The role of non-stabilizerness in Bell's inequalities violations

S. Cusumano

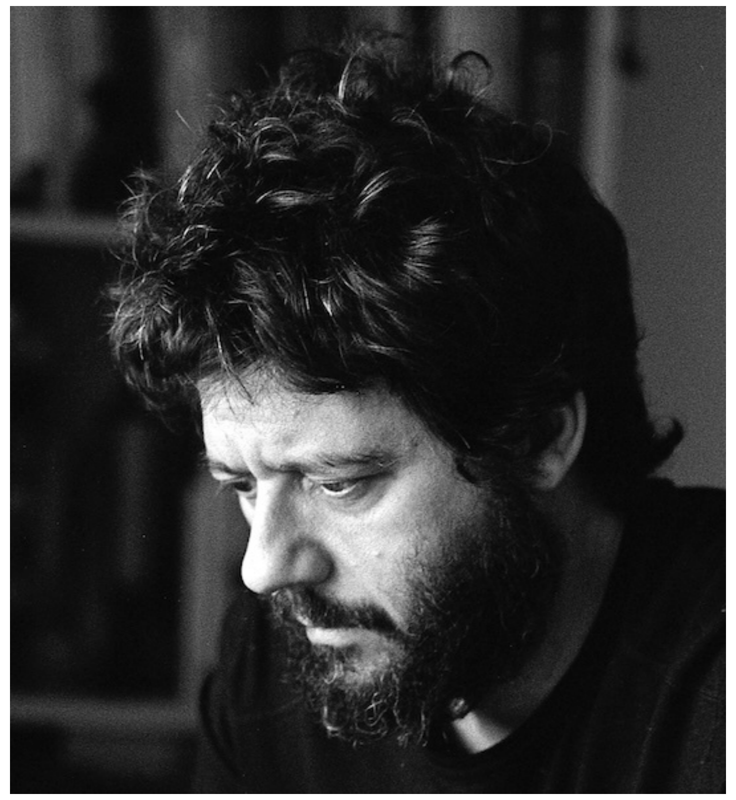
Quantum Computing@INFN, Padova 31/10/2024

Università degli Studi di Napoli Federico II

NQSTI National Quantum Science and Technology Institute

INFN Napoli
Istituto Nazionale di Fisica Nucleare
Sezione di Napoli

The Quantum Information group @Naples



Prof. A. Hamma

Dr. S. Cusumano

Dr. L. Campos-Venuti

Dr. J. Odavic

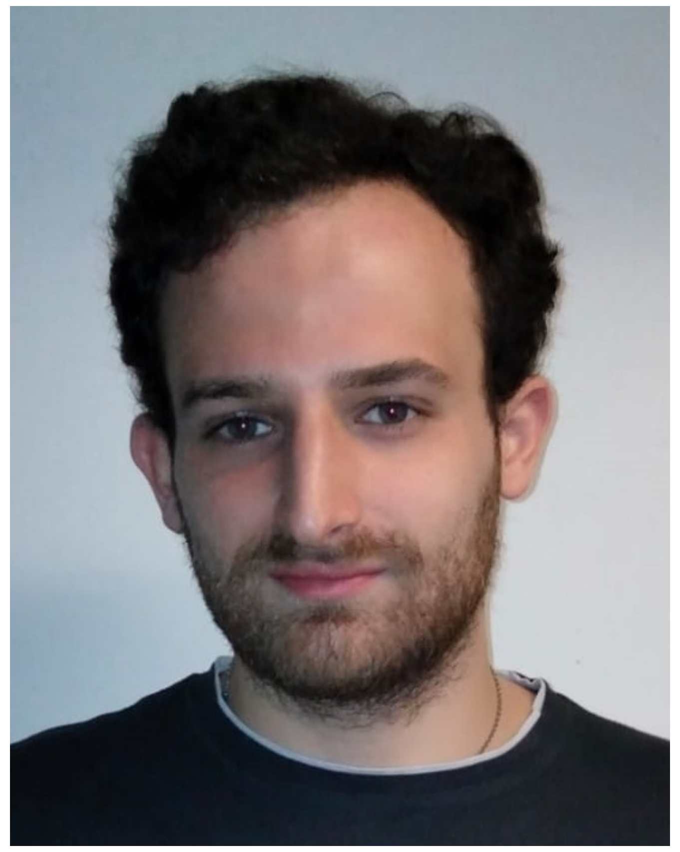
Dr. A. Amabile

B. Jasser
PhD

G. Esposito
PhD

D. Iannotti
PhD

S. Cepollaro
PhD



M. Viscardi
PhD

M. Nazzaro
PhD

Quantum Information @Naples

-Quantum Complexity
-Quantum Algorithms
-Quantum Chaos
-Quantum Thermodynamics
-Quantum Many Body Theory
-Quantum Spacetime
-Outreach and didactics of physics

Today @INFN Padova

- Brief summary of Bell's inequalities
- Non-stabilizerness and Stabilizer Rényi Entropy
- Isospectral twirling over Clifford and Unitary group
- Bell's violations in random quantum circuits

https://nqsti.it/news/qtris-il-gioco-della-meccanica-quantistica

Università degli Studi di Napoli Federico II

NQSTI National Quantum Science and Technology Institute

INFN Napoli
Istituto Nazionale di Fisica Nucleare
Sezione di Napoli

The Bell's inequality setting

Bell's inequality allows for the discrimination between quantum theory and a hidden variable theory, excluding the latter

Bell's inequality violations imply the presence of entanglement, i.e. violations can be used as entanglement witnesses

Usual CHSH setting to test Bell's inequalities

$$A_1 B_1 + A_2 B_1 - A_1 B_2 + A_2 B_2$$

If observer A chooses to measure along directions:

$$A_1 = X \qquad A_2 = Z$$

Then observer B has to measure along the directions:

$$B_1 = -\frac{X + Z}{\sqrt{2}} \qquad B_2 = \frac{-X + Z}{\sqrt{2}}$$

in order to observe the maximal violation, i.e. to achieve the Tsirelson bound.

No violations without entanglement or magic

It is well known that without entanglement it is impossible to violate Bell's inequalities

$$\mathrm{Tr}\left[(X_1X_2 + Z_1X_2 - X_1Z_2 + Z_1Z_2)(U_1 \otimes U_2)|00\rangle\langle 00|(U_1 \otimes U_2)^\dagger\right] \leq 2$$

It is also possible to prove that entanglement alone is not sufficient to violate the bound

$$\mathrm{Tr}\left[(X_1X_2 + Z_1X_2 - X_1Z_2 + Z_1Z_2)C|00\rangle\langle 00|C^\dagger\right] \leq 2$$

True also for mixed states

To violate Bell's inequalities we need a unitary operator which is neither Clifford nor local

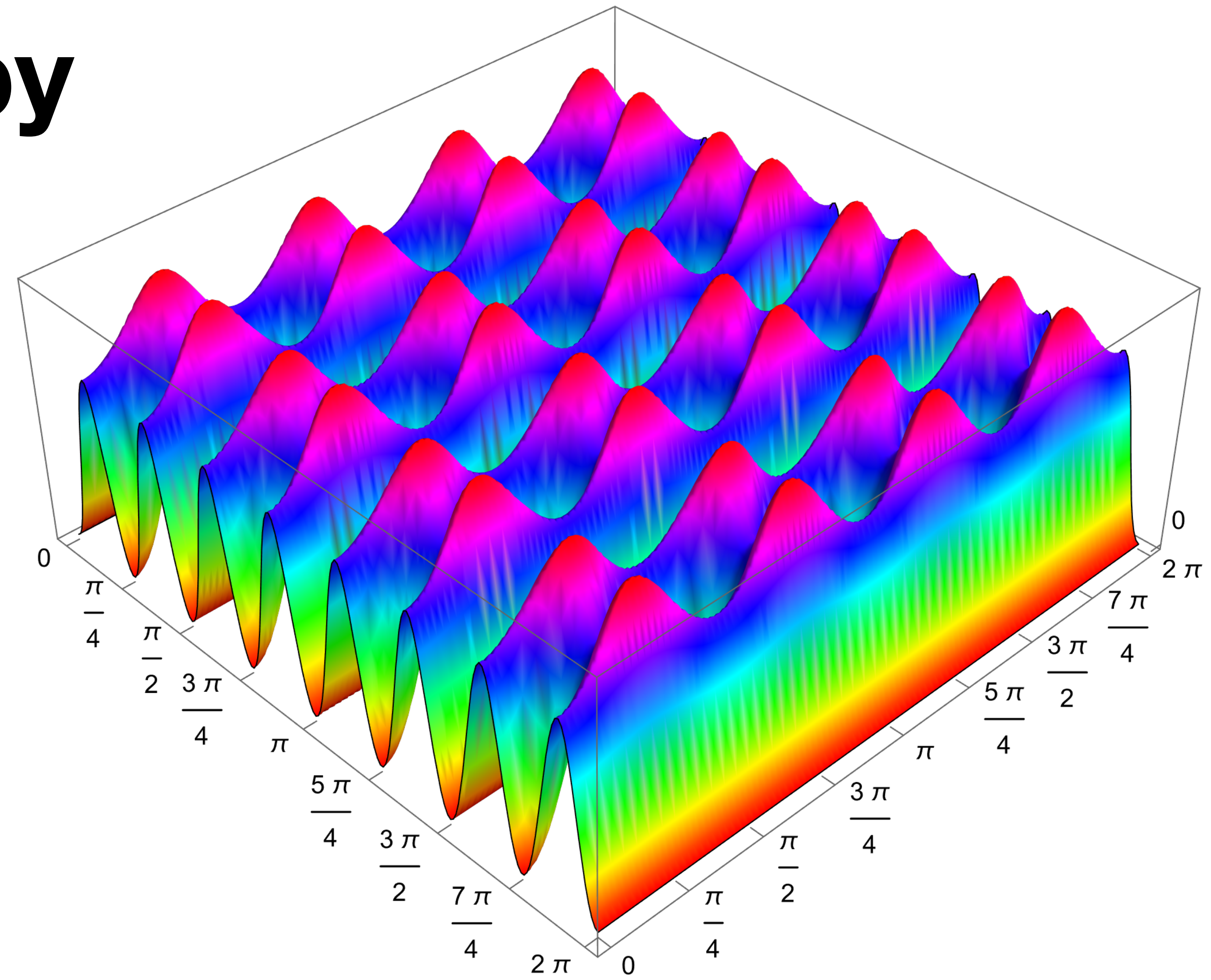
$$\mathrm{Tr}\left[(X_1X_2 + Z_1X_2 - X_1Z_2 + Z_1Z_2)U|00\rangle\langle 00|U^\dagger\right]$$

Università degli Studi di Napoli Federico II

NQSTI National Quantum Science and Technology Institute

INFN Napoli Istituto Nazionale di Fisica Nucleare Sezione di Napoli

The Stabilizer Rényi Entropy

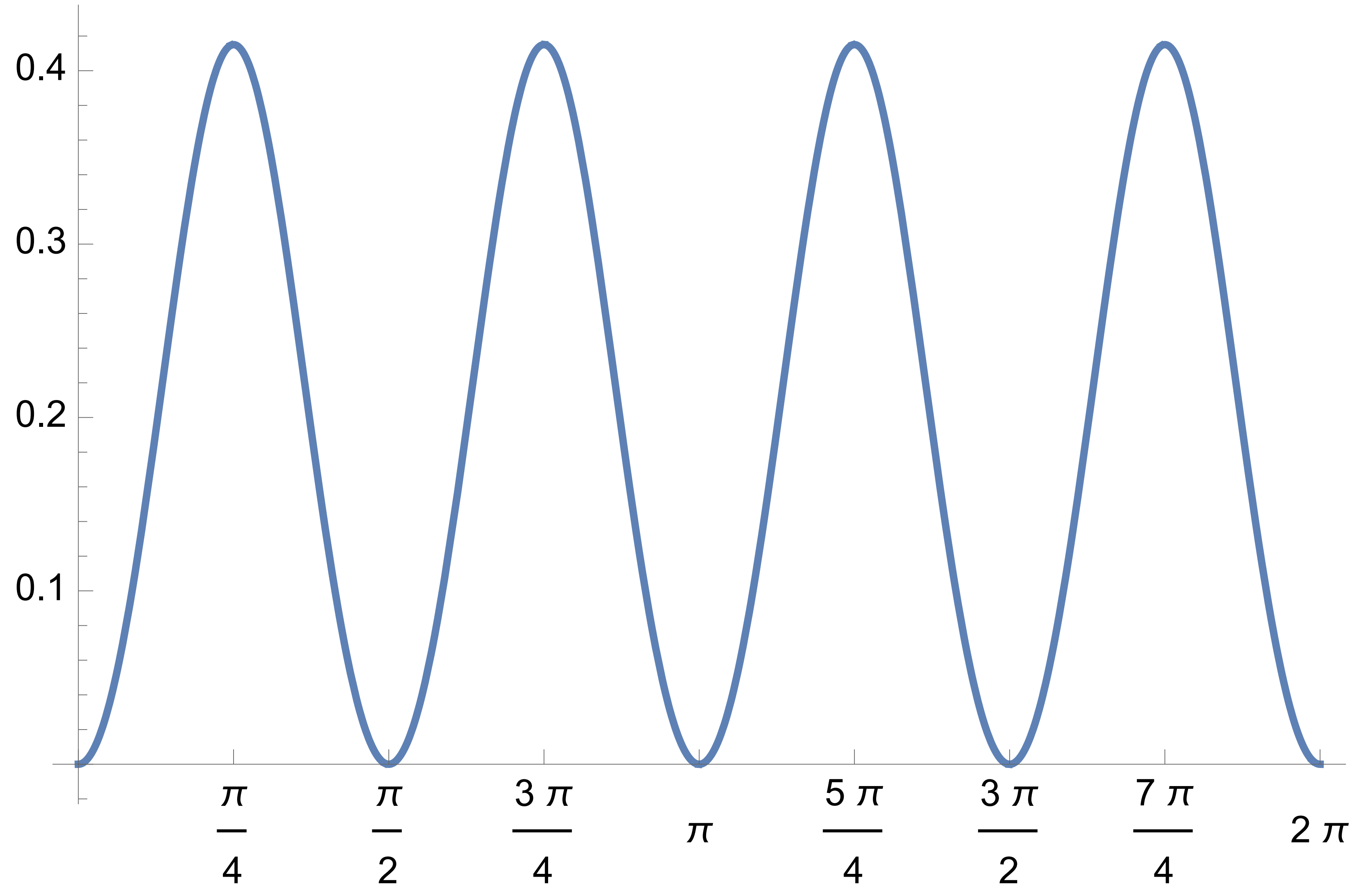


Leone, Oliviero, Hamma, Phys. Rev. Lett. 128, 050402 (2022)

$$\mathcal{P}_n = \{I, X, Y, Z\}^{\otimes n} \qquad \Xi_P(|\psi\rangle) = \frac{\langle\psi|P|\psi\rangle^2}{d} \qquad \sum_{P\in\mathcal{P}_n} \Xi_P = 1$$

$$M_\alpha(|\psi\rangle) = (1-\alpha)^{-1}\log\sum_{P\in\mathcal{P}_n}\Xi_P^\alpha(|\psi\rangle) - \log d$$

$$M_2(|\psi\rangle) = -\log\sum_{P\in\mathcal{P}_n}\Xi_P^2(|\psi\rangle) - \log d$$



For a generic 1 qubit state: $|\psi\rangle = \dfrac{\cos\theta|0\rangle + e^{-i\varphi}\sin\theta|1\rangle}{2}$

Università degli Studi di Napoli Federico II

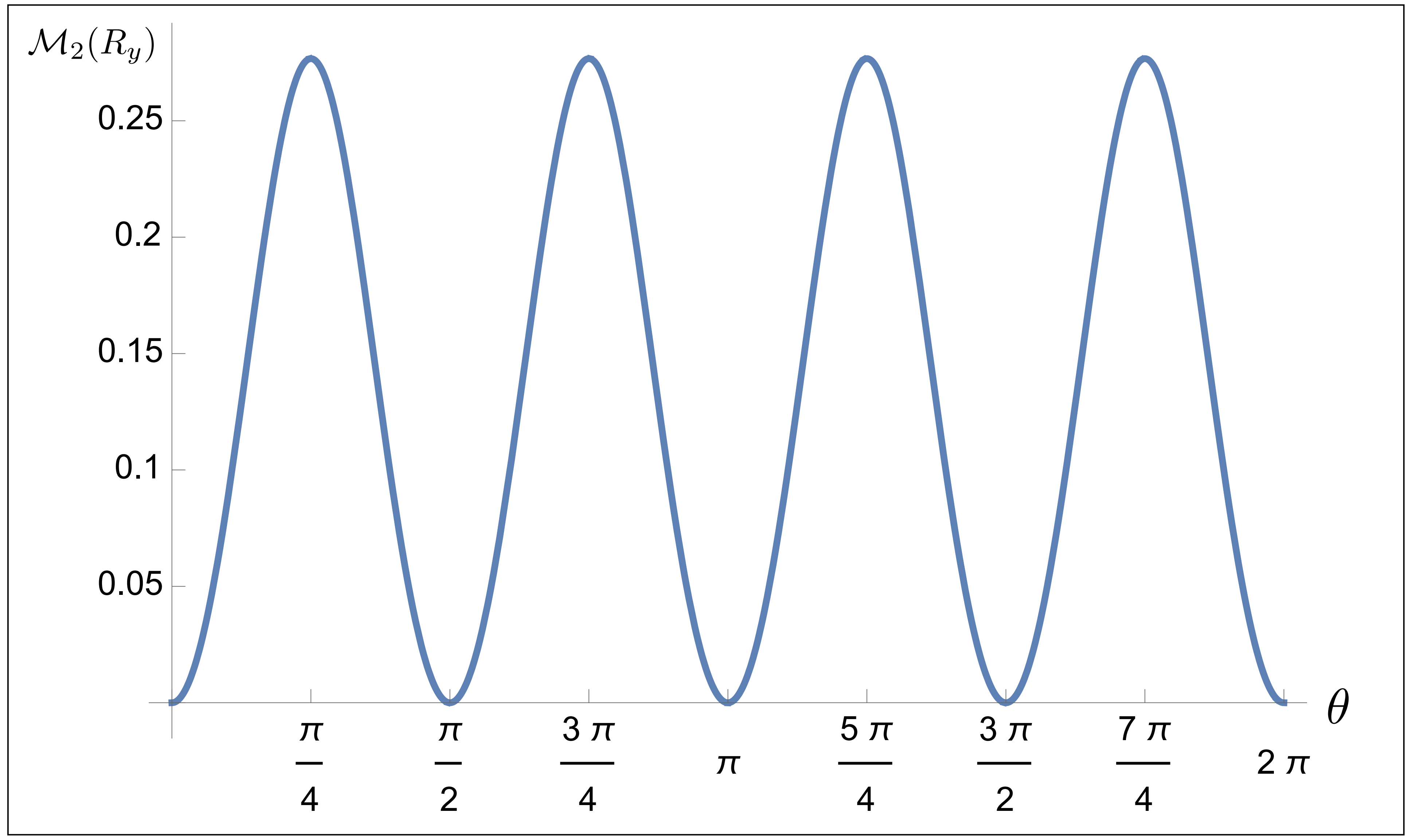
NQSTI National Quantum Science and Technology Institute

INFN Napoli Istituto Nazionale di Fisica Nucleare Sezione di Napoli

Magic power of an operator

The magic power of an operator is defined as the average of the SRE over the set of Clifford states:

$$\mathcal{M}_2 = \frac{1}{|\mathrm{STAB}|} \sum_{|\psi\rangle \in \mathrm{STAB}} M_2(U|\psi\rangle)$$



As an example, one can consider the 1-qubit rotation operator around the y direction:

$$R_y(\theta) = \exp[-i\frac{\theta}{2}Y] = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Y$$

Computing the magic power of the rotation operator one obtains: $\mathcal{M}_2(R_y) = -\frac{2}{3}\log\left(\frac{7+\cos(4\theta)}{8}\right)$

Università degli Studi di Napoli Federico II
NQSTI National Quantum Science and Technology Institute
INFN Napoli Istituto Nazionale di Fisica Nucleare Sezione di Napoli

Bell's violations need magic!

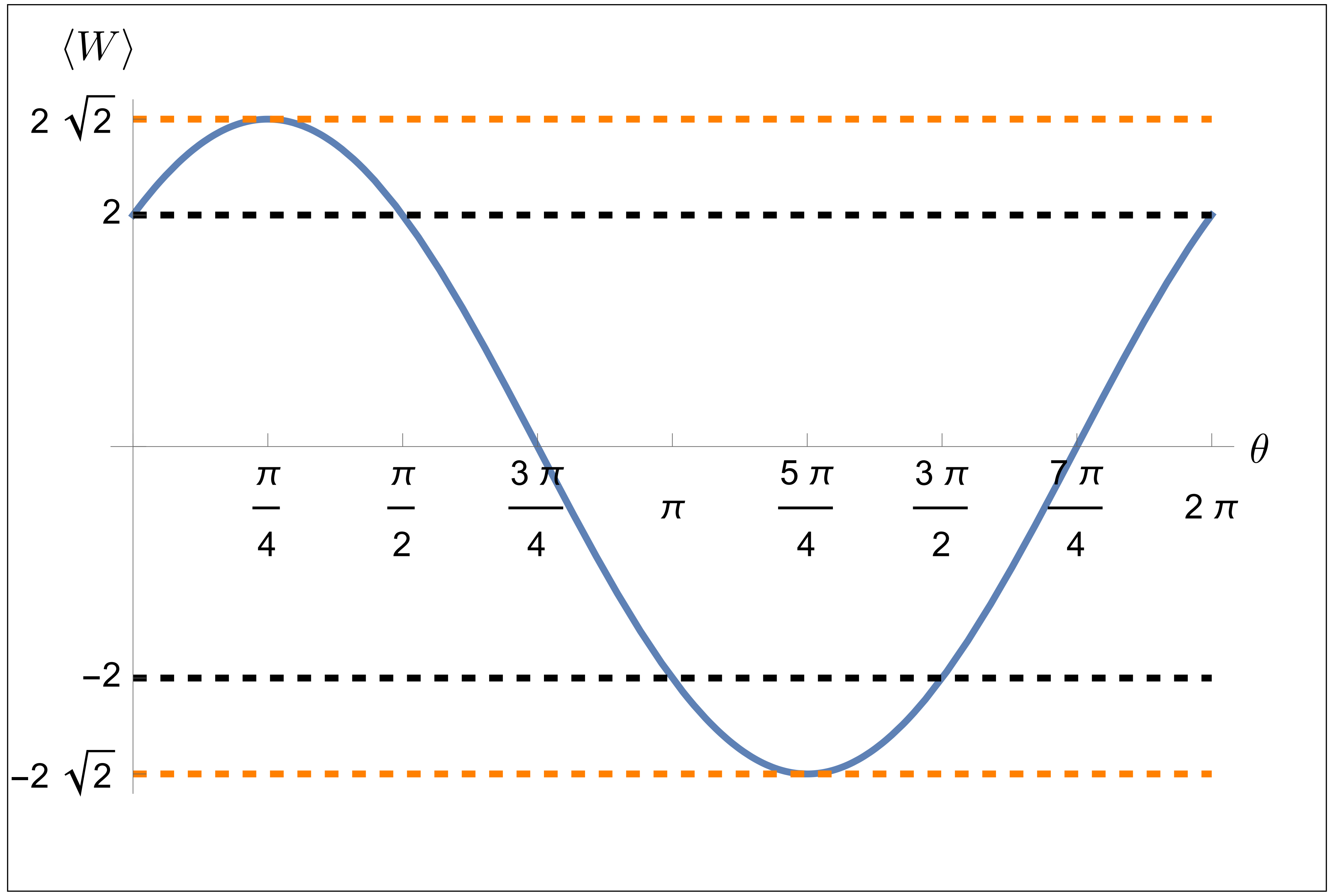
One can see that magic is a necessary ingredient to violate Bell's inequalities

$$-\frac{X+Z}{\sqrt{2}} = R_y^\dagger\left(\frac{\pi}{4}\right) X R_y\left(\frac{\pi}{4}\right) \qquad \frac{-X+Z}{\sqrt{2}} = R_y^\dagger\left(\frac{\pi}{4}\right) Z R_y\left(\frac{\pi}{4}\right)$$

Angles for which the rotation operator is magic free do not lead to violations

$$\langle W \rangle = \mathrm{Tr}\left[\mathrm{R_y^\dagger}(\theta)(\mathrm{X_1X_2 + X_1Z_2 - Z_1X_2 + Z_1Z_2})\mathrm{R_y}(\theta)|\Phi^+\rangle\langle\Phi^+|\right]$$

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$



Maximal violations are obtained for maximum magic power!

Università degli Studi di Napoli Federico II

NQSTI National Quantum Science and Technology Institute

INFN Napoli Istituto Nazionale di Fisica Nucleare Sezione di Napoli

Violations with random pure states

The first thing we want to compute is the probability of violating the CHSH inequality with a random pure state

Using the technique in L. Campos Venuti and P. Zanardi, Phys. Lett. A, 377 1854 (2013):

$$P_W(x) = \langle \delta(\langle\psi|W|\psi\rangle - x)\rangle_\psi = -\frac{3}{8}\mathrm{sign(x)x} + \sum_{\pm} \frac{3(2\sqrt{2} \pm \mathrm{x})^2 \mathrm{sign}(2\sqrt{2} \pm \mathrm{x})}{64\sqrt{2}} \qquad W = X_1X_2 + Z_1X_2 - X_1Z_2 + Z_1Z_2$$

Then one can compute the probability of violating the CHSH inequality with a random pure state as:

$$P_{\mathrm{viol}} = 2\int_2^{2\sqrt{2}} dx\; P_W(x) = \frac{10 - 7\sqrt{2}}{4} \simeq 2.51\%$$

Isospectral twirling

Consider the expectation value:

$$W = \mathrm{Tr}\left[\mathrm{W}\mathrm{V}|00\rangle\langle 00|\mathrm{V}^\dagger\right] = \mathrm{Tr}\left[\mathrm{T}_{(12)}(\mathrm{V}\otimes\mathrm{V}^\dagger)(|00\rangle\langle 00|\otimes\mathrm{W})\right]$$

Given the spectrum of U, we want to average its action over all possible eigenvectors over a set of unitaries:

$$\mathcal{R}_{\mathcal{G}}^{(2k)}(V) = \int_{\mathcal{G}} dG\, G^{\dagger\otimes 2k}(V^{\otimes k}\otimes V^{\dagger\otimes k})G^{\otimes 2k}$$

One can average W over the given group:

$$\langle W\rangle_{\mathcal{G}} = \mathrm{Tr}\left[\mathrm{T}_{(12)}\mathcal{R}_{\mathcal{G}}^{(2)}(\mathrm{V})(|00\rangle\langle 00|\otimes\mathrm{W})\right]$$

Similarly for the standard deviation:

$$W^2 = \mathrm{Tr}\left[\mathrm{W}\mathrm{V}|00\rangle\langle 00|\mathrm{V}^\dagger\right]^2 = \mathrm{Tr}\left[\mathrm{T}_{(13)(24)}(\mathrm{V}^{\otimes 2}\otimes\mathrm{V}^{\dagger\otimes 2})(|00\rangle\langle 00|^{\otimes 2}\otimes\mathrm{W}^{\otimes 2})\right]$$

$$\langle W^2\rangle_{\mathcal{G}} = \mathrm{Tr}\left[\mathrm{T}_{(13)(24)}\mathcal{R}_{\mathcal{G}}^{(4)}(\mathrm{V})(|00\rangle\langle 00|^{\otimes 2}\otimes\mathrm{W}^{\otimes 2})\right] \qquad \sigma_W^{(\mathcal{G})} = \sqrt{\langle W^2\rangle_{\mathcal{G}} - \langle W\rangle_{\mathcal{G}}^2}$$

Università degli Studi di Napoli Federico II

NQSTI National Quantum Science and Technology Institute

INFN Napoli Istituto Nazionale di Fisica Nucleare Sezione di Napoli

Clifford vs Unitary averages

We want to compare the action of a unitary operator when averaged over the Clifford and the unitary group

We notice that the average of W depends on the order 2 (k=1) isospectral twirling. As the Clifford group is a 3-design, there is no difference between the average over the Clifford group and the full unitary group.

$$\langle W \rangle_{\mathcal{U}} = \langle W \rangle_{\mathcal{C}} = \frac{c_2 - 1}{15} \leq 1 \qquad V = \sum_i v_i |v_i\rangle\langle v_i| \qquad c_2 = \sum_{i,j} v_i v_j^*$$

The situation is different when looking at the fluctuations, which depend on the order 4 (k=2) isospectral twirling

$$\langle W^2 \rangle_{\mathcal{U}} = \frac{-4c_2 + c_2' + c_3^* + c_3 + c_4 + 1344}{1680} \qquad \langle W^2 \rangle_{\mathcal{C}} = \frac{7}{9} + \frac{|c_{V^2}|^2 + 2\Re[c_V^{*2} c_{V^2}] + |c_V|^4}{80} - \frac{|c_{VV^\dagger}|^2 + c_{(VV^\dagger)^2}}{72}$$

One then expects differences between the Clifford and unitary averages when looking at fluctuations of W

UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II

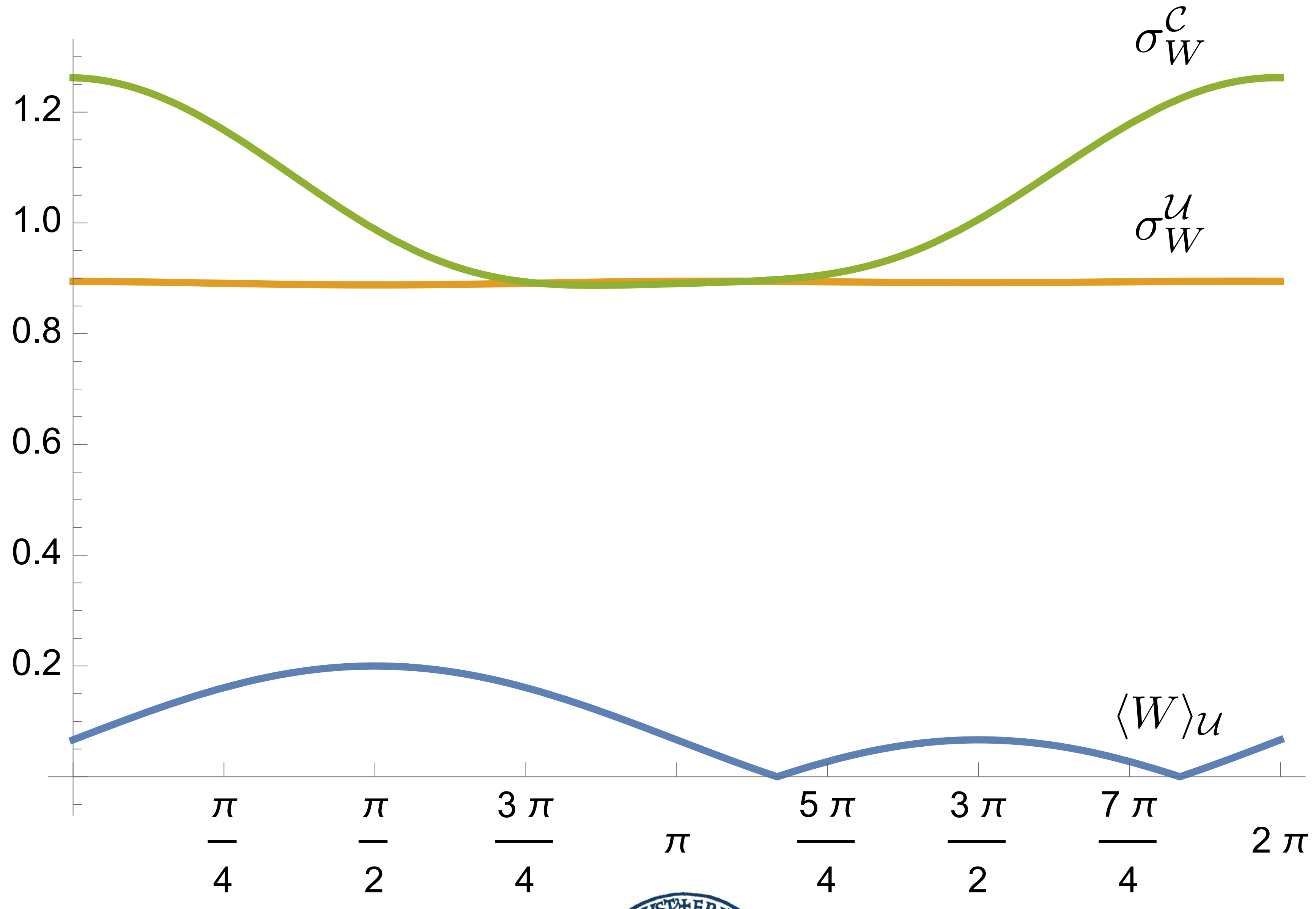
NQSTI National Quantum Science and Technology Institute

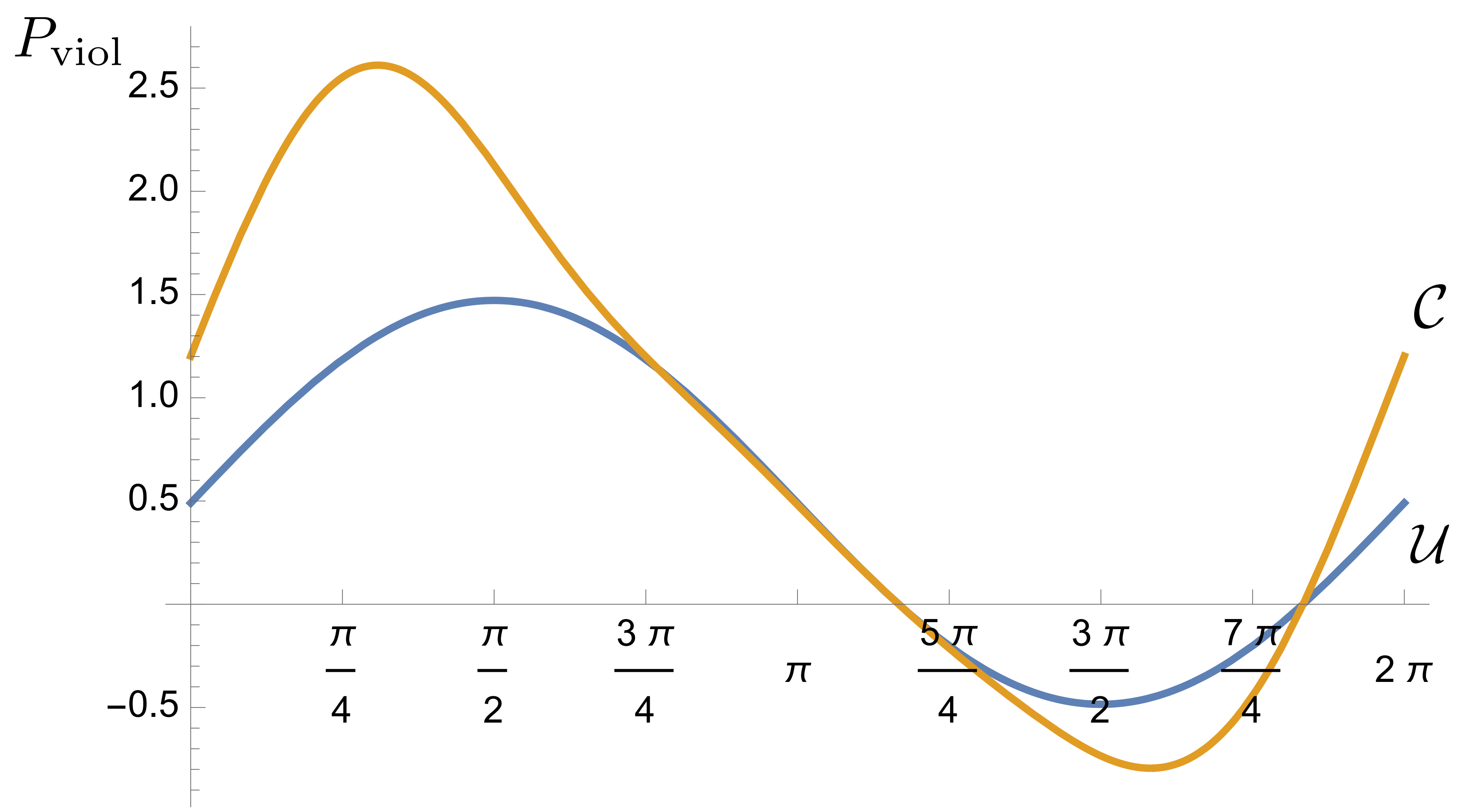
INFN NAPOLI Istituto Nazionale di Fisica Nucleare Sezione di Napoli

Probabilistic violations of the bound

A measure of the probabilistic violation of the CHSH inequality is given by the number of standard deviations between the average and the minimum violation

$$V = R_y(\theta) \circ \text{CNOT} \circ \text{Hadamard}$$





Università degli Studi di Napoli Federico II

NQSTI National Quantum Science and Technology Institute

INFN Napoli Istituto Nazionale di Fisica Nucleare Sezione di Napoli

Conclusions and outlook

What we know

-Magic is a necessary ingredient for Bell's inequality violations

-Results suggest that sampling a unitary with a given spectrum with eigenvectors from the Clifford group is better than sampling from the full unitary group

- Bell's violations are due to the interplay between entanglement and magic

What we want to know

-Averaging for unitaries with spectra taken from important ensembles, e.g. the Gaussian Diagonal Ensemble or the Gaussian Unitary Ensemble

-Scaling of the results with the dimension of the Hilbert space and with the number of parties

-Quantifying the complexity of Bell's inequality

- Fully understanding the interplay between magic and entanglement leading to Bell's violations

Università degli Studi di Napoli Federico II
NQSTI National Quantum Science and Technology Institute
INFN Napoli Istituto Nazionale di Fisica Nucleare Sezione di Napoli

Thank you for the attention!

Università degli Studi di Napoli Federico II

NQSTI National Quantum Science and Technology Institute

INFN NAPOLI Istituto Nazionale di Fisica Nucleare Sezione di Napoli