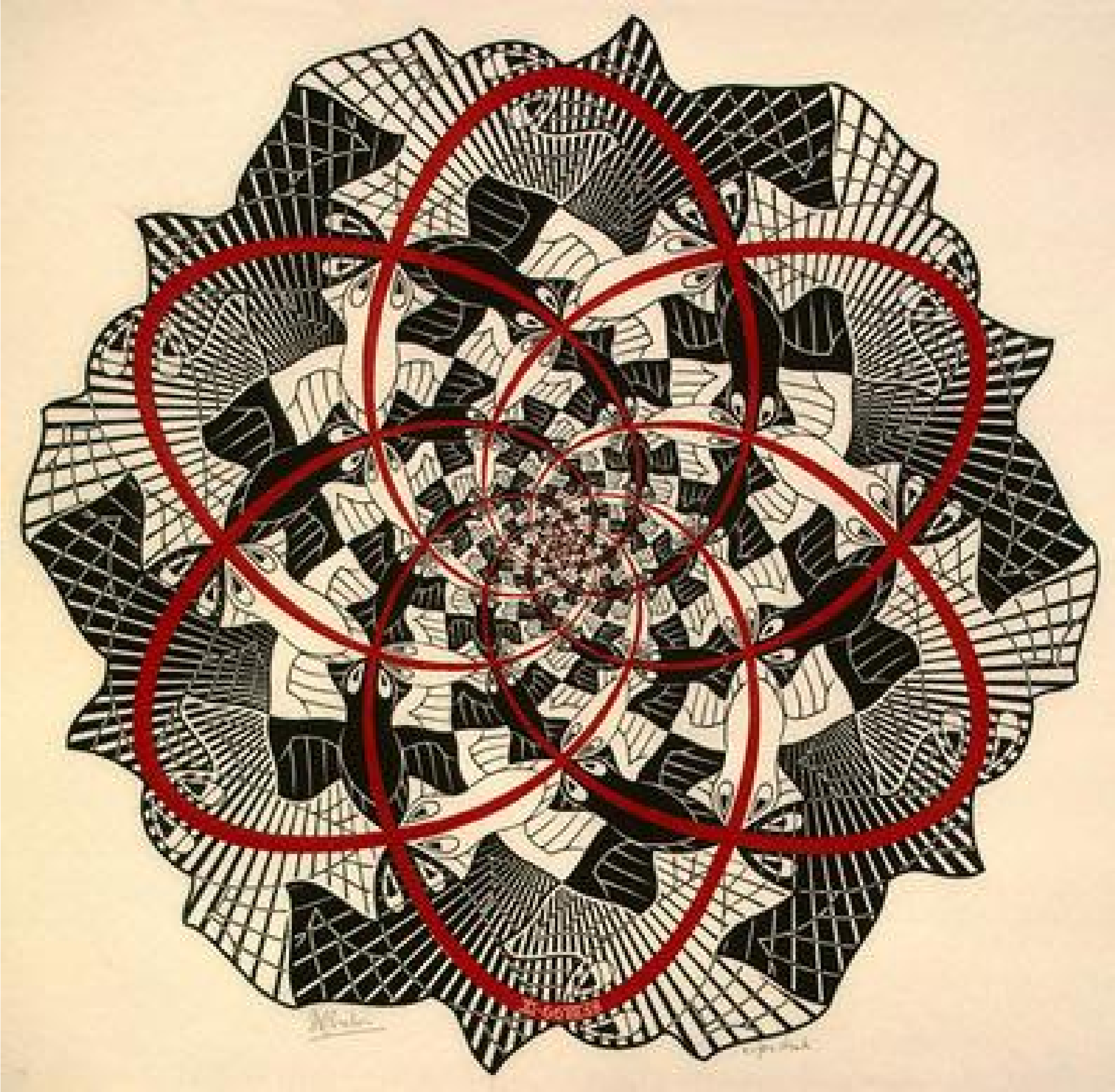
Bijker & Iachello, PRL 122, 162501 (2019)

Who's Who?

- UNAM, Mexico
- Adrian Santana
- Omar Díaz

- UABJ, Mexico
- Emiliano Villavicencio

- Yale
- Valeria Della Rocca
- Francesco Iachello



Pauli Principle

- ACM: effective α-α interaction of the Morse type
- CSM similar to Nilsson model
- Nucleon moves in the deformed field generated by the cluster of α particles